Development of the Adomian's Method for Solving non-linear Fredholm-Fredholm Integral Equations

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الملخص:

اقترحت طريقة جديدة مباشرة ومقبولة لحساب حل عددي للمعادلة التكاملية غير الخطية من نوع فريدهولم–فريدهولم.

أساس الطريقة هي حل متسلسلة لكمية غير معرفة بطريقة ادمون. حصلنا على نتائج أفضل مقارنة بطريقة ادمون. بعض الأمثلة قدمت لتوضيح دقة وحساب الفعالية لهذه الطريقة.

Abstract:

A new and effective direct method to determine the numerical solution of non- linear Fredholm-Fredholm integral equations is proposed.

The method is based on a series solution for the unknown quantity by Adomian's method. We obtained very good results compared with classic Adomian's method Some numerical examples are provided to illustrate accuracy and computational efficiency of the method.

1. Introduction

Over several decades, numerical method in electromagnetic have been the subject of extensive researches. many problems in electromagnetic can be modeled by integral and integro-differential equation for example, electric field integral equation (EFIE) and magnetic field integral equation (MFIE)[3]. Some papers studied the problem of existence and uniqueness of solution of non-linear Volterra-Fredholm integral equations see [1,6,7]. Some authors use Adomian's method for solving this equations[4] .

In this article we present Adomian's method solution solving non-linear integral equation of Fredholm-Fredholm type:-

$$\phi(x) = f(x) + \int_a^b k_1(x,t)\, g_1(\phi(t))\, dt + \int_a^b k_2(x,t)\, g_2(\phi(t))\, dt \qquad x,t \in [a,b] \quad \text{……..(1.1)}$$

where $g_1(\phi(x)) = \phi^{p_1}(x)$, $g_2(\phi(x)) = \phi^{p_2}(x)$, are non-linear function of ϕ, $p_1 \neq p_2 \geq 2$ are positive integers, $k_1(x,t)$ *and* $k_2(x,t)$ are referred to as the kernel and $f(x)$ a given function, g , k *and* f are non functions.

2. Analysis:

In this section, we first describe the algorithm of Adomian's method as it applies to a general non-linear equation of the form[2]:

$$\phi = f + N(\phi) \qquad \text{………………………….(2.1)}$$

or

$$\phi - N(\phi) = f \qquad \text{………………………..(2.2)}$$

where N is a non-linear operator on a Hilbert space H and f is a known element of H. we assume that for a given f .

It is well known that (AM) considers $\phi(x)$ as an infinite sum of components $\phi_n(x)$ defined by:

$$\phi(x) = \sum_{n=0}^{\infty} \phi_n(x) = \phi_o + \phi_1 + \phi_2 + \phi_3 + \ldots. \qquad \text{…………………..(2.3)}$$

and the non-linear operator N can be decomposed into

$$A_o(\phi_o) + A_1(\phi_o,\phi_1) + A_2(\phi_o,\phi_1,\phi_2) + A_3(\phi_o,\phi_1,\phi_2,\phi_3) + \ldots\ldots\ldots \qquad \text{...(2.4)}$$

that is :

$$N(\phi) = \sum_{n=0}^{\infty} A_n(\phi_o,\phi_1,\phi_2,..\phi_n) \qquad \text{..……………………………(2.5)}$$

where A_n are the so-called Adomian polynomial's.

Substituting (2.3) and (2.5) into the functional equation (2.2) yields:

$$\sum_{n=0}^{\infty} \phi_n(x) - \sum_{n=0}^{\infty} A_n = f \qquad \text{..……………………………(2.6)}$$

If the series in (2.6) is convergent, then (2.6) holds upon setting:

$$\begin{aligned} \phi_o &= f \\ \phi_1 &= A_o(\phi_o) \\ \phi_2 &= A_1(\phi_o,\phi_1) \qquad \text{..………………………..(2.7)} \\ &\vdots \\ \phi_n &= A_{n-1}(\phi_o,\phi_1,\phi_2,\ldots\ldots.\phi_{n-1}) \end{aligned}$$

Thus, one can recursively determine every term of the series $\sum_{n=o}^{\infty}\phi_n$. The convergence of this series has been established. The two hypotheses necessary for proving convergence of the Adomian's method as given in [5] are as follows.

