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On The Intersection of Young's Diagrams Core

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الملخص:

في هذا البحث الذي يدخل ضمن موضوع التمثيل الرياضي سنقدم مفهوم "القيمة الدليل" وكذلك "المخطط الرئيسي" لأية قلب– e والتي تؤخذ من التجزئة μ للعدد الصحيح الموجب r ومن خلالهما نقدم فكرة تقاطع هذه المخططات.

Abstract:

In this paper, we will introduce the method of intersection of β-numbers for any partition μ of a non-negative integer r. The results of this intersection are represented and specificated the exactly position according to a "*guide value*" and a "*main diagram*".

By using the same method we will create a new way for this intersection after finding the core of each "guide".

1. Introduction:

Let r be a non-negative integer. A *composition* μ of r is a sequence $(\mu_1, \mu_2, ..., \mu_n)$ of non-negative integers such that

$$|\mu| = \sum_{j=1}^{n} \mu_j = r.$$

A composition μ is a *partition* if $\mu_j \geq \mu_{j+1}$ for all $j \geq 1$.

In this paper, we fix μ as a partition. Choose an integer b greater than or equal the number of parts of a partition μ, and define: $\beta_i = \mu_i + b - i$ for $1 \leq i \leq b$. The set $\{ \beta_1, ..., \beta_b \}$ is said to be a set of *β-number* for μ. For example, if μ=(7,5,5,3,2,2,2,1) then the number of parts of μ is 8, then β-numbers are (15,12,11,8,6,5,4,2,0) for $b = 9$.
For more detail of β-numbers, see [2,4].

In the following diagram; see [5], we will represent β-numbers by many runners depending on $e \geq 2$, as follows:

$$\left.\begin{array}{cccc} \underline{\text{run.1}} & \underline{\text{run.2}} & & \underline{\text{run.e}} \\ 0 & 1 & \ldots & e\text{-}1 \\ e & e+1 & \ldots & 2e\text{-}1 \\ 2e & 2e+1 & \ldots & 3e\text{-}1 \\ . & . & & . \\ . & . & & . \\ . & . & & . \end{array}\right\} \ldots \text{(A).}$$

Also, we can create a new abacus configuration by moving all beads as high as possible on each runner, this operation is called the *e-core* of μ. From the above example, if $e = 2$ then diagram (A) and $e - core$ are represented by the following:

$$\begin{array}{cc} 0 & 1 \\ 2 & 3 \\ 4 & 5 \\ 6 & 7 \\ 8 & 9 \\ 10 & 11 \\ 12 & 13 \\ 14 & 15 \end{array} \xrightarrow[b=9]{\beta\text{-number}} \begin{array}{cc} \bullet & - \\ \bullet & - \\ \bullet & \bullet \\ \bullet & - \\ \bullet & - \\ - & \bullet \\ \bullet & - \\ - & \bullet \end{array} \xrightarrow{2\text{-core}} \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & - \\ \bullet & - \\ \bullet & - \\ - & - \\ - & - \end{array} \;,$$

and if e =3

$$\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \\ 12 & 13 & 14 \\ 15 & 16 & 17 \end{array} \xrightarrow[b=9]{\beta\text{-number}} \begin{array}{ccc} \bullet & - & \bullet \\ - & \bullet & \bullet \\ \bullet & - & \bullet \\ - & - & \bullet \\ \bullet & - & - \\ \bullet & - & - \end{array} \xrightarrow{3\text{-core}} \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & - & \bullet \\ \bullet & - & \bullet \\ \bullet & - & \bullet \\ - & - & - \\ - & - & - \end{array} \;.$$

A partition μ of r is called $w - regular$; $w \geq 2$, if there does not exist $i \geq 1$ such that $\mu_i = \mu_{i+w-1} > 0$. Also, μ is $w - restricted$ if $\mu_i - \mu_{i+1} < w$, for all i, or equivalently if the conjugate partition μ'; which is given by:

$$\mu_i' = \left|\{ j \geq 1 : \mu_j \geq i \}\right| \text{ is } \textit{w-regular}.$$

