VARIANCES OF MAXIMUM LIKELIHOOD ESTIMATORS OFVARIANCE COMPONENTS FOR THE RANDOMEFFECT MODELS

ALAA H.ABDULLAH
Department of mathematics, college of science
University of Basrah, Basrah-Iraq

Received 11/12/2005,Accepted 16/3/2006

ABSTRACT
In this study we find the information matrix for the random effect models which is used to find the variance of maximum likelihood estimators of variance components of these models under normality condition.

Key words: random effect model, covariance matrix. information matrix

(1-1) Introduction

The matter of finding the information matrix has been studied by many authors because of its importance in finding the estimation of variance components for balanced or unbalanced linear model but these studies for unbalanced data are more difficult than the balanced once because of their complexities in regard with the calculations and the elements of the information matrix for the variance components of the linear model involve, under normality,the invers of the covariance matrix of vector of observations which is essential to finding the information matrix.

Searle (1970) [5] developed general method to find the explicit expression for elements of information matrix of variance components under normality condition and used this matrix in finding the variance of maximum likelihood estimators of variance components and he displayed this result for 2-way nested classification random model . Rudan and Searle (1971) [6] used these results for 3-way nested classification random effect model . Abdullah (1997) [1] used them for 3-way random effect with unbalanced data .

The aim of this paper is to find the information matrix for the random effect models which are have all measurements have the same variance and the correlation for observations at different level k is ρ_3, the correlation for observations at different level j is ρ_2 and the correlation for observations at different level j and k is ρ_1 and used this matrix to find the the variance of maximum likelihood estimators of variance components of these models and results of this paper can be used to find the information matrix for the certain 2-way random effect model, for the certain 3-way random effect model and for the certain 4-way mixed effect model and we can find the variance of maximum likelihood estimators of variance components of these models.

(1-2) Notation

Let I_n be the $n\times m$ identity matrix, let $J_{n\times m}$ be the $n\times m$ matrix of 1 in every position. If A is $n\times m$ matrix and $B=(b_{ij})$ is $p\times q$ matrix then the Kronecker product of A and B written as $A\otimes B$ is the $np\times mq$ matrix (Ab_{ij}) [4]. If A is an $n\times m$ matrix the tr(A) is the sum of elements of leading diagonal .Clearly $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$ and $tr(A\otimes B)=tr(A)\cdot tr(B)$.

(1-3) The method

Suppose that $\sigma^2 = (\sigma_1^2, \sigma_2^2, \cdots, \sigma_t^2)$ is the vector of variance components of random effect in the linear models and let $\hat{\sigma}^2$ is the vector corresponding to the maximum likelihood estimator.To find the variance of $\hat{\sigma}^2$ we must find the information matrix T for these models where the explicit expression for elements of this matrix T can be found by useing the form $t_{ij} = tr(V^{-1} V_i V^{-1} V_j)$ .When V^{-1} is the covariance matrix and V_i is the partial derivative of V with respect to σ_i^2 ,therefore to find this variance we will use Searle"s form (1970) [5] $Var(\hat{\sigma}^2) = 2T^{-1}$

2.The Model

Suppose that in an experiments there are n treatments and every one of these treatments has d_i of experiment unit with r_{ij} of observations. Let y_{ijk} is the observation k of the unit j from treatments i with $i = 1,2,\cdots,n$, $j = 1,2,\cdots,d_i$ and $k = 1,2,\cdots,r_{ij}$ and let

$$y_{ij} = \begin{bmatrix} y_{ij1} \\ y_{ij2} \\ \vdots \\ y_{ijr_{ij}} \end{bmatrix} , \; y_i = \begin{bmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{id_i} \end{bmatrix} , \; y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Now suppose that all measurements have the same variance and every pair of measurements came from:

(1) Same treatments of same experiment unit with different observations .
(2) Same treatments of different experiment unit with same observations .
(3) Same treatments of different experiment unit with different observations .

