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TWO-WAY MULTIVARIATE REPEATED MEASUREMENTS ANALYSIS OF VARIANCE MODEL

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Abstract

The two-way multivariate repeated measurements analysis of variance (2-way MRM ANOVA) model for complete data is studied by considering the case of multivariate response variables. The test statistics of various hypotheses on between-units factors, within-units factors and the interaction between them are given.

Key Words: Two – way Multivariate Repeated Measurements, Analysis of Variance, Likelihood Ratio Criterion.

1.Introduction

Repeated measurements analysis is widely used in many fields, for example, the health and life sciences, epidemiology, biomedical research and so on. Many literatures have been given to the univariate repeated measurements analysis of variance (RM ANOVA), see Crowder and Hand (1990)[3], Vonesh and Chinchilli (1997)[9]. The focus of this paper is the multivariate repeated measurements analysis of variance (RM ANOVA) model for complete data. A complete MRM design means that measurements are available at each time point for each experimental unit [9]. The MRM generalizes RM in the sense that it allows a vector of observations at each measurement. Al-Mouel and Wang (2004) [1], studied the one – way multivariate repeated measurements analysis of variance (1-way MRM). The terminology we use for the MRM design in this paper is a two-way MRM ANOVA which refers to the situation with only two within-units factor.

2. Two –Way MRM Design

There is a variety of possibilites for the between units factors in a two-way design. In a randomized two-way MRM experiment, the experimental units are randomized to two or more between units factors or groups. The response variables are measured on each of p occasions, which are regarded as p levels of a within – unit factor, which we lable as "Time" for convenience. We consider the case of a multivariate response variables and two between-units factors. Also, we assume that we have two within – units factors (suppose we call them "Time" and "Day") and two groups or treatment factors (factor A and factor B).

For convenience we use the following notation:

p = # of responses = $t.d$;

t = # of levels of Times ;

d = # of levels of Days ;

q = # of groups = $a.b$;

a = # of levels of between-units factor A ;
b = # of levels of between-units factor B ;
n_{jk} = # of experimental units assigned to level (j,k) of (A,B) ;

$n = n_{11} + \cdots + n_{ab}$ is total sample size ;

where "#" means the number.

For convenience, we consider the following linear model and parameterization for the two-way MRM design with two between-units factor incorporating two univariate random effects:

$$Y_{ijklm} = \mu + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \delta_{i(j)} + \rho_{i(k)} + \tau_l + \gamma_m + (\alpha\tau)_{jl} + (\alpha\gamma)_{jm} + (\beta\tau)_{kl} + (\beta\gamma)_{km} + (\tau\gamma)_{lm} + (\alpha\beta\tau)_{jkl} + (\alpha\beta\gamma)_{jkm} + (\alpha\tau\gamma)_{jlm} + (\beta\tau\gamma)_{klm} + (\alpha\beta\tau\gamma)_{jklm} + \in_{ijklm} \quad (2.1)$$

where

$i = 1,\ldots, n_{jk}$ is an index for experimental unit of level (j,k) of the treatment factors (A,B) ,

$j = 1,\ldots, a$ is an index for levels of the between–units factor (Group A),
$k = 1,\ldots, b$ is an index for levels of the between –units factor (Group B),

$l = 1,\ldots, t$ is an index for levels of the within-units factor (Time) ,
$m = 1,\ldots, d$ is an index for levels of the within – units factor (Day),
$Y_{ijklm} = [Y_{ijklm1},\ldots,Y_{ijklmr}]'$ is the response measurements of within-units factors (Time-Day) for unit i within treatment factors (A,B) ,

$\mu = [\mu_1,\ldots,\mu_r]'$ is the overall mean vector,
$\alpha_j = [\alpha_{j1},\ldots,\alpha_{jr}]'$ is the added effect of the j^{th} levels of the treatment factor A,

$\beta_k = [\beta_{k1},\ldots,\beta_{kr}]'$ is the added effect of the k^{th} levels of the treatment factor B,
$\delta_{i(j)} = [\delta_{i(j)1},\ldots,\delta_{i(j)r}]'$ is the random effect due to experimental unit i of the j^{th} level of treatment factor A ,