Condition 2.1: The non- linear function equation (2.2) has a series solution $\sum_{n=o}^{\infty}\phi_n$ such that $\sum_{n=o}^{\infty}(1+\in)^n|\phi_n|<\infty$. Where $\in>o$ may be very small.

Condition (2.2): The non-linear operator $N(\phi)$ can be developed in series $N(\phi)=\sum_{n=o}^{\infty}\alpha_n\phi^n$.

These hypotheses, for proving convergence, are generally satisfied in physical problems [5].

The modified Adomian method [4] may be roughly described as a reassignment of the initial approximates ϕ_o *and* ϕ_1. In particular, if f is split into two functions, say, $f=f1+f2$, then we may rewrite (2.7) as:

$$\begin{aligned}
\phi_o &= f1\\
\phi_1 &= f2+A_o(\phi_o)\\
\phi_2 &= A_1(\phi_o,\phi_1)\\
&\vdots\\
\phi_n &= A_{n-1}(\phi_o,\phi_1,\phi_2,.......\phi_{n-1})
\end{aligned} \qquad \text{.................................(2.8)}$$

The choice of how to assign ϕ_o *and* ϕ_1 is experimental, yet it leads to less computational and does accelerate the convergence.

To compute Adomian polynomials we as a new method mentioned in [4]. Consider the equation (1.1), to solve (1.1) by AM, we write

$$\phi(x)=\sum_{n=0}^{\infty}\phi_n(x) \qquad \text{..(2.9)}$$

and

$$\begin{aligned}
\phi^{p1}(x) &= \sum_{n=o}^{\infty}A_n(\phi_o,\phi_1,\phi_2,.....)\\
& \qquad p_1\neq p_2\geq 2 \qquad \text{..................(2.10)}\\
\phi^{p2}(x) &= \sum_{n=o}^{\infty}A_n(\phi_o,\phi_1,\phi_2,.....)
\end{aligned}$$

and substituted the series (2.3) , (2.9) and (2.10) in to (1.1) giving

$$\sum_{n=o}^{\infty}\phi_n(x) = f(x) + \int_a^b K_1(x,t)(\sum A_n(\phi_o(t),\phi_1(t),......\phi_n(t)))\ dt + \qquad \text{.....(2.11)}$$

$$\int_a^b K_2(x,t)(\sum A_n(\phi_o(t),\phi_1(t),......\phi_n(t)))\ dt$$

or

$$\phi_o + \phi_1 + \phi_2 + \phi_3 + ... = f(x) + \int_a^b K_1(x,t)(A_o + A_1 + ...)dt + \qquad \text{...........(2.12)}$$

$$\int_a^b K_2(x,t)(A_o + A_1 + ...)dt$$

If the series is convergent, then we can determined each term of the series $\sum_{n=o}^{\infty}\phi_n(x)$ recursively:

$$\phi_o = f(x)$$

$$\phi_1 = \int_a^b K_1(x,t)\,A_o(\phi_o)\ dt + \int_a^b K_2(x,t)\ A_o(\phi_o)\ dt$$

$$\phi_2 = \int_a^b K_1(x,t)\,A_1(\phi_o,\phi_1)\ dt + \int_a^b K_2(x,t)\,A_1(\phi_o,\phi_1)\ dt \qquad \text{.........(2.13)}$$

$$\vdots$$

$$\phi_n = \int_a^b K_1(x,t)\,A_{n-1}(\phi_o,......\phi_{n-1})\ dt + \int_a^b K_2(x,t)\ A_{n-1}(\phi_o,......\phi_{n-1})\ dt$$

The algorithm in (2.13) determines the ϕ_n's and hence the solution ϕ can be determined by (2.3). we will also apply the modified decomposition by writing $f = f1 + f2$ with appropriate choice for ϕ_o *and* ϕ_1.

