On the θ-g- Continuity* in topological spaces.

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Abstract

In this paper, we study certain types of continuous functions in topological spaces, where we defined it by using θ-g- neighbourhood .and some properties of these concepts are proved.

المستخلص

فـي هـذا البحـث درسـنا بعـض انـواع الـدوال المسـتمره فـي الفضـاءات التوبولوجيـه وهـي الـدوال (ثيتا – جي – المستمره*) وذلك باستخدام الجوار من النوع (ثيتا – جي – جوار)وقمنا باثبات بعض الخصـائص لهذه الدوال .

1-Introduction and preliminaries

Before we present the θ-g- continuous* mapping we give a historical notations about it ,The subject of θ-closed sets was first studied in 1966 by Velicko [8] ,In 1970, Levine [5] introduced the notion of generalized closed sets in topological spaces as a generalization of closed sets. Since then, many concepts related to generalized closed sets were defined and investigated . The generalizations of generalized closed and generalized Continuity were intensively studied in recent years by Balachandran, Devi ,Maki and Sundaram [4] ,The aim of this paper is to introduce the notions of θ-generalized-continuous* (briefly ,θ-g-cont.*) function , θ-generalized-
homeomorphisms * and study some of their simple properties.

Definition(1-1) [3]: let (x,t) be a topological space and let Y be a subset of X. The t-relative topology for Y is the collection t_y given by $t_y=\{G\cap Y:G \in t\}$.

Definition(1-2)[3] : if $f:X\to Y$ and $A\subset X$, then the mapping $g: A\to Y$ Defined by $g(x)=f(x)$ $x \in X$ is called restriction of f to A and is denoted by $f|A$ or f_A it is evident that $f|A=f\cap(A\times Y)$.

Definition (1-3)[1]: Apoint $x \in X$ is said to be ***θ-adherent point*** of $A\subset X$, if $cl(U) \cap A \neq \emptyset$ for every open U of $x \in X$(such that cl(u) represent the closure of U .The set of all θ-adherent point of A is Denoted by ***clθ(A)*** or θ-cl(A).

Definition(1-4) [1] : A set A is said to be ***θ-closed*** if A =clθ(A) or A= θ-cl A. The complement of a θ-closed set is called ***θ-open*** set.

Definition(1-5) :The set N is ***θ-nhd*** of x if there exist an θ-open $G \ni x \in G\subset N$.

Example(1-6): let X={a,b,c} and τ ={φ, {a}, {b,c},X} , consider the subset A={a} of X clearly {a} is the only θ-adherent point of A Hence ***A is θ-closed***.The complete of A is {b,c} is θ-open.

Example(1- 7): Let X={a,b,c,d,e} and let τ ={φ,{ b},{d,e},{b,d,e},{a,c,d,e},X} Be a topology on X ,consider the subset A={b,c,d} then a, b ,c ,d and e are θ-adherent points of A ,Then the set of all θ-adherent of A is {a, b, c, d, e} = clθ(A) Then A ≠ clθ(A) , hence ***A is not θ-closed set*** .

Remark(1-8) : every ***θ-closed*** sets are ***closed*** but the converse is not true as the following example.

Example (1-9): Let X ={d,e,f} and let τ ={∅,{d},{d,e},{d,f},X} consider the subset B ={f},since the complement of τ are X, ∅,{e,f},{f} and {e} then ***B is closed but not θ-closed*** since the set of all θ-adherent points are {d,e,f}=cl θ(B)≠B .

Definition (1-10) [2]: A subset A of a space (X,T) is called ***θ-g-closed*** if clθ(A) ⊂ U whenever A⊂U and U is open in X . the Complement of ***θ-g-closed*** is ***θ-g-open***.

Example(1-11): As example(1-6)Let X ={d,e,f} and letT={∅,{d},{d,e},{d,f},X}. Consider the subset B ={f} ,***B is closed but not θ-g-closed*** since if consider U ={d,f}. Note that X = Clθ(B) ⊄U ∈ τ .

Remark(1-12) : every ***θ-closed*** sets are ***θ-g-closed*** but the converse is not true as the following example.
Example (1-13): Let X ={d,e,f} and let τ ={∅,{d,e},X}. Consider the subset D ={d,f}. Since the only open subset of D is X, D is clearly θ-generalized closed. But it is easy to see that D is not θ-closed.