2. The intersection of β-numbers:

In this section, we will produce the principle of intersection of a "*mains diagrams*" or "*guides diagrams*" which correspond to the values of "b guides". The results of this operation are specified the number and the position of beads in diagram (A). By choosing e then b we will see that diagram (A) will be the <u>same</u> as the above with another value of b, since $b \geq n.$ Such repetition occurs when the second b is greater than the first. Diagram (A) will be as "Down-shifted", as follows:
Let $\mu = (5^2,2,1)$ and e =4 then :

e =4	b =8	b =12	b =16
0 1 2 3	• • • •	• • • •	• • • •
4 5 6 7	- • - •	• • • •	• • • •
8 9 10 11	- - - •	- • - •	• • • •
12 13 14 15	• - - -	- - - •	- • - •
16 17 18 19	- - - -	• - - -	- - - •
20 21 22 23	- - - -	- - - -	• - - -

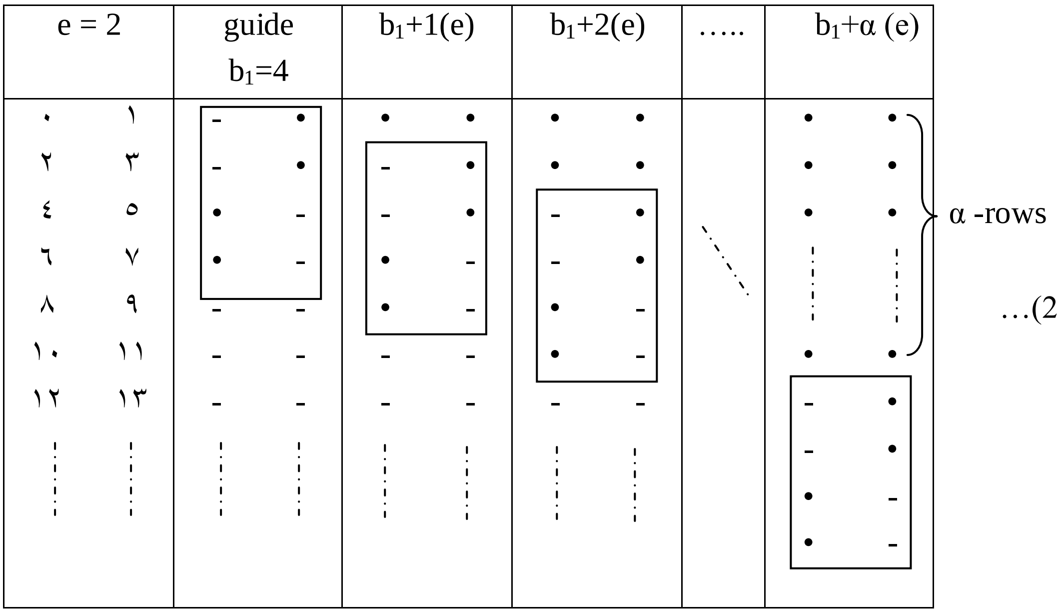
….(2.1)

Since the difference between $b_1 = 8$ and $b_2 = 12$ is 4 which the value of e, then "down-shifted" will decrease one row of diagram (A), and the difference between $b_1 = 8$ and $b_3 = 16$ is $2e$, which explain why the "down - shifted" occurs twice.
It may happens an "Up-shifted" will occurs in diagram (A), if we choose b_1 then $b_2 < b_1$ where $b_1 - b_2 = ne, \quad n = 0,1,....$ In (2.1) by taking $b = 4$, canceling the first row in $b_1 = 8$ and repeating the remainder part of $b_1 = 8$ in $b = 4$.