Then the Adomian polynomial A_o depends up on ϕ_o with order 0, A_1 depends up on ϕ_o *and* ϕ_1 with order 1 and so forth, see[4], we will have A_n as follows:

$$A_o = g(\phi_o)$$

$$A_1 = \phi_1 \; g^{(1)}(\phi_o)$$

$$A_2 = \phi_2 \; g^{(1)}(\phi_o) + \frac{\phi_1^2}{2!} g^{(2)}(\phi_o)$$

$$A_3 = \phi_3 \; g^{(1)}(\phi_o) + \phi_1 \phi_2 \, g^{(2)}(\phi_o) + \frac{\phi_1^3}{3!} \; g^{(3)}(\phi_o)$$

$$A_4 = \phi_4 g^{(1)}(\phi_o) + \phi_1 \phi_3 \, g^{(2)}(\phi_o) + \frac{\phi_2^2}{2!} g^{(2)}(\phi_o) + \frac{\phi_1^2}{2} \phi_2 \, g^{(3)}(\phi_o)$$

$$A_5 = \phi_5 \, g^{(1)}(\phi_o) + \phi_1 \phi_4 \, g^{(2)}(\phi_o) + \phi_2 \phi_3 \, g^{(2)}(\phi_o) + \frac{\phi_1^2}{2} \phi_3 g^{(3)}(\phi_o) + \frac{1}{2} \phi_2^2 \phi_1 g^{(3)}(\phi_o)$$

$$A_6 = \phi_6 \, g^{(1)}(\phi_o) + \phi_1 \phi_5 g^{(2)}(\phi_o) + \phi_2 \phi_4 \, g^{(2)}(\phi_o) + \phi_1 \phi_2 \phi_3 g^{(3)}(\phi_o) + \frac{\phi_3^2}{2} g^{(2)}(\phi_o) + \frac{\phi_1^2}{2} \phi_4 g^{(3)}(\phi_o) + \frac{\phi_2^3}{3!} g^{(3)}(\phi_o)$$

$$A_7 = \phi_7 \, g^{(1)}(\phi_o) + \phi_1 \phi_6 g^{(2)}(\phi_o) + \phi_2 \phi_5 g^{(2)}(\phi_o) + \phi_3 \phi_4 g^{(2)}(\phi_o) + \frac{\phi_3^2}{2} \phi_1 g^{(3)}(\phi_o) + \frac{\phi_1^2}{2} \phi_5 g^{(3)}(\phi_o) + \frac{\phi_2^2}{2} \phi_3 g^{(3)}(\phi_o)$$

Go on this course, we will get A_n

3. Numerical Examples:

Here we list the results of approximating some problems solved by Adomian's method and modified Adomian's method.

Example 1: we apply the standard Adomian's method

$$\phi(x) = f(x) + \int_0^1 K_1(x,t)\, g_1(\phi(t))\, dt + \int_0^1 K_2(x,t)\, g_2(\phi(t))\, dt \qquad x, t \in [0,1]$$

where

$f(x) = x^2 - 0.023809 \;\; x - 0.14$ $\qquad \phi(x) = x^2$

$K_1(x,t) = 0.7$ $\qquad g_1(\phi(x)) = \phi^2(x)$

$K_2(x,t) = x/6$ $\qquad g_2(\phi(x)) = \phi^3(x)$

and solved by two methods Adomian's and modification method, and the results will be compared.

A) classic Adomian's method

In this example, the non-linear term is $g_1(\phi(x))$ *and* $g_2(\phi(x))$ using the algorithm $\phi_o(x) = f(x) = x^2 - 0.023809\ x - 0.14$, then

$$\phi_1(x) = \int_0^1 (0.7)\ \phi_o^2(t)\ dt + \int_0^1 (x/6)\ \phi_o^3(t)\ dt$$

$$\phi_2(x) = \int_0^1 (0.7)\,2\phi_o(t)\ \phi_1(t)\ dt + \int_0^1 (x/6)\,3\phi_o^2(t)\ \phi_1(t)\ dt$$

$$\phi_3(x) = \int_0^1 (0.7)[2\phi_o(t)\ \phi_2(t) + \phi_1^2(t)]dt + \int_0^1 (x/6)[3\phi_o^2(t)\ \phi_2(t) + 3\phi_o(t)\phi_1^2(t)]\ dt$$

$$\phi_4(x) = \int_0^1 (0.7)[2\phi_o(t)\ \phi_3(t) + 2\phi_1(t)\phi_2(t)]dt + \int_0^1 (x/6)[3\phi_o^2(t)\ \phi_3(t) + 6\phi_o\,\phi_1(t)\phi_2(t) + \phi_1^3]\ dt$$