$\rho_{i(k)} = [\rho_{i(k)1},\ldots,\rho_{i(k)r}]'$ is the random effect due to experimental unit i of the k^{th} level of treatment factor B,
$(\alpha\beta)_{jk} = [(\alpha\beta)_{jk1},\ldots,(\alpha\beta)_{jkr}]'$ is the added effect of the interaction between the factors A and B at levels j,k ,

$\tau_l = [\tau_{l1},\ldots,\tau_{lr}]'$ is the added effect of the l^{th} levels of Time ,

$\gamma_m = [\gamma_{m1},\ldots,\gamma_{mr}]'$ is the added effect of the m^{th} levels of Day ,

$(\alpha\tau)_{jl} = [(\alpha\tau)_{jl1},\ldots,(\alpha\tau)_{jlr}]'$ is the added effect of the interaction between the treatment factor A and Time at levels j,l ,

$(\beta\tau)_{kl} = [(\beta\tau)_{kl1},\ldots,(\beta\tau)_{klr}]'$ is the added effect of the interaction between the treatment factor B and Time at levels k,l ,

$(\alpha\gamma)_{jm} = [(\alpha\gamma)_{jm1},...,(\alpha\gamma)_{jmr}]'$ is the added effect of the interaction between the treatment factor A and Day at levels j,m ,

$(\beta\gamma)_{km} = [(\beta\gamma)_{km1},...,(\beta\gamma)_{kmr}]'$ is the added effect of the interaction between the treatment factor B and Day at levels k,m ,

$(\tau\gamma)_{lm} = [(\tau\gamma)_{lm1},...,(\tau\gamma)_{lmr}]'$ is the added effect of the interaction between the Time and Day at levels l,m ,

$(\alpha\beta\tau)_{jkl} = [(\alpha\beta\tau)_{jkl1},...,(\alpha\beta\tau)_{jklr}]'$ is the added effect of the interaction between the treatment factors A and B, and Time at their j^{th}, k^{th}, and l^{th} levels respectively ,

$(\alpha\beta\gamma)_{jkm} = [(\alpha\beta\gamma)_{jkm1},...,(\alpha\beta\gamma)_{jkmr}]'$ is the added effect of the interaction between the treatment factors A and B, and Day at their j^{th}, k^{th} and m^{th} levels respectively ,

$(\alpha\tau\gamma)_{jlm} = [(\alpha\tau\gamma)_{jlm1},...,(\alpha\tau\gamma)_{jlmr}]'$ is the added effect of the interaction between the treatment factor A and Time and Day at their j^{th}, l^{th},and m^{th} levels respectively ,

$(\beta\tau\gamma)_{klm} = [(\beta\tau\gamma)_{klm1},...,(\beta\tau\gamma)_{klmr}]'$ is the added effect of the interaction between the treatment factor B and Time and Day at the k^{th}, l^{th} and m^{th} levels respectively ,

$(\alpha\beta\tau\gamma)_{jklm} = [(\alpha\beta\tau\gamma)_{jklm1},...,(\alpha\beta\tau\gamma)_{jklmr}]'$ is the added effect of the interaction between the treatment factors A, B and Time, Day at their j^{th} , k^{th}, l^{th}, and m^{th} respectively, and

$\in_{ijklm} = [\in_{ijklm1},...,\in_{ijklmr}]'$ is the random error of within-units factors (Time, Day) at their levels (l,m) for unit i within between-units factors A, B at their levels (j,k) respectively.

where "′" means the transpose.

For the parameterization to be of full rank, we impose the following set of conditions:

$$\sum_{j=1}^{a}\alpha_j=0\ ;\ \sum_{k=1}^{b}\beta_k=0\ ;\ \sum_{l=1}^{t}\tau_l=0\ ;\ \sum_{m=1}^{d}\gamma_m=0\ ;$$

$$\sum_{j=1}^{a}(\alpha\tau)_{jl}=0=\sum_{l=1}^{t}(\alpha\tau)_{jl}\ ,\qquad \sum_{j=1}^{a}(\alpha\beta)_{jk}=0=\sum_{k=1}^{b}(\alpha\beta)_{jk}$$

$$\sum_{j=1}^{a}(\alpha\gamma)_{jm}=0=\sum_{m=1}^{d}(\alpha\gamma)_{jm}\ ,\qquad \sum_{k=1}^{b}(\beta\tau)_{kl}=0=\sum_{l=1}^{t}(\beta\tau)_{kl}$$