Have the covariance $\sigma^2\rho_3$, $\sigma^2\rho_2$ and $\sigma^2\rho_1$

Therefore

$$cov(Y_{ijk}, Y_{i'j'k'}) = \begin{cases} \sigma^2 & ; \; i = i', j = j', k = k' \\ \sigma^2\rho_3 & ; \; i = i', j = j', k \neq k' \\ \sigma^2\rho_2 & ; \; i = i', i \neq j', k = k' \\ \sigma^2\rho_1 & ; \; i = i', j \neq j', k \neq k' \\ 0 & ; \; i \neq i' \end{cases} \quad \cdots (1)$$

Suppose y"s are independent and have the covariance matrix $F_i > 0$ for all i , where F_i is positive definite matrix [2] so the covariance matrix for measurements vector is :

$$V = var(y) = \begin{bmatrix} F_1 & 0 & \cdots & 0 \\ 0 & F_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & F_n \end{bmatrix}$$

Now we will set up the covariance matrix by useing (1).To explain the structure of this matrix ,we suppose that n=2 and so

j	j=1	j=2	j=3
r_{ij}	3	3	3

Therefore we will get measurements vector to this assumption as follows:

$$Y = (y_{i11}, y_{i12}, y_{i13}, y_{i21}, y_{i22}, y_{i23}, y_{i31}, y_{i32}, y_{i33})$$

so the covariance matrix V for the vector Y is

$$V = \sigma^2 F$$

Where

F=

1	ρ_3	ρ_3	ρ_2	ρ_1	ρ_1	ρ_2	ρ_1	ρ_1
ρ_3	1	ρ_3	ρ_1	ρ_2	ρ_1	ρ_1	ρ_2	ρ_1
ρ_3	ρ_3	1	ρ_1	ρ_1	ρ_2	ρ_1	ρ_1	ρ_2
ρ_2	ρ_1	ρ_1	1	ρ_3	ρ_3	ρ_2	ρ_1	ρ_1
ρ_1	ρ_2	ρ_1	ρ_3	1	ρ_3	ρ_1	ρ_2	ρ_1
ρ_1	ρ_1	ρ_2	ρ_3	ρ_3	1	ρ_1	ρ_1	ρ_2
ρ_2	ρ_1	ρ_1	ρ_2	ρ_1	ρ_1	1	ρ_3	ρ_3
ρ_1	ρ_2	ρ_1	ρ_1	ρ_2	ρ_1	ρ_3	1	ρ_3
ρ_1	ρ_1	ρ_2	ρ_1	ρ_1	ρ_2	ρ_3	ρ_3	1

The matrix F can be partioned into d^2 of block matrices,where in the diagonal matrices of order $r_{ij} \times r_{ij}$ defined as:

$$F_{ij} = (1-\rho_3) I_{r_{ij}} + \rho_3 J_{r_{ij}} \qquad \cdots(2)$$

whereas the nondiagonal matrices of order $r_{ij} \times r_{ij'}$.
We can extend the matrix F from special form to general form as follows:

$$F = (F_{ij,ij'}) \qquad i, i' = 1,2,\cdots,n$$
$$j, j' = 1,2,\cdots,d_i \qquad \cdots(3)$$

where

$$F_{ij,ij'} = \begin{cases} (1-\rho_3) I_{r_{ij}} + \rho_3 J_{r_{ij}} & i = i' \ , \ j = j' \\ (\rho_2 - \rho_1) I_{r_{ij}} + \rho_1 J_{r_{ij}} & i = i' \quad j \neq j' \end{cases} \qquad \cdots(4)$$

also we can get that

$$F_i = (1-\rho_3-\rho_2+\rho_1) I_{d_i r_{ij}} + (\rho_2 - \rho_1) I_{r_{ij}} \otimes J_{d_i} + (\rho_3 - \rho_1) J_{r_{ij}} \otimes I_{d_i} + \rho_1 J_{d_i r_{ij}} \qquad \cdots(5)$$