$$\phi_5(x) = \int_0^1 (0.7)[2\phi_o(t)\ \phi_4(t) + 2\phi_1(t)\phi_3(t) + \phi_2^2(t)]dt \quad + \int_0^1 (x/6)[3\phi_o^2(t)\ \phi_4(t) + 6\phi_o(t)\phi_1(t)\phi_3(t) + 3\phi_o(t)\phi_2^2(t)\ + 3\ \phi_1^2(t)\phi_2(t)]dt$$

$$\phi_6(x) = \int_0^1 (0.7)[2\phi_o(t)\ \phi_5(t) + 2\phi_2(t)\phi_3(t) + 2\phi_1(t)\phi_4(t)\,]dt + \int_0^1 (x/6)[3\phi_o^2(t)\ \phi_5(t) + 6\phi_o(t)\phi_2(t)\phi_3(t) + 6\phi_o(t)\phi_1(t)\phi_4(t) + 3\ \phi_1^2(t)\phi_3(t) + 3\phi_2^2(t)\ \phi_1(t)]dt$$

$$\phi_7(x) = \int_0^1 (0.7)[2\phi_o(t)\,\phi_6(t) + 2\phi_2(t)\phi_4(t) + 2\phi_1(t)\phi_5(t) + \phi_3^{\,2}(t)]dt +$$

$$\int_0^1 (x/6)[3\phi_o^{\,2}(t)\,\phi_6(t) + 6\phi_o(t)\phi_2(t)\phi_4(t) + 6\phi_o(t)\phi_1(t)\phi_5(t) +$$

$$3\,\phi_o(t)\phi_3^{\,2}(t) + 6\phi_1(t)\phi_2(t)\phi_3(t) + 3\phi_1^{\,2}(t)\phi_4(t) + \phi_2^{\,3}(t)]dt$$

$$\phi_8(x) = \int_0^1 (0.7)[2\phi_o(t)\,\phi_7(t) + 2\phi_1(t)\phi_6(t) + 2\phi_2(t)\phi_5(t) + 2\phi_3(t)\phi_4(t)]dt +$$

$$\int_0^1 (x/6)[3\phi_o^{\,2}(t)\,\phi_7(t) + 6\phi_o(t)\phi_1(t)\phi_6(t) + 6\phi_o(t)\phi_2(t)\phi_5(t) + 6\phi_o(t)\phi_3(t)\phi_4(t) +$$

$$3\,\phi_1(t)\phi_3^{\,2}(t) + 3\phi_1^{\,2}(t)\phi_5(t) + 3\phi_2^{\,2}(t)\,\phi_3(t)]dt$$

$\phi(x)$ is approximated by using eight terms of Adomian's method

$\phi(x) \cong \phi_o(x) + \phi_1(x) + \phi_2(x) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (2.16)

Figure (1.1) shows the approximate solution, it is obvious from that the approximation is good.

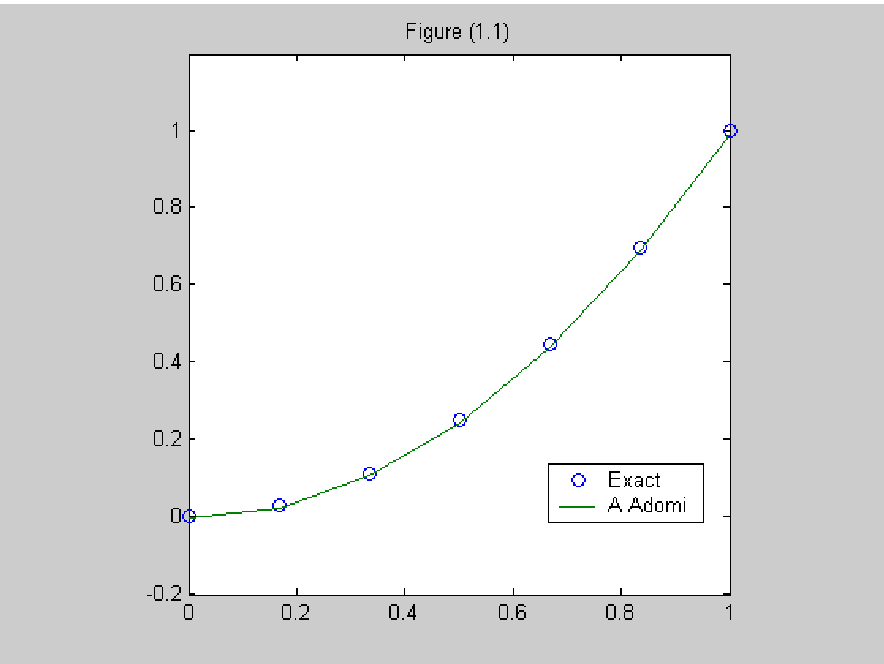


Figure (1.1) comparison of convergence rate for classic Adomian's method and exact solution x^2