$$\sum_{k=1}^{b}(\beta\gamma)_{km}=0=\sum_{m=1}^{d}(\beta\gamma)_{km}\ ,\qquad \sum_{l=1}^{t}(\tau\gamma)_{lm}=0=\sum_{m=1}^{d}(\tau\gamma)_{lm}$$

$$\sum_{j=1}^{a}(\alpha\beta\tau)_{jkl}=0=\sum_{k=1}^{b}(\alpha\beta\tau)_{jkl}=\sum_{l=1}^{t}(\alpha\beta\tau)_{jkl}$$

$$\sum_{j=1}^{a}(\alpha\beta\gamma)_{jkm}=0=\sum_{k=1}^{b}(\alpha\beta\gamma)_{jkm}=\sum_{m=1}^{d}(\alpha\beta\gamma)_{jkm}$$

$$\sum_{j=1}^{a}(\alpha\tau\gamma)_{jlm}=0=\sum_{l=1}^{t}(\alpha\tau\gamma)_{jlm}=\sum_{m=1}^{d}(\alpha\tau\gamma)_{jlm}$$

$$\sum_{k=1}^{b}(\beta\tau\gamma)_{klm}=0=\sum_{l=1}^{t}(\beta\tau\gamma)_{klm}=\sum_{m=1}^{d}(\beta\tau\gamma)_{klm}$$

$$\sum_{j=1}^{a}(\alpha\beta\tau\gamma)_{jklm}=0=\sum_{k=1}^{b}(\alpha\beta\tau\gamma)_{jklm}=\sum_{l=1}^{t}(\alpha\beta\tau\gamma)_{jklm}=\sum_{m=1}^{d}(\alpha\beta\tau\gamma)_{jklm}$$

We assume that $\in_{ijklm}$'s , $\delta_{i(j)}$'s, and $\rho_{i(k)}$'s are independent with (2.2)

$$\in_{ijklm}=[\in_{ijklm1},...,\in_{ijklmr}]'\sim i.i.d.\,N_r(0,\Sigma_\in)\,, \qquad (2.3)$$

$$\delta_{i(j)}=[\delta_{i(j)1},...,\delta_{i(j)r}]'\sim i.i.d.\,N_r(0,\Sigma_\delta)\ \text{, and} \qquad (2.4)$$

$$\rho_{i(k)}=[\rho_{i(k)1},...,\rho_{i(k)r}]'\sim i.i.d.\,N_r(0,\Sigma_\rho)\,, \qquad (2.5)$$

where N_r is denoted to the multivariate-normal distribution, and, $\Sigma_\in,\Sigma_\delta$, and Σ_ρ are all $r\times r$ positive definite matrices.

Let $Y_{ijk}=[Y_{ijk1},Y_{ijk2},\ldots,Y_{ijkp}]$, i.e.,

$$Y_{ijk} = \begin{bmatrix} Y_{ijk11} & Y_{ijk12} & \ldots\ldots & Y_{ijkp1} \\ Y_{ijk12} & Y_{ijk22} & \ldots\ldots & Y_{ijkp2} \\ \vdots & \vdots & \vdots & \vdots \\ Y_{ijk1r} & Y_{ijk2r} & \ldots\ldots & Y_{ijkpr} \end{bmatrix} \qquad (2.6)$$

Let the covariance matrix of $\vec{Y}_{ijk}$ is denoted by Σ, where $\vec{A} = Vec(A)$. The $Vec(.)$ operator creates a column vector from a matrix A by simply stacking the column vectors of A below one another [9]. It follows from the random effects that Σ satisfies the assumption of compound symmetry, i.e.,

$$\Sigma = \mathrm{I}_p \otimes \Sigma_{\in} + J_p \otimes \Sigma_{\delta} + J_p \otimes \Sigma_{\rho} \ , \qquad (2.7)$$

where I_p denote the $p \times p$ identity matrix, J_p denote $p \times p$ matrix of one's, and $\otimes$ be the Kroncker product operation of two matrices.