Without the loss of generatily,we supposes that $d_i = d$, $r_{ij} = r_j$ and $F_i = F$ for all i ,where this assumption changes the order of the matrix and does not affect the structure of F; therefore

$$F = (1-\rho_3-\rho_2+\rho_1) I_{dr_j} + (\rho_2 - \rho_1) I_{r_j} \otimes J_d + (\rho_3 - \rho_1) J_{r_j} \otimes I_d + \rho_1 J_{dr_j} \qquad \cdots(6)$$

whereas covariance matrix is

$$V = \sigma^2 F \otimes I_n$$
$$= \sigma^2 (1-\rho_3-\rho_2+\rho_1) I_{ndr_j} + \sigma^2 (\rho_2 - \rho_1) I_{r_j} \otimes J_d \otimes I_n + \sigma^2 (\rho_3 - \rho_1) J_{r_j} \otimes I_{nd} + \sigma^2 \rho_1 J_{dr_j} \otimes I_n \quad \cdots(7)$$

3.The inverse of the covariance matrix

In this section,we give the inverse of the covariance matrix V ,where $(r_j = r)$. Gabbara (1994) [3] found that the inverse of the covariance matrix V for this model is by useing tearing method and the expression of this inverse is

$$V^{-1} = F^{-1} \otimes I_n \qquad \cdots(8)$$

Where

$$F^{-1} = \psi_1 I_{dr} + \psi_2 I_r \otimes J_d + \psi_3 J_r \otimes I_d + \psi_4 J_{dr} \qquad \cdots(9)$$

in which

$$\psi_1 = \frac{1}{\lambda_1} \;,\; \psi_2 = \frac{-(\rho_2 - \rho_1)}{\lambda_1\lambda_2} \;,\; \psi_3 = \frac{-(\rho_3 - \rho_1)}{\lambda_1\lambda_2} \;,$$

$$\psi_4 = \frac{[(\rho_3 - \rho_1)(\lambda_3\lambda_4 - \lambda_1\lambda_2) - d\rho_1\lambda_1\lambda_2]}{d\lambda_1\lambda_2\lambda_3\lambda_4} \qquad \cdots(10)$$

$$\lambda_1 = 1 - \rho_3 - \rho_2 + \rho_1 \;,\; \lambda_2 = \lambda_1 + r(\rho_3 - \rho_1) \;,\; \lambda_3 = \lambda_1 + d(\rho_2 - \rho_1)$$

and $\lambda_4 = \lambda_3 + r(\rho_3 - \rho_1) + dr\rho_1$ $\cdots(11)$

4.The information matrix

To find the information matrix for these models which have the covariance in (1), we use the Searle"s form

$t_{ij} = tr(V^{-1}V_iV^{-1}V_j)$, where V^{-1} is the inverse of the covariance matrix V and V_i is the partial derivative of V with respect to σ_i^2.

Let

$$a = \sigma^2\rho_1, b = \sigma^2(\rho_3 - \rho_1),\; c = \sigma^2(\rho_2 - \rho_1) \text{ and } e = \sigma^2(1 - \rho_3 - \rho_2 + \rho_1) \qquad \cdots(12)$$

Therefore from (9)

$$V_a = \frac{\partial V}{\partial a} = J_{dr} \otimes I_n \qquad \cdots(13)$$

$$V_b = \frac{\partial V}{\partial b} = J_r \otimes I_{dn} \qquad \cdots(14)$$

$$V_c = \frac{\partial V}{\partial c} = I_r \otimes J_d \otimes I_n \qquad \cdots(15)$$

$$V_e = \frac{\partial V}{\partial e} = I_{drn} \qquad \cdots(16)$$

Now by the Searle's form we find the elements of the information matrix T which is defined as

$$T = \begin{bmatrix} t_{aa} & t_{ab} & t_{ac} & t_{ae} \\ t_{ba} & t_{bb} & t_{bc} & t_{be} \\ t_{ca} & t_{cb} & t_{cc} & t_{ce} \\ t_{ea} & t_{eb} & t_{ec} & t_{ee} \end{bmatrix} \qquad \cdots(17)$$

This matrix is symmetric $(t_{ij} = t_{ji})$ so we will evaluate only the elements of the main diagonal and the elements of uppre or lower main diagonal.
Now from (8), (9), and (13)

$$t_{aa} = tr(V^{-1}V_a V^{-1}V_a) = tr[(V^{-1}V_a)^2] = tr[((V_1^{-1} \otimes I_n)(J_{dr} \otimes I_n))^2]$$

$$= nr^2 d^2 (\psi_1 + d\psi_2 + r\psi_3 + dr\psi_4)^2 = \frac{nd^2 r^2}{\lambda_4{}^2} \qquad \cdots(18)$$