B) modified Adomian's method

assume that $f = f1 + f2$

then

$f1(x) = x^2 - 0.023809\ x \qquad f2(x) = -0.14$

let

$\phi_o(x) = f1 = x^2 - 0.023809\ x \quad then$

$$\phi_1(x) = f2 + \int_0^1 (0.7)\phi_o{}^2(t)dt + \int_0^1 (x/6)\phi_o{}^3(t)dt$$

$$\phi_2(x) = \int_0^1 (0.7)2\phi_o(t)\phi_1(t)dt + \int_0^1 (x/6)3\phi_o{}^2(t)\phi_1(t)dt$$

⋮

We can see also from Figure (1.2) that this modified is very good.

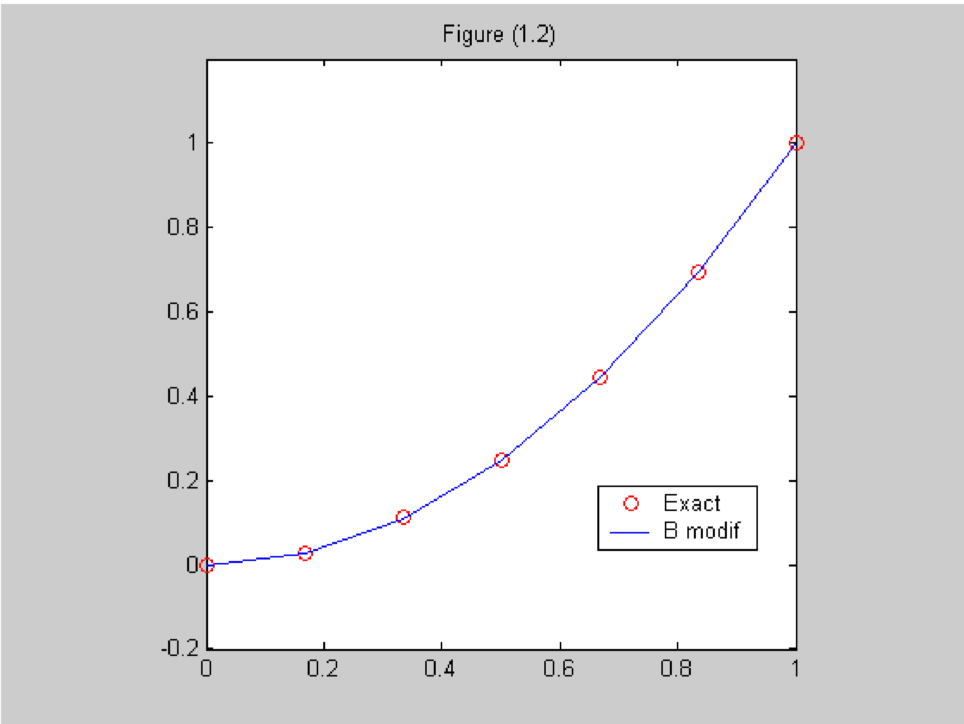


Figure (1.2): comparison of convergence rate for modify Adomian's method and exact solution x^2.This method was applied at the 6th iteration.

A comparison of the approximate solution between classic and the modified Adomian's method with the exact solution $\phi(x) = x^2$ of the integral equation at $x = 0, 1/6, 2/6, 3/6, 4/6, 5/6$ *and* 1 yield the errors displayed in table (1.1).