3. Analysis of Variance (ANOVA)

Let U_* be any $p \times p$ orthogonal matrix is partitioned as follows :

$$U_* = [\ p^{-\frac{1}{2}} j_p \quad U_T \quad U_D \quad U_{T \times D}\ ], \qquad (3.1)$$

where j_p denote the $p \times 1$ vector of one's, U_T is $p \times (t-1)$ matrix, U_D is

$p \times (d-1)$ matrix, and $U_{T \times D}$ is $p \times (t-1)(d-1)$ matrix, $U'_T\, j_p = 0$, $U'_T U_T = I_{t-1}$, $U'_D\, j_p = 0$, , $U'_{TxD}\, j_p = 0$, $U'_{TxD} U_{TxD} = I_{(t-1)(d-1)}$,

$$U_{TxD} = \begin{bmatrix} w_{11} & \cdots & w_{1,(t-1)(d-1)} \\ \vdots & & \vdots \\ w_{p1} & & w_{p,(t-1)(d-1)} \end{bmatrix} \qquad (3.2)$$

Let $Y^*_{ijk} = Y_{ijk} U_*$

where $Y^*_{ijk} = [\, Y^*_{ijk1}, Y^*_{ijk2}, \ldots, Y^*_{ijkp}\ ]$

$$COV(\overrightarrow{Y^*_{ijk}}) = COV(\overrightarrow{Y_{ijk}\, U_*}) = (U'_* \otimes \mathrm{I}_r)\Sigma(U_* \otimes \mathrm{I}_r)$$

$$= (U'_* \otimes \mathrm{I}_r)(\mathrm{I}_p \otimes \Sigma_{\in} + J_p \otimes \Sigma_{\delta} + J_p \otimes \Sigma_{\rho})(U_* \otimes \mathrm{I}_r) \qquad (3.3)$$

$$= \mathrm{I}_p \otimes \Sigma_{\in} + U'_* J_p U_* \otimes \mathrm{I}_r\, \Sigma_{\delta}\, \mathrm{I}_r + U'_* J_p U_* \otimes \mathrm{I}_r\, \Sigma_{\rho}\, \mathrm{I}_r$$

We can write (3.3) above in the following matrix form :

$$Cov\left(\overrightarrow{Y_{ijk}^{*}}\right)=\begin{bmatrix}\Sigma_{\in}+p(\Sigma_{\delta}+\Sigma_{\rho}) & 0 & \ldots\ldots & 0\\ 0 & \Sigma_{\in} & \ldots\ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & & \Sigma_{\in}\end{bmatrix}_{rpxrp} \quad (3.4)$$

Now, $Y_{ijk1}^{*}=Y_{ijk}\quad p^{-\frac{1}{2}}j_{p}$,i.e.

$$\begin{bmatrix}Y_{ijk11}^{*}\\ Y_{ijk12}^{*}\\ \vdots\\ Y_{ijk1r}^{*}\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{p}}\sum_{l=1}^{t}\sum_{m=1}^{d}Y_{jklm1}\\ \frac{1}{\sqrt{p}}\sum_{l=1}^{t}\sum_{m=1}^{d}Y_{jklm2}\\ \vdots\\ \frac{1}{\sqrt{p}}\sum_{l=1}^{t}\sum_{m=1}^{d}Y_{jklmr}\end{bmatrix} \quad (3.5)$$

From (2.1), we obtain :

$$\begin{aligned}Y_{ijk1}^{*}&=p^{-\frac{1}{2}}\sum_{l=1}^{t}\sum_{m=1}^{d}Y_{ijklm}\\ &=p^{\frac{1}{2}}\mu+p^{\frac{1}{2}}\alpha_{j}+p^{\frac{1}{2}}\beta_{k}+p^{\frac{1}{2}}(\alpha\beta)_{jk}+p^{\frac{1}{2}}\delta_{i(j)}+p^{\frac{1}{2}}\rho_{i(k)}\\ &+p^{-\frac{1}{2}}\sum_{l=1}^{t}\sum_{m=1}^{d}\in_{ijklm}\end{aligned} \quad (3.6)$$

Then the set of vectors

$$\left[Y_{1111}^{*},\ldots,Y_{n_{11}111}^{*}\right]',\left[Y_{1211}^{*},\ldots,Y_{n_{21}211}^{*}\right]',\ldots,\left[Y_{1a11}^{*},\ldots,Y_{n_{a1}11}^{*}\right]',\ldots,\left[Y_{1ab1}^{*},\ldots,Y_{n_{ab}ab1}^{*}\right]'$$

have mean vectors :