Now let

$$A = \frac{ndr}{\lambda_4^2} \qquad \cdots(19)$$

$$B_{ijk} = \frac{nr^i d^j ((r-1)^{1-i}(d-1)^{1-j}\lambda_4^2 + \lambda_2^{2k}\lambda_3^{2(1-k)})}{\lambda_2^{2k}\lambda_3^{2(1-k)}\lambda_4^2} \qquad \cdots(20)$$

$$C = ndr[\frac{\lambda_2^2\lambda_3^2 - \lambda_2^2(\rho_2-\rho_1)(\lambda_1+\lambda_3) + (\rho_3-\rho_1)(2\lambda_2(\rho_2-\rho_1) - r\lambda_3(\rho_3-\rho_1)}{\lambda_1^2\lambda_2^2\lambda_3^2}$$

$$+\frac{(\rho_3-\rho_1)(\lambda_3\lambda_4 - \lambda_1\lambda_2) - d\rho_1\lambda_1\lambda_3}{d\lambda_1\lambda_2\lambda_3\lambda_4}(\frac{2\lambda_4 - r(\rho_3-\rho_1) - rd\rho_1}{\lambda_3\lambda_4})] \qquad \cdots(21)$$

Therefore

$$t_{aa} = rdA \qquad \cdots(22)$$

Similarly the other elements of the matrix are

$$\frac{1}{r}t_{ab} = \frac{1}{d}t_{ac} = t_{bc} = t_{ae} = A \qquad \cdots(23)$$

$$\frac{1}{r}t_{bb} = t_{be} = B_{101} \qquad \cdots(24)$$

$$\frac{1}{d}t_{cc} = t_{ce} = B_{010} \qquad \cdots(25)$$

$$t_{ee} = C \qquad \cdots(26)$$

Now the information matrix is

$$T = \begin{bmatrix} rdA & rA & dA & A \\ rA & B_{101} & A & B_{101} \\ dA & A & dB_{010} & B_{010} \\ A & B_{101} & B_{010} & C \end{bmatrix} \qquad \cdots(27)$$

After finding the inverse of T we can find the variance of the maximum likelihood estimator of variance components σ_a^2, σ_b^2, σ_c^2 and σ_e^2 by useing Searle"s form

$V(\hat{\sigma}^2) = 2T^{-1} = 2(t_{ij})^{-1}$,which can be written by the matrix as:

$$\begin{bmatrix} v(\hat\sigma_a^2) & cov(\hat\sigma_a^2,\hat\sigma_b^2) & cov(\hat\sigma_a^2,\hat\sigma_c^2) & cov(\hat\sigma_a^2,\hat\sigma_e^2) \\ cov(\hat\sigma_b^2,\hat\sigma_a^2) & v(\hat\sigma_b^2) & cov(\hat\sigma_b^2,\hat\sigma_c^2) & cov(\hat\sigma_b^2,\hat\sigma_e^2) \\ cov(\hat\sigma_c^2,\hat\sigma_a^2) & cov(\hat\sigma_c^2,\hat\sigma_b^2) & v(\hat\sigma_c^2) & cov(\hat\sigma_c^2,\hat\sigma_e^2) \\ cov(\hat\sigma_e^2,\hat\sigma_a^2) & cov(\hat\sigma_e^2,\hat\sigma_b^2) & cov(\hat\sigma_e^2,\hat\sigma_c^2) & v(\hat\sigma_e^2) \end{bmatrix} = 2\begin{bmatrix} t_{aa} & t_{ab} & t_{ac} & t_{ae} \\ t_{ba} & t_{bb} & t_{bc} & t_{be} \\ t_{ca} & t_{cb} & t_{cc} & t_{ce} \\ t_{ea} & t_{eb} & t_{ec} & t_{ee} \end{bmatrix}^{-1} \cdots(28)$$