Table(1.1)

x	0	0.16667	0.33333	0.5	0.66667	0.83333	1
Error by classic	0.0067	0.0069	0.0072	0.0074	0.0077	0.0079	0.0082
Error by MA	4.468e-4	4.634e-4	4.8 e-4	4.966 e-4	5.132 e-4	5.298 e-4	5.464 e-4

Example 2:

We apply the standard Adomian's method

$$\phi(x) = f(x) + \int_0^1 K_1(x,t)\, g_1(\phi(t))\, dt + \int_0^1 K_2(x,t)\, g_2(\phi(t))\, dt \qquad x \in [0,1]$$

where

$f(x) = x^3 - 0.076923\,x - 0.00909$ $\qquad$ $\phi(x) = x^3$

$K_1(x,t) = 0.1t$ $\qquad$ $K_2(x,t) = x$

$g_1(\phi(x)) = \phi^3(x)$ $\qquad$ $g_2(\phi(x)) = \phi^4(x)$

A) classic Adomian's method

$$\phi_o(x) = x^3 - 0.076923\,x - 0.00909$$

$$\phi_1(x) = \int_0^1 (0.1\,t)\, \phi_o{}^3(t)\, dt + \int_0^1 x\, \phi_o{}^4(t)\, dt$$

$$\phi_2(x) = \int_0^1 (0.1\,t) 3\phi_o{}^2(t)\, \phi_1(t)\, dt + \int_0^1 x\, 4\phi_o{}^3(t)\, \phi_1(t)\, dt$$

$\vdots$

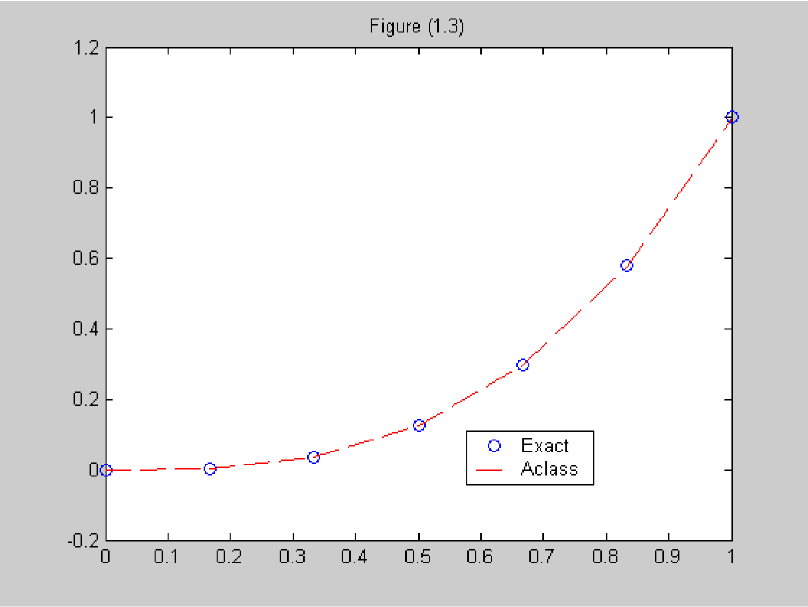


Figure (1.3): classic Adomian's method versus exact solution for $\phi(x) = x^3$

B) modified Adomian's method

assume that

$$f = f1 + f2$$

$$then \qquad f1 = x^3 - 0.076923\,x \qquad f2 = -0.00909$$

Let

$$\phi_o = f1 = x^3 - 0.076923\,x$$

$$\phi_1(x) = f2 + \int_0^1 (0.1\,t)\phi_o{}^3(t)dt + \int_0^1 x\,\phi_o{}^4(t)dt$$

$$\phi_2(x) = \int_0^1 (0.1\,t)\;3\;\phi_o{}^2(t)\phi_1(t)dt + \int_0^1 x\;4\;\phi_o{}^3(t)\phi_1(t)dt$$