$\sqrt{p}\mu+\sqrt{p}\alpha_{1}+\sqrt{p}\beta_{1}+\sqrt{p}\alpha_{1}\beta_{1},\sqrt{p}\mu+\sqrt{p}\alpha_{2}+\sqrt{p}\beta_{1}+\sqrt{p}\alpha_{2}\beta_{1},\ldots,$
$\sqrt{p}\mu+\sqrt{p}\alpha_{a}+\sqrt{p}\beta_{1}+\sqrt{p}\alpha_{a}\beta_{1},\sqrt{p}\mu+\sqrt{p}\alpha_{1}+\sqrt{p}\beta_{2}+\sqrt{p}\alpha_{1}\beta_{2},$
$\sqrt{p}\mu+\sqrt{p}\alpha_{2}+\sqrt{p}\beta_{2}+\sqrt{p}\alpha_{2}\beta_{2},\ldots,\sqrt{p}\mu+\sqrt{p}\alpha_{a}+\sqrt{p}\beta_{2}+\sqrt{p}\alpha_{a}\beta_{2},$
……., $\sqrt{p}\mu+\sqrt{p}\alpha_{a}+\sqrt{p}\beta_{b}+\sqrt{p}\alpha_{a}\beta_{b}$, respectively, and each of them has covariance matrix $p\Sigma_{\delta}+p\Sigma_{\delta}+\Sigma_{\in}$.

So, the null hypothesis of the same treatment effects are :

H_{01} : $\alpha_1 = \alpha_2 = \ldots = \alpha_a = 0$

H_{02} : $\beta_1 = \beta_2 = \ldots = \beta_b = 0$

H_{O3} : $\alpha_1\beta_1 = \alpha_2\beta_1 = \ldots = \alpha_a\beta_1 = \alpha_1\beta_2 = \alpha_2\beta_2 = \ldots = \alpha_a\beta_2 = \ldots =$

$\alpha_1\beta_b = \alpha_2\beta_b = \ldots = \alpha_a\beta_b = 0$

The ANOVA based on the set of transformed observations above the Y^*_{ijk1} 's provides the ANOVA for between –units effects. This leads to the following form for the sum square terms

$$S_A = \sum_{j=1}^{a} n_{jk}(Y^*_{jk1} - \overline{Y^*_1})(Y^*_{jk1} - \overline{Y^*_1})' \quad , k=1,\ldots,b \qquad (3.7)$$

$$S_B = \sum_{k=1}^{b} n_{jk}(Y^*_{jk1} - \overline{Y^*_1})(Y^*_{jk1} - \overline{Y^*_1})' \quad , j=1,\ldots,a \qquad (3.8)$$

$$S_{AxB} = \sum_{j=1}^{a}\sum_{k=1}^{b} n_{jk}(Y^*_{jk1} - \overline{Y^*_1})(Y^*_{jk1} - \overline{Y^*_1})' , k=1,\ldots,b \qquad (3.9)$$

$$S_{U(AxB)} = \sum_{j=1}^{a}\sum_{k=1}^{b}\sum_{i=1}^{n_{jk}} (Y^*_{ijk1} - \overline{Y^*_{jk}})(Y^*_{ijk1} - \overline{Y^*_{jk}})' , \qquad (3.10)$$

where

$$\overline{Y^*_{jk}} = \sum_{j=1}^{a}\sum_{k=1}^{b}\sum_{i=1}^{n_{jk}} Y^*_{ijk1} / n_{jk} \quad , \qquad (3.11)$$

$$\overline{Y^*_1} = \frac{1}{n}\sum_{j=1}^{a}\sum_{k=1}^{b}\sum_{i=1}^{n_{jk}} Y^*_{ijk1} \qquad (3.12)$$

Thus : (1) $S_{U(A\times B)} \sim W_r(n - ab, (p(\Sigma_\delta + \Sigma_\rho) + \Sigma_\in))$,

(2) $S_A \sim W_r(a-1, (p(\Sigma_\delta + \Sigma_\rho) + \Sigma_\in))$,

(3) $S_B \sim W_r(b-1, (p(\Sigma_\delta + \Sigma_\rho) + \Sigma_\in))$,

(4) $S_{A\times B} \sim W_r((a-1)(b-1), (p(\Sigma_\delta + \Sigma_\rho) + \Sigma_\in))$,

where W_r denote the multivariate-Wishart distribution.