Proposition(1-14)[2] : A finite union of θ-g-closed sets is always a θ-g-closed set.

Theorem (1-15)[2] : If A is θ-g-open in (X,τ) and B is θ-g-open in (Y,σ), then A×B is θ-g-open in the product space (X×Y,τ×σ).

Remark (1-16): Every θ-g-open sets are open but the converse is not true as the following example.

Example (1-17): take the complement to the subset B in Example(1-11) it is easily to see that ***B is open but not θ-g***-open.

Definition (1-18): Let x be a point of a topological space X. A subset N of X is said to be ***θ-g- neighbourhood*** of x in X if there exists a θ-g-open Set U⊂X ∋ x ϵ U ⊆ N .
Theorem(1-19): A subset of topological space is *θ-g*-open iff it is a θ-g- neighbourhood of each of its point .
proof :let a subset G of a topological space be θ-g-open then for every x ϵ G, x ϵ G ⊆ G and therefore G is θ-g- neighbourhood of each of its point conversely ,let G be θ-g- neighbourhood of each of its point , then to each x ϵ G there exist an *θ-g*-open set G_x such that x ϵ G_x ⊆ G it follows

that G=U{ G_x:x ϵ G}(by take the complement Proposition(1-14) hence G is *θ-g-* open being a union of *θ-g*-open sets ∎

Definition(1-20) : Let (X, *τ*) be a topological space & A⊆X ,the $\theta - g -$closure of A Is $\theta - g - cl(A) = \cap$ { f:f is θ-g-closed set , f ⊃ A} .

2- θ-g-continuous* mapping

Definition (2-1): let (X, τ_1) and (Y, τ_2) be topological spaces. A mapping f: X→Y is said to be θ-g-continuous* at x_0 ϵ X iff to every θ-g-nhd M of f(x) there exists a θ-g-nhd N of x such that f[N] ⊂M . so f is said to be (τ_1- τ_2)θ-g-cont.*(or simply θ-g-cont. *) iff it is θ-g-cont. * to every points of X it follows from this definition that f is θ-g- continuous* at x_0 ϵ X iff to every τ_2- θ-g-open H containing f(x_o) there exist τ_1- θ-g-open G containing x_o such that f(G) ⊂H .

Definition (2-2) : let (X,t_1) and (Y,t_2) be topological spaces and f be a mapping of X into Y then

1) f is said to be an θ-g- open mapping iff f(G) is t_2-θ-g-open whenever G is t_1-θ-g-open
2) f is said to be a θ-g- homeomorphism* iff
   i) f is bijective
   ii) f is t_1-t_2 θ-g-continuous*
   iii) f^{-1} is t_2-t_1 θ-g-continuous *

Theorem(2-3): Let X and Y be topological spaces. A mapping f: X→Y is θ-g-continuous* if and only if the inverse image under f of Every θ-g-open set in Y is θ-g-open in X.

Proof : Assume that f is θ-g- Continuity* and let H be any θ-g-open set in Y. We want to show that f^{-1}[H] is θ-g-open in X. If f^{-1}[H] = φ, There is nothing to prove. So let f^{-1}[H] ≠ φ and let x ϵ f^{-1}[H] So that f(x)ϵ H. By θ-g-ontinuity* of f, there exists a θ-g-open set G_x in X such that x ϵ G_x and f [G_x] ⊂H, that is, x ϵ G_x ⊂ f^{-1}[H].This shows that f^{-1}[H] is a θ-g-nhd of each of its points and so by Theorem(1-19) it is θ-g-open in X . Conversely, let f^{-1}[H] be θ-g-open in X for every θ-g-open set H in Y .We shall show that f is θ-g-cont.* at x ϵ X. let H be any θ-g-open Set in Y such that f(x)ϵ H so that x ϵ f^{-1}[H] . By hypothesis f^{-1}[H] Is θ-g-open in X . If f^{-1}[H]=G , then G is an θ-g-open set in X Containing x such that f[G]= f[f^{-1}[H]] ⊂H, Hence f is a θ-g-continuous* function ∎

Corollary (2-4): : let X and Y be topological spaces, A mapping $f: X \to Y$ is θ-g-continuous* if and only if the inverse image under f of every θ-g-closed set in Y is θ-g-closed in X.