$\vdots$

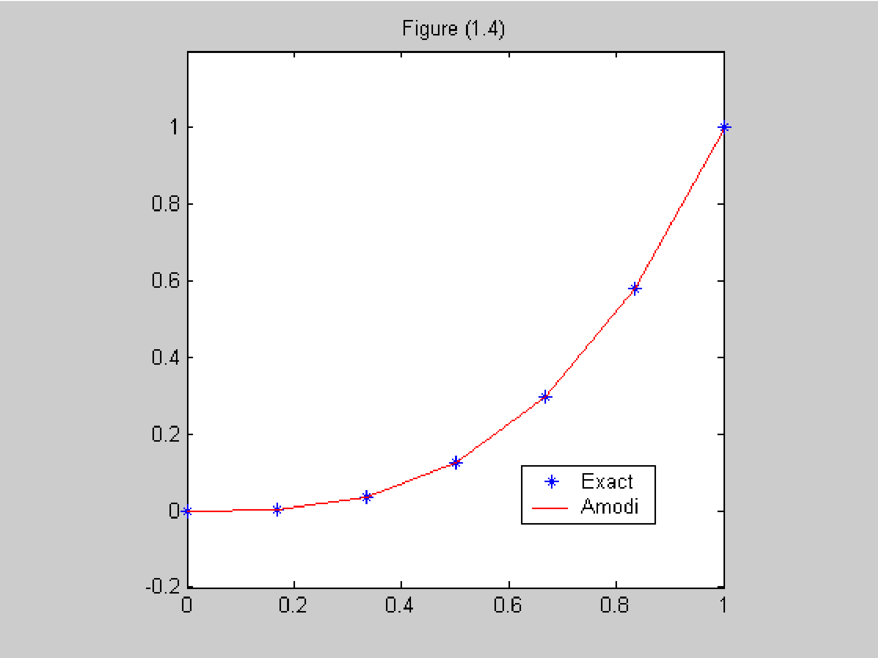


Figure (1.4): classic modify Adomian's method versus exact solution for $\phi(x) = x^3$

A comparison of the approximate solution between classic and the modified Adomian's method with the exact solution $\phi(x) = x^3$ of the integral equation at $x = 0\,,1/6\,,2/6\,,3/6\,,4/6\,,5/6$ *and* 1 yield the errors displayed in table (1.2).

Table (1.2)

x	0	0.16667	0.33333	0.5	0.66667	0.83333	1
Error by classic	0.0001	0.0003	0.0005	0.0006	0.0008	0.0010	0.0012
Error by AM	5.74e-4	2.059e-4	3.545e-4	5.031e-4	6.517e-4	8.003e-4	9.489e-4

Example :3

Consider the problem

$$\phi(x) = f(x) + \int_0^1 K_1(x,t)\, g_1(\phi(t))\, dt + \int_0^1 K_2(x,t)\, g_2(\phi(t))\, dt \qquad x,t \in [0,1]$$

$f(x) = x^4 - 0.047619\, x - 0.071429$ $\qquad \phi(x) = x^4$

$K_1(x,t) = t$ $\qquad K_2(x,t) = x^2$

$g_1(\phi(x)) = \phi^3(x)$ $\qquad g_2(\phi(x)) = \phi^5(x)$

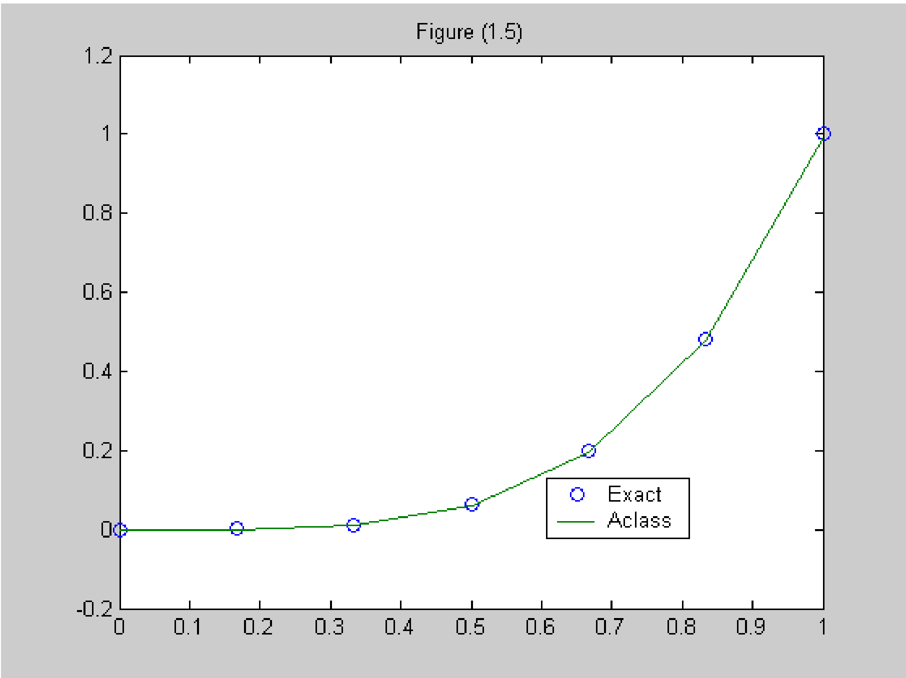
A) by classic Adomian's method

$$\phi_o(x) = x^2 - 0.047619\, x - 0.071429$$

$$\phi_1(x) = \int_0^1 t\, \phi_o{}^3(t)\, dt + \int_0^1 x^2\, \phi_o{}^5(t)\, dt$$