Let $S_1 = S_A, S_2 = S_B, S_3 = S_{A\times B}$ .Then the test statistics are as follows :

(1) The multivariate Wilks test (Wilks, 1932)[10]:

$$T_W = \frac{\left|S_{U(A\times B)}\right|}{\left|S_{U(A\times B)} + S_k\right|} \text{ , when } H_{0k} \text{ is true , for } k = 1,2,3. \qquad (3.13)$$

(2) The Lawley-Hotelling trace (Lawley, 1938; Hotelling, 1947)[5,4]:

$T_{LH} = trace\,(S_k\, S^{-1}_{U(A\times B)})$, when H_{0k} is true , for $k = 1,2,3.$ (3.14)

(3) The Bartlett-Nanda-Pillai trace (Bartlett, 1939, Nanda, 1950; pillai, (1955)[2,6,7] :

$T_{BNP} = trace\{S_k\, S^{-1}_{U(A\times B)}\,(I + S_k\, S^{-1}_{U(A\times B)})^{-1}\}$, when H_{0k} is true ,for $k = 1,2,3.$ (3.15)

(4) Roy's Union-Intersection (UI) test or largest root (Roy, 1953)[8]:

$T_R =$ largest characteristics root of $(S_k\, S^{-1}_{U(A\times B)})$, when H_{0k} is true , for $k = 1,2,3.$

(3.16)

The ANOVA based on the set of transformed observation the Y^*_{ijk1m} 's for each l=2,…,t, and m=2,…,d has the model which is partitioned as follows:

$$Y^*_{ijklm} = \sum_{l'=2}^{t} \sum_{m'=2}^{d} Y_{ijkl'm'}\, u_{k',l'-1} \quad \text{, for each } k'=1,\dots,p, \qquad (3.17)$$

$$Y^*_{ijklm} = \sum_{l'=2}^{t} \sum_{m'=2}^{d} Y_{ijkl'm'}\, v_{k',m'-1} \quad \text{, for each } k'=1,\dots,p, \qquad (3.18)$$

$$Y^*_{ijklm} = \sum_{l'=2}^{t} \sum_{m'=2}^{d} Y_{ijkl'm'}\, w_{k',(l'-1)(m'-1)} \text{, for each } k'=1,\dots,p. \ (3.19)$$

Now $[Y^*_{ijk2},\dots,Y^*_{ijkp}] = Y_{ijk}\,U = [Y_{ijk}\,U_T \quad Y_{ijk}\,U_D \quad Y_{ijk}\,U_{T\times D}]$ (3.20)

Now, from (3.17) we obtain :

$$Y^*_{ijklm} = [u_{t-1,k'}][\sum_{m'=2}^{d} Y_{ijklm'}] \qquad (3.21)$$

From (2.1) and (3.21), we get:

$$Y^*_{ijklm} = (t-1)(d-1)\{\mu + \alpha_j + \beta_k + \delta_{i(j)} + \rho_{i(k)} + (\alpha\beta)_{jk}\}$$
$$[u_{1k'} + u_{2k'} + \cdots + u_{t-1,k'}] + (d-1)\{u_{1k'}\,\tau_2 + u_{2k'}\,\tau_3 + \cdots +$$
$$u_{t-1,k'}\,\tau_t\} + (d-1)\{u_{1k'}\,(\alpha\tau)_{j2} + u_{2k'}\,(\alpha\tau)_{j3} + \cdots +$$
$$u_{t-1,k'}\,(\alpha\tau)_{jt}\} + (d-1)\{u_{1k'}\,(\beta\tau)_{k2} + u_{2k'}\,(\beta\tau)_{k3} + \cdots +$$
$$u_{t-1,k'}\,(\beta\tau)_{kt}\} + (d-1)\{u_{1k'}\,(\alpha\beta\tau)_{jk2} + u_{2k'}\,(\alpha\beta\tau)_{jk3} +$$
$$\cdots + u_{t-1,k'}\,(\alpha\beta\tau)_{jkt}\} + \{u_{1k'} + u_{2k'} + \cdots + u_{t-1,k'}\}$$

$$[\sum_{m'=2}^{d}\{\gamma_{m'}+(\alpha\gamma)_{jm'}+(\beta\,\gamma)_{km'}+(\beta\gamma)_{km'}\}]+$$