By use the matlab program we can find the inverse of the information matrix and we will get :

$$K = r^2d^2AB_{101}B_{010}C + 3rdA^2B_{101}B_{010} + rdA^3C - r^2dAB_{101}B_{010}^2 - rd^2AB_{101}^2B_{010}$$
$$- r^2dA^2B_{010}C - 2rA^3B_{010} + r^2A^2B_{010}^2 + d^2A^2B_{101}^2 - rd^2A^2B_{101}C - 2dA^3B_{101} + A^4 \cdots(29)$$

Then

$$V(\hat\sigma_a^2) = \frac{1}{K}(rdB_{101}B_{010}C + 2AB_{101}B_{010} - A^2C - rB_{101}B_{010}^2 - dB_{101}^2B_{010}) \qquad \cdots (30)$$

$$V(\hat\sigma_b^2) = \frac{A}{K}(rd^2B_{010}C + dAB_{010} - d^2AC + rdB_{010}^2) \qquad \cdots(31)$$

$$V(\hat\sigma_c^2) = \frac{A}{K}(r^2dB_{101}C + rAB_{101} - r^2AC + rdB_{101}^2) \qquad \cdots(32)$$

$$V(\hat\sigma_e^2) = \frac{A}{K}(r^2d^2B_{101}B_{010} + rdA^2 - r^2dAB_{010} + rd^2AB_{101}) \qquad \cdots(33)$$

5.Applications

In this section,we give some applications which are taken from models which have the covariance in (1) to show how our resultes can be applied.

5.1 Consider the 2-way random effect model

$$Y_{ijk} = \theta + a_i + b_{ij} + (ab)_{ik} + e_{ijk} \qquad ...(34)$$

With i=1,2,…,n, j=1,2,…,d and k=1,2,…,r and θ unknown parameter and $a_i \approx n(0,\sigma_a^2)$, $b_{ij} \approx n(0,\sigma_b^2)$, $(ab)_{ik} \approx n(0,\sigma_{ab}^2)$, $e_{ijk} \approx n(0,\sigma_e^2)$, a_i and b_{ij} are random effects such the secod effect is nested in the first effect and the secod effect is iteraction with the first effect with

$$cov(Y_{ijk}, Y_{i'j'k'}) = \begin{cases} \sigma^2 & ;\ i=i', j=j', k=k' \\ \sigma^2\rho_3 & ;\ i=i', j=j', k\neq k' \\ \sigma^2\rho_2 & ;\ i=i', i\neq j', k=k' \\ \sigma^2\rho_1 & ;\ i=i', j\neq j', k\neq k' \\ 0 & ;\ i\neq i' \end{cases}$$

Therefore the covariance matrix of this model has same structure of (7)with

$\sigma_a^2 = \sigma^2\rho_1,\ \sigma^2\rho_3 = \sigma_a^2 + \sigma_b^2\ ,\sigma^2\rho_2 = \sigma_a^2 + \sigma_{ab}^2$ and $\sigma^2 = \sigma_a^2 + \sigma_b^2 + \sigma_{ab}^2 + \sigma_e^2$

therefore the information matrix of this model has same structure of (27).Also the maximum likelihood estimator of variance components σ_a^2 , σ_b^2 , σ_{ab}^2 and σ_e^2 have same structure of (30), (31),(32)and(33)

5.2 Consider the 3-way random effect model

$$Y_{ijk} = \theta + a_i + b_{ij} + c_{ik} + e_{ijk} \qquad ...(35)$$

With i=1,2,…,n , j=1,2,…,d and k=1,2,…,r and θ is unknown parameter .The a_i, b_{ij}, c_{ik} and e_{ijk} are an observed independent random variable with zero mean and variance $\sigma_a^2, \sigma_b^2, \sigma_c^2$ and σ_e^2 respectively and the effects b_{ij} and c_{ik} are nested in the effect a_i. The covariance matrix of this model has same structure of (7) with

$$\sigma_a^2 = \sigma^2\rho_1,\ \sigma^2\rho_3 = \sigma_a^2 + \sigma_b^2\ ,\sigma^2\rho_2 = \sigma_a^2 + \sigma_c^2 \text{ and } \sigma^2 = \sigma_a^2 + \sigma_b^2 + \sigma_c^2 + \sigma_e^2$$

therefore the information matrix of this model has same structure of (27). Also the maximum likelihood estimator of variance components σ_a^2 , σ_b^2 , σ_c^2 and σ_e^2 have same structure of(30),(31),(32)and(33)