Proof : Assume that f is θ-g-continuous* and let F be any θ-g-closed set in Y. To show that f^{-1}[F] is θ-g-closed in X. since f is θ-g-continuous* and Y-F is θ-g-open in Y, it follows from theorem(2-3) that f^{-1}[Y-F] = X-f^{-1}[F] is θ-g-open in X, that is , f^{-1}[F] is θ-g-closed in X.

Conversely, let $f^{-1}[F]$ be θ-g-closed in X for every θ-g-closed set F in Y. We want to show that f is a θ-g-continuous* function. Let G be any θ-g-open set in Y. then Y-G is θ-g-closed in Y and so by hypothesis, $f^{-1}[Y-G]=X-f^{-1}[G]$ is θ-g-closed in X, that is , $f^{-1}[G]$ is θ-g-open in X, Hence f is θ-g-continuous* by theorem (2-3) ■

Theorem(2-5): A mapping f from a space X into another space Y is θ-g-Continuous* if and only if $f[\theta-g-cl(A)] \subset \theta-g-cl[f(A)]$ for every A⊂X. Or f is θ-g-continuous* iff for every x ϵ X arbitrarily θ-g-close to A , f(x) is arbitrarily θ-g-close to f[A] .

Proof : let f be θ-g-continuous*. Since $\theta-g-cl[f(A)]$ is θ-g-closed in Y, $f^{-1}[\theta-g-cl[f(A)]$ is θ-g-closed in X [Corollary (2-4)] and therefore

$\theta-g-cl[f^{-1}[\theta-g-cl[f(A)]]]= f^{-1}[\theta-g-cl[f(A)]]$ --------(1)

$Now \quad f[A] \subset \theta-g-cl[f(A)] \Rightarrow A \subset f^{-1}[f[A]] \subset f^{-1}[\theta-g-cl[f(A)]] \Rightarrow$
$\theta-g-\bar{A} \subset \theta-g-cl[f^{-1}[\theta-g-cl[f(A)]]] = f^{-1}[\theta-g-cl[f(A)]]$ by (1)
$\Rightarrow f[\theta-g-cl(A)] \subset \theta-g-cl[f(A)]$

Conversely, let $f[\theta-g-cl(A)] \subset \theta-g-cl[f(A)]$ for every A⊂X. Let F be any θ-g-closed set in Y so that $\theta-g-cl(F) = F$. Now $f^{-1}[F]$ is a subset of X so that by hypothesis

$f[\theta-g-cl[f^{-1}[F]] \subset \theta-g-cl[f[f^{-1}[F]]] \subset \theta-g-cl(F) = F.$

Therefore $\theta-g-cl[f^{-1}[F]] \subset f^{-1}[F]$. But $f^{-1}[F] \subset \theta-g-cl[f^{-1}[F]]$ Always. Hence $\theta-g-cl[f^{-1}[F]] = f^{-1}[F]$ and so $f^{-1}[F]$ is θ-g-closed in X. Hence f is θ-g-continuous* by Corollary (2-4) ■

Theorem(2-6): A mapping f of a space X into another space Y is θ-g-Continuous* if and only if $\theta-g-cl[f^{-1}[B]] \subset f^{-1}[\theta-g-cl(B)]$ for every B⊂Y.

Proof : let f be θ-g-continuous*, since $\theta-g-cl(B)$ is θ-g-closed in Y, $f^{-1}[\theta-g-cl(B)]$ is θ-g-closed in X [Theorem(2-5)] and Therefore

$\theta-g-cl[f^{-1}[\theta-g-cl(B)]] = f^{-1}[\theta-g-cl(B)]$--------(1)

Now $\quad B \subset \theta-g-cl(B) \Rightarrow f^{-1}[B] \subset f^{-1}[\theta-g-cl(B)]$
$\Rightarrow \theta-g-cl[f^{-1}[B]] \subset \theta-g-cl[f^{-1}[\theta-g-cl(B)] = f^{-1}[\theta-g-cl(B)]$
by (1). Conversely, let the condition hold and let F be any θ-g-closed set in Y so that $\theta-g-cl(F) = F$. By hypothesis. $\theta-g-cl[f^{-1}[F]] \subset f^{-1}[\theta-g-cl(F)] = f^{-1}[F]$. But $f^{-1}[F] \subset \theta-g-cl[f^{-1}[F]]$ always.