$$u_{1k'}\sum_{m'=2}^{d}(\tau\gamma\}_{2m'}+u_{2k'}\sum_{m'=2}^{d}(\tau\gamma)_{3m'}+\cdots+u_{t-1,k'}\sum_{m'=2}^{d}(\tau\gamma)_{tm'}+$$

$$u_{1k'}\sum_{m'=2}^{d}(\alpha\tau\gamma)_{j2m'}+u_{2k'}\sum_{m'=2}^{d}(\alpha\tau\gamma)_{j3m'}+\cdots+$$

$$u_{t-1,k'}\sum_{m'=2}^{d}(\alpha\tau\gamma)_{jtm'}+u_{1k'}\sum_{m'=2}^{d}(\beta\tau\gamma)_{k2m'}+u_{2k'}\sum_{m'=2}^{d}(\beta\tau\gamma)_{k3m'}$$

$$+\cdots+u_{t-1,k'}\sum_{m'=2}^{d}(\beta\tau\gamma)_{ktm'}+u_{1k'}\sum_{m'=2}^{d}(\alpha\beta\tau\gamma)_{jk2m'}+$$

$$u_{2k'}\sum_{m'=2}^{d}(\alpha\beta\tau\gamma\}_{jk3m'}+\cdots+u_{t-1,k'}\sum_{m'=2}^{d}(\alpha\beta\tau\gamma\}_{jktm'}+$$

$$u_{1k'}\sum_{m'=2}^{d}\in_{ijk2m'}+u_{2k'}\sum_{m'=2}^{d}\in_{ijk3m'}+\cdots+u_{t-1,k'}\sum_{m'=2}^{d}\in_{ijktm'}. \tag{3.22}$$

Because U* is an orthogonal matrix (3.22) becomes:

$$Y^{*}_{ijklm}=(d-1)\sum_{l'=2}^{t}u_{l'-1,k'}\{\tau_{l'}+(\alpha\tau)_{jl'}+(\beta\tau)_{kl'}+(\alpha\beta\tau)_{jkl'}\}+$$

$$\sum_{l'=2}^{t}\sum_{m'=2}^{d}u_{l'-1,k'}\{(\tau\gamma)_{l'm'}+(\alpha\tau\gamma)_{jl'm'}+(\beta\tau\gamma)_{kl'm'}+$$

$$(\alpha\beta\tau\gamma)_{jkl'm'}+\in_{ijkl'm'}\}, \tag{3.23}$$

because the components of $[u_{k'1},u_{k'2},\ldots,u_{k',t-1}]$ sum to zero for each k′ =1,…,p.

Now, form (3.18) and (2.1), we obtain :

$$Y^{*}_{ijkl}=\{v_{1k'}+v_{2k'}+\cdots+v_{d-1,k'}\}[(t-1)\{\mu+\alpha_j+\beta_k+\delta_{i(j)}+\rho_{i(k)}+$$

$$(\alpha\beta)_{jk}\}]+(t-1)[\sum_{m'=2}^{d}v_{m'-1,k'}\{\gamma_{m'}+(\alpha\gamma)_{jm'}+(\beta\gamma)_{km'}+$$

$$(\alpha\beta\gamma)_{jkm'}\}]+\{v_{1k'}+v_{2k'}+\cdots+v_{d-1,k'}\}[\sum_{l'=2}^{t}\{\tau_{l'}+(\alpha\tau)_{jl'}$$

$$+(\beta\tau)_{kl'}+(\alpha\beta\tau)_{jkl'}\}]+\sum_{m'=2}^{d}\sum_{l'=2}^{t}v_{m'-1,k'}\{(\beta\tau\gamma)_{kl'm'}+$$

$$+(\alpha\beta\tau\gamma)_{jkl'm'}+\in_{ijkl'm'}\} \tag{3.24}$$

because U* is an orthogonal matrix then the components of $[v_{k'1},v_{k'2},\ldots,v_{k',d-1}]$ sum to zero for each k′=1,…,p. Then (3.24) becomes

$$Y^*_{ijklm}=(t-1)\sum_{m'=2}^{d}v_{m'-1,k'}\{\gamma_{m'}+(\alpha\gamma)_{jm'}+(\beta\gamma)_{km'}+(\alpha\beta\gamma)_{jkm'}\}$$

$$+\sum_{m'=2}^{d}\sum_{l'=2}^{t}v_{m'-1,k'}\{(\tau\gamma)_{l'm'}+(\alpha\tau\gamma)_{jl'm'}\}+(\beta\tau\gamma)_{kl'm'}$$