$$\phi_2(x) = \int_0^1 t\, 3\phi_o{}^2(t)\, \phi_1(t)\, dt + \int_0^1 x^2\, 5\phi_o{}^4(t)\, \phi_1(t)\, dt$$

$\vdots$



Figure(1.5) comparison of convergence rate for modify Adomian's method and exact solution $\phi(x) = x^4$.

B) by modification method

assume that :

$$f = f1 + f2$$

$$then \qquad f1 = x^4 - 0.047619\, x \qquad f2 = -0.071429$$

$$let \qquad \phi_o = f1 = x^4 - 0.047619\, x$$

then

$$\phi_1(x) = f2 + \int_0^1 t\ \phi_o{}^3(t)dt + \int_0^1 x^2\ \phi_o{}^5(t)dt$$

$$\phi_2(x) = \int_0^1 t\ 3\ \phi_o{}^2(t)\phi_1(t)dt + \int_0^1 x^2\ 5\ \phi_o{}^4(t)\phi_1(t)dt$$

$$\vdots$$

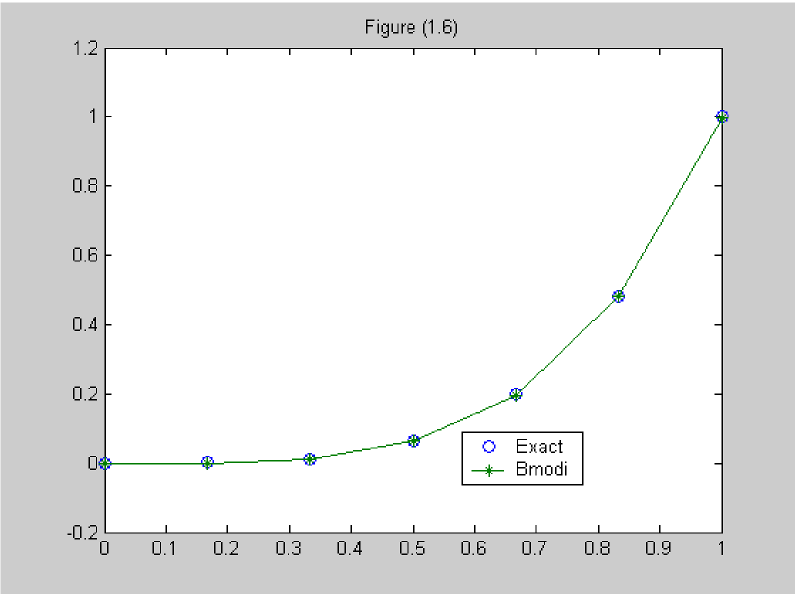


Figure (1.6): comparison of convergence rate for modify Adomian's method and exact solution $\phi(x) = x^4$. This method was applied at the 6th iteration.

A comparison of the approximate solution between classic and the modified Adomian's method with the exact solution $\phi(x) = x^4$ of the integral equation at $x = 0, 1/6, 2/6, 3/6, 4/6, 5/6$ *and* 1 yield the errors displayed in table (1.3).

Table (1.3)

X	0	0.16667	0.33333	0.5	0.66667	0.83333	1
Error by classic	0.003	0.0031	0.0033	0.0038	0.0045	0.0035	0.0064
Error by AM	3.656e-4	3.744e-4	4.010e-4	4.453e-4	5.073e-4	5.87e-4	6.845e-4

4. Conclusion:

The Adomian's and modified Adomian's method are applied to solve the Fredholm-Fredholm integral equations. The method is based upon changing the non- linear operator in to finite series. Hence this Development for Adomian's is much faster than classic Adomian's method and keeping the accuracy of the solution. The numerical examples support this claim.

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