Hence $\theta-g-cl[f^{-1}[F]] = f^{-1}[F]$ and so $f^{-1}[F]$ is θ-g-closed in X. It follows from Corollary (2-4) that f is θ-g-continuous* ■

Theorem(2-7): let X,Y and Z, be topological spaces and the mappings $f: X \to Y$ and $g: Y \to Z$ be θ-g-continuous* . Then the composition map $gof: X \to Z$ is θ-g-continuous*.

Proof : let G be any θ-g-open set in Z . since g is θ-g-continuous* , $g^{-1}[G]$ is θ-g-open in Y by theorem(2-3). Again since f is θ-g-continuous*, $f^{-1}[g^{-1}[G]]$ is θ-g-open in X [theorem(2-3)]. But $f^{-1}[g^{-1}[G]] = (f^{-1} o\, g^{-1})[G] = (gof)^{-1}[G]$. Thus the inverse image under gof of very θ-g-open set in Z is θ-g- open in X and therefore gof is θ-g-continuous* by theorem(2-3) ■

Theorem (2- 8): Let X and Y be topological spaces and A a non empty subset of X if f :X→Y is θ-g- continuous* then the restriction f_A:A→Y f_A of f to A is θ-g- continuous* where A has relative topology .

Proof : by definition (1-2) let G be any be any open subset of Y then by definition of f_A it is evident that $f_A^{-1}(G) = A \cap f^{-1}(G)$. Since f is θ-g-continuous*, $f^{-1}(G)$ is θ-g- open is θ-g- open in X theorem (2-3) hence by definition (1-1) $A \cap f^{-1}(G)$ is open in A . It follows by theorem (2-3) that f_A is θ-g- continuous* function ■

Theorem (2-9): The projection h : (X × Y,τ × σ) → (X,τ) is a θ-g-cont.* map.

Proof: By definition(1-10) and Theorem (1-15), for a θ- generalized closed set d of (X,τ), $h^{-1}(x\backslash d)$ = (X\d)×Y is θ-g-open in (X×Y,τ×σ). Therefore, $h^{-1}(d)$ = F×Y =X×Y\(h^{-1}(X\d)) is θ-generalized closed ■

Theorem(2 -10): let (X,t_1) and (Y,t_2) be topological spaces and let f be A bijective mapping of X to Y. then the following statements are equivalent:

1) f is a θ-g- homeomorphism*
2) f is θ-g-continuous* and θ-g- open
3) f is θ-g-continuous* and closed

Proof : 1↔2 : asumme(1) let g be the inverse mapping of f so that f=g and g^{-1}= f since f is one to one onto ,g is one to one onto . let G be t_1-θ-g- open set .since g is θ-g-continuous* g^{-1}(H) is t_1-θ-g- Open but g^{-1}=f so that g^{-1} (G) = f(G) is t_1-θ-g- open It follows that f is an θ-g-open mapping . also f is θ-g-continuous* by hypothesis .Hence (1)→(2) Conversely, assume (2) that is let f be a bijective ,θ-g-continuous* and θ-g- open . To prove that g=f^{-1} is θ-g-continuous* . Let G be any t_1-θ-g- open set , then f(G) is t_2-θ-g- open by hypothsis , that is, g(G) is t_2-θ-g- open and so g=f^{-1} is θ-g-continuous* hence (2)→(1) (1)→(3) assume (1) let h be any closed set then X-H is θ-g- open since g=f^{-1} is θ-g-continuous* it follows that g^{-1}(X-H) is t_2-θ-g- open but g^{-1}(X-H) =Y- g^{-1}(H) hence Y- g^{-1}(H) is t_2-θ-g- open that is g^{-1}(H)=f(H) is t_2-closed thus it is shown that H is t_1-closed implies f(H) is t_2-closed hence f is closed mapping thus (1)→(3) now assume (3) to prove that g=f^{-1} is θ-g-continuous* let G be any t_1-θ-g- open then X-G is t-closed since f is closed mapping f(X-G) = g^{-1}(X-G)=Y- g^{-1}(G) is y-closed , that is , g^{-1}(G) is t_2–θ-g- open thus inverse image g of every t_1-θ-g- open set is θ-g-open hence g=f^{-1} is θ-g-continuous* and so (1)→(3) ■

Remark(2-11): for more details about the relations between θ-continuous and ,g-continuous you can see [2],[4],[5],[6] and [7] .

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