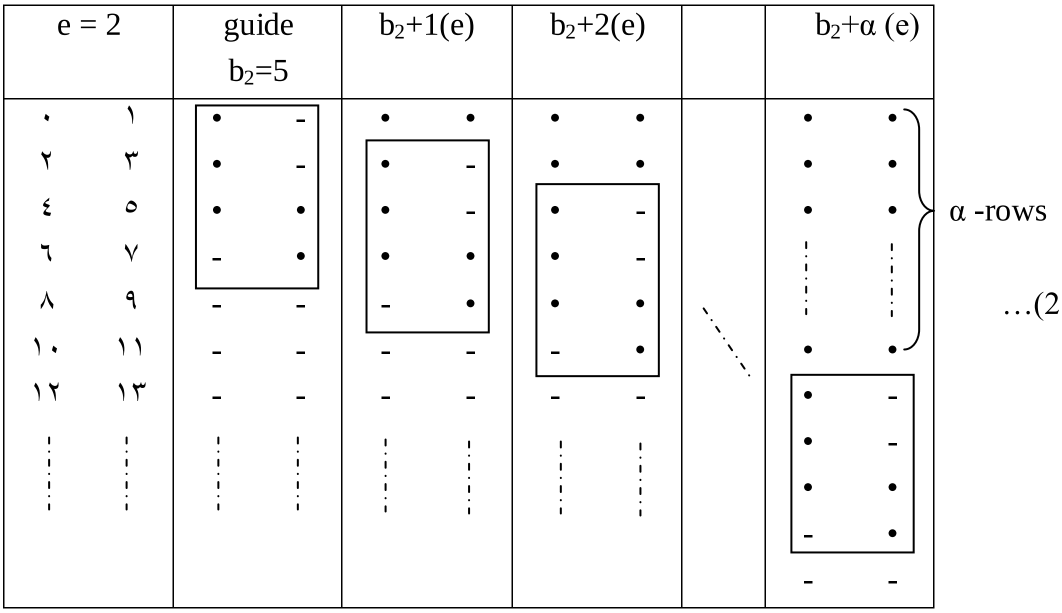
Depending on the above we can assume the following definition:

Definition (2.2): For any diagram (A) and $\beta - numbers$, the values of $b = n, n+1,..., n+(e-1)$ are called the "*guides*" where n is the number of parts of the partition μ of r.

For example, the guides of μ =(3,2,2,1) are b_1=4 and b_2=5 if e=2, this will be illustrated by the following:



…(2.3)



…(2.4)

We will define any diagram of (A) that corresponds any b guide, as a "Main diagram" or "Guide diagram".

Theorem (2.5): There is e of mains diagrams (A) for any partition μ of r.
<u>Proof:</u> By using the mathematical induction. Let e=2, then we have 2 guides $b_1 = n$ and $b_2 = n+1$, and each guide corresponds to a special diagram (A). This means, we have two different diagrams because if we choose any value of b, then our diagram corresponds to $(b_1 + \alpha e)$ or

$(b_2 + \gamma e)$ where α and γ are e multiples which will fill α rows in diagram (A) and then a "down-shifted" for b_1 and γ rows in diagram (A) and then do a "down-shifted" for b_2 respectively.
Now, let e=3, then it is very clear the values of guides are $b_1 = n,$ $b_2 = n+1,$ and $b_3 = n+2.$ Repeating the same operation of the first part in this proof, so we have only three different diagrams (A).
Considering the condition of number of diagrams are satisfied if $e = z.$ Suppose we have $(e+1)$-diagrams (A) for $e = z+1$ and

$$b_1 = n,\ b_2 = n+1,\ \ldots\ b_e = n+z \text{ and } b_{e+1} = n+z+1.$$

This means that there exists a case with $b < n$we take a case for $b < n$ which is a contradiction. Since $b \geq n$ always. So we have only e-diagrams (A)■

The reason of taking the intersection of the main diagrams (A) is to know the intersection locations of them and in which e may exist intersection points or not. Then we want the following:

i) Let τ be a number of redundant part of the partition μ of r, then: $\mu = (\mu_1, \mu_2, .., \mu_n) = \left(\mu_1^{\tau_1}, \mu_2^{\tau_2}, ..., \mu_m^{\tau_m}\right),$ such that $r = \sum_{j=1}^{n} \mu_j = \sum_{k=1}^{m} \mu_k^{\tau_k}$.

ii) We will denote the intersection of main diagrams by $\bigcap_{s=1}^{e} m.d._{b_s}.$

iii) The result of the intersection will be a numerical value which denote the existence of many common locations. If such location dose not exist we denote it by ϕ.

Theorem (2.6): For any $e \geq 2,$ the following are holds:

1) $\bigcap_{s=1}^{e} m.d._{b_s} = \phi$ if $\tau_k = 1$ for all $1 \leq \mathrm{k} \leq \mathrm{m}$.

2) Let Ω be the number of parts of μ which satisfies the condition $\tau_k \geq e$ for some k , then $\bigcap_{s=1}^{e} m.d._{b_s} = (\sum_{\omega=1}^{\Omega} \tau_\omega - \Omega(e-1))$.