5.3 Consider the 4-way mixed effect model

$$Y_{ijk} = \theta + a_i + b_{ij} + c_{ik} + d_{jk} + e_{ijk} \qquad ...(36)$$

With i=1,2,…,n , j=1,2,…,d and k=1,2,…,r and θ,d are unknown parameters such that $\sum d_{jk} = 0$ and $a_i \approx n(0,\sigma_a^2)$, $b_{ij} \approx n(0,\sigma_b^2)$, $(c)_{ik} \approx n(0,\sigma_c^2)$, $e_{ijk} \approx n(0,\sigma_e^2)$.

The covariance matrix of this model has same structure of (7)with

$$\sigma_a^2 = \sigma^2\rho_1, \sigma^2\rho_3 = \sigma_a^2 + \sigma_b^2\ ,\sigma^2\rho_2 = \sigma_a^2 + \sigma_c^2 \text{ and } \sigma^2 = \sigma_a^2 + \sigma_b^2 + \sigma_c^2 + \sigma_e^2$$

therefore the information matrix of this model has same structure of (27).Also the maximum likelihood estimator of variance components σ_a^2 , σ_b^2 , σ_c^2 and σ_e^2 has same structure of (30), (31),(32)and(33

6.Conclusions

The information matrix is used to find variance of maximum likelihood estimator of variance components for the linear model which has random effects in the linear model which depend upon normal distribution and note that the results which are obtain in this study not for general statistical models but it is for models with the covariance given in (1).

We can get the one-way random effect model by ignoring the b"s and c"s in (34) and putting $\sigma_b^2 \equiv 0$ and $\sigma_{ab}^2 \equiv 0$ the effect of this on the t"s is to ignore $t_{ba}, t_{bb}, t_{bab}, t_{be}$, t_{aba}, t_{abab} and t_{abe}. Also we can obtain the 2-way random effect model by ignoring the c"s in (35) and puting $\sigma_c^2 \equiv 0$ the effect of this on the t"s is to ignore t_{ac}, t_{bc}, t_{cc}, and t_{ce} .

We can use these results in finding the information matrix for the models satisfied (1) by the addition of the fixed effect to this model which is not interactive with random effect.

REFERENCES

[1] Abdullah, A.H. "Large sample varinces of maximum likelihood estimators of variance components for 3- way random effect model with unbalanced data". M.Sc. Thesis, Unversity of Basrah, Basrah-Iraq. (1997).

[2] Arnold ,S.F. The theory of linear models and multivariate analysis" Wiley,New York . .(1981)."

[3] Gabbara ,S.D. ,"The inverse of patterned matrix of a special structure"Basrah j.Science,A,Vo1 ,No.2,189-200. .(1994).

[4] Graybill,F.A."Matrices with applications in statistics" Wadsworth international group,Belmont,California. .(1983).

[5] Searle, S.R.",Large sample variances of maximum likelihood estimators of variance components using unbalanced data".Biometrics 26,505-554. (1970).

[6] Searle, S.r. and Rudan J.W." Large sample variances of maximum likelihood estimators ofvariance components in 3-way nested classiffication random model with unbalanced data ".,Biometrics 27,1087. (1971).

تباينات مقدرات الامكان الاعظم لمركبات التباين لنماذج التاثيرات العشوائية

علاء حسن عبد الله

قسم الرياضات – كلية العلوم
جامعةالبصرة, البصرة – العراق

المستخلص

في هذه الدراسة تم ايجاد مصفوفة المعلومات لنماذج التاثيرات العشوائية والتي تستخدم لحساب تباينات مقدرات الامكان الاعظم لمركبات التباين للمؤثرات في هذه النماذج تحت شروط التوزيع الطبيعي.