Proof:

1) Since every part of μ of r has repeated just for one time; $n = m$, and by choosing b_1, the location of μ_1 will be different of the location of

μ_1 in $b_2, b_3, ..., b_e$ respectively where $b_1 < b_2 < ... < b_e$. So there will be not any common location for the other parts of μ which has been repeated for just one time and then $\bigcap_{s=1}^{e} m.d._{b_s} = \phi$.

2) The different case of the first condition of this theorem will lead us to one of the following:

   i) Some of these repetitions are less than the chosen e and let it be $(e-a)$ for some μ_m and this part will be equal to $b_1, b_2, ..., b_{e-a}$. Thus there is no effect of this location with the intersection's results for all $1 \le s \le e$ and this result will be ϕ.

   ii) By i) some repetitions will be equal to the chosen e. Since there exists just one case for all μ_m which verify this condition, then $\bigcap_{s=1}^{e} m.d._{b_s} = \sum_{\omega=1}^{\Lambda} \tau_\omega$ where Λ is the sum of all parts which verify the condition.

   iii) Let Ω be the number of the remaining repetitions which be greater than the chosen e for some μ_m. Since the minimum value will be canceled of β_σ where $1 \le \sigma \le m$ of the parts μ_m of $e = 2,$ if $e = 3$ the minimum value and the above will be canceled and so on, then $\bigcap_{s=1}^{e} m.d._{b_s} = (\sum_{\omega=1}^{\Omega} \tau_\omega - \Omega(e-1))$.

Examples:

1) let μ =(3,2,1), then β-numbers for b_1=3 are (5,3,1) and for b_2=4 are (6,4,2,0); then $\bigcap_{s=1}^{2} m.d._{b_s} = \phi$.

2) let μ =(3,2^2,1), if we choose b_1=4 then β-numbers are (6,4,3,1) and (7,5,4,2,0) if b_2=5; then $\bigcap_{s=1}^{2} m.d._{b_s} = 1$. Taking e=3, implies that $\bigcap_{s=1}^{3} m.d._{b_s} = \phi$.

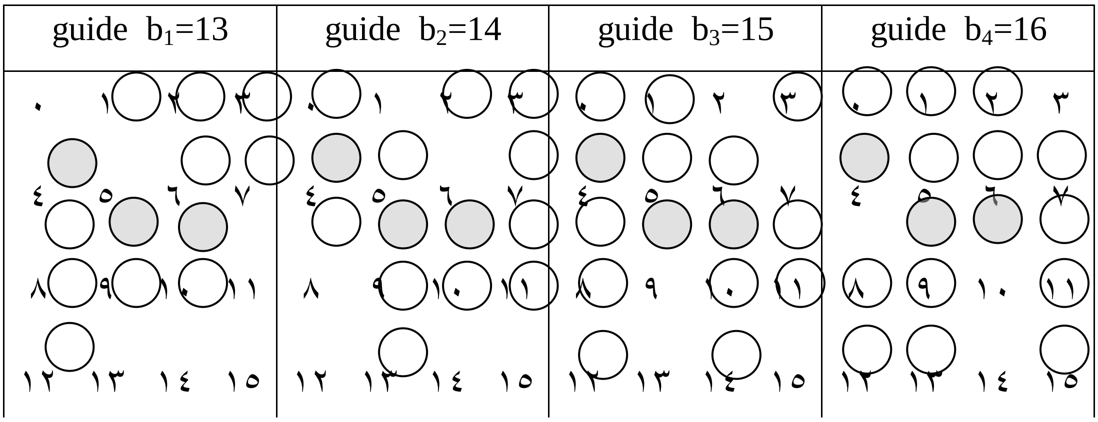
3) let μ =(4,3^3,2^5,1^4), then:

$$\bigcap_{s=1}^{2} m.d._{b_s} = 9, \quad \bigcap_{s=1}^{3} m.d._{b_s} = 6, \quad \bigcap_{s=1}^{4} m.d._{b_s} = 3, \quad \bigcap_{s=1}^{5} m.d._{b_s} = 1$$

and

$$\bigcap_{s=1}^{e} m.d._{b_s} = \phi \text{ when } e \ge 6.$$

To illustrate the meaning of $\bigcap_{k=1}^{4} m.d._{b_s} = 3$, then:



…(2.7)

and

$$=\bigcap_{s=1}^{4} \text{m.d.}_{b_s} \quad \begin{array}{cccc} - & - & - & - \\ \bullet & - & - & - \\ - & \bullet & \bullet & - \\ - & - & - & - \\ - & - & - & - \end{array} .$$

Remark (2.9): The main diagram (A) in the case $b_1 = n$ play a main role in the intersection's result, because any bead in location f of $m.d._{b_1}$ will be in the location $(f+1)$ in $m.d._{b_2}$ …and in the location $(f+(e-1))$ in $m.d._{b_e}$ respectively. By many repetitions we can obtain the intersection result. (See the following):

In (2.7), the $m.d._{b_1}$ is

e =4				b_1 =13				
0	1	2	3	-	•	•	•	I
4	5	6	7	•	-	•	•	II
8	9	10	11	•	•	•	-	
12	13	14	15	•	•	•	-	III
16	17	18	19	•	-	-	-	IV

….(2.10)

In I, there are four beads, since $e=4$ then the last bead in location 4 will be considered. In II, there are five beads which means that we will take the 9th and 10th beads respectively. In III and IV which have less than the chosen e, we see that there is no existence of common beads. So the result of this intersection will be only 3 beads.

By using theorem (2.6) and remark (2.9), we have the following theorems:

Theorem (2.10): Let μ be a partition of r be a $w-$regular, then:

$$\bigcap_{s=1}^{e} m.d._{b_s} = \begin{cases} value & if \quad e < w, \\ \phi & if \quad e \geq w. \end{cases}$$

Theorem (2.11): A partition μ of r be a $h-$restricted, then:

$$\bigcap_{s=1}^{e} m.d._{b_s} = \begin{cases} value & if \quad e < h \quad or \quad (e = h \quad and \quad h < w), \\ \phi & if \quad e > h \quad or \quad (e = h \quad and \quad h \geq w). \end{cases}$$

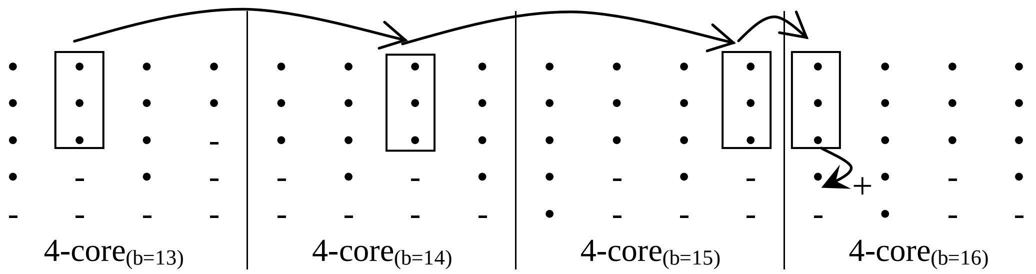
3. The intersection of e -core:

We will use the same previous technique to find the intersection of e-core for all guides. By the definition of e-core and the definition (2.2), each guide corresponds to its one e-core.

Remark (3.1):

According to all notes about e-core in [1] and [5] and the second section of this paper, then: "All beads in all runner of core_{b_i} will be right-shifted in the next $\text{core}_{b_{i+1}}$, under consideration that the last runner (run.e) will be right-shifted to (run.1) adding 1 extra bead."

From above example, μ =(4,3^3,2^5,1^4) and (2.7), then:



…(3.2)

Remark (3.3): For any $e \geq 2$ and n be the number of parts of partition μ of r, the only guide of $e-core$ if $b = n$.

Theorem (3.4): The intersection of e-cores for any partition μ of r; $\bigcap_{c=1}^{e} e - core_{b_c}$ depends on the guide of the e-core, represented by every bead in position 0,1,…. and stopped in the first absence of the bead.

Proof:

By using the right-shifted in remark (3.1) and the guide of e-core in remark (3.3), then any absence of beads in guide of e-core will effect to the result of this intersection.

From the above example in (3.2), then

$$= \bigcap_{j=1}^{4} 4\text{-}\mathrm{core}_{bj} \qquad \begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & - \\ - & - & - & - \\ - & - & - & - \end{array} \quad .$$

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