$$+(\alpha\beta\tau\gamma)_{jkl'm'}+\in_{ijkl'm'}\} \tag{3.25}$$

Now, from (3.19) and (2.1), we obtain:

$$Y^*_{ijklm}=(t-1)(d-1)\{\mu+\alpha_j+\beta_k+\delta_{i(j)}+\rho_{i(k)}+(\alpha\beta)_{jk}\}[w_{1k'}+$$

$$w_{2k'}+\cdots+w_{t-1,k'}+\cdots+w_{d-1,k'}+\cdots+w_{(t-1)(d-1),k'}]+(t-1)[$$

$$\sum_{l'=2}^{t}\sum_{m'=2}^{d}w_{(m'-1)(l'-1),k'}\{\gamma_{m'}+(\alpha\gamma)_{jm'}+(\beta\gamma)_{km'}+(\alpha\beta\gamma)_{jkm'}\}]$$

$$+\sum_{l'=2}^{t}w_{l'-1,k'}\{\tau_{l'}+(\alpha\tau)_{jl'}+(\beta\tau)_{kl'}+(\alpha\beta\gamma)_{jkm'}\}+\sum_{l'=2}^{t}w_{l'-1,k'}$$

$$\{\tau_{l'}+(\alpha\tau)_{jl'}+(\beta\tau)_{kl'}+(\tau\gamma)_{l'2}+(\alpha\beta\tau)_{jkl'}+(\alpha\tau\gamma)_{jl'2}+$$

$$(\beta\tau\gamma)_{kl'2}+(\alpha\beta\tau\gamma)_{ijkl'2}+\in_{ijkl'2}\}+\sum_{l'=2}^{t}w_{2(l'-1),k'}\{\tau_{l'}+(\alpha\tau)_{jl'}$$

$$+(\beta\tau)_{kl'}+(\tau\gamma)_{l'3}+(\alpha\beta\tau)_{jkl'}+(\alpha\tau\gamma)_{jl'3}+(\beta\tau\gamma)_{kl'3}$$

$$+(\alpha\beta\tau\gamma)_{jkl'3}+\in ijkl'3\}+\sum_{l'=2}^{t}w_{(l'-1)(d-1),k'}\{\tau_{l'}+(\alpha\tau)_{jl'}$$

$$+(\beta\tau)_{kl'}+(\tau\gamma)_{l'd}+(\alpha\beta\tau)_{jkl'}+(\alpha\tau\gamma)_{jl'd}+(\beta\tau\gamma)_{kl'd}$$

$$+(\alpha\beta\tau\gamma)_{jkl'd}+\in_{ijkl'd}\}. \tag{3.26}$$

Because U_* is an orthogonal matrix then the components of $[w_{k'1}, w_{k'2}, \ldots, w_{k',(t-1)(d-1)}]$ sum to zero for each k′=1,…,p. Then (3.26) becomes :

$$Y^*_{ijklm} = \sum_{l'=2}^{t} \sum_{m'=2}^{d} w_{(m'-1)(l'-1),k'} [(t-1)\{\gamma_{m'} + (\alpha\gamma)_{jm'} + (\beta\gamma)_{km'} + (\alpha\beta\gamma)_{jkm'}\}$$

$$+ \tau_{l'} + (\alpha\tau)_{jl'} + (\beta\tau)_{kl'} + (\alpha\beta\tau)_{jkl'} + (\tau\gamma)_{l'm'} + (\alpha\tau\gamma)_{jl'm'} +$$

$$(\beta\tau\gamma)_{kl'm'} + (\alpha\beta\tau\gamma)_{jkl'm'} + \in_{ijkl'm'}]. \qquad (3.27)$$

Then from above analysis we test the following hypothesis :

$H_{04}: \tau_1 = \tau_2 = \cdots = \tau_t = 0.$

$H_{05}: \gamma_1 = \gamma_2 = \cdots = \gamma_d = 0.$

$H_{06}: (\alpha\tau)_{j1} = (\alpha\tau)_{j2} = \cdots = (\alpha\tau)_{jt} = 0.$

$H_{07}: (\alpha\gamma)_{j1} = (\alpha\gamma)_{j2} = \cdots = (\alpha\gamma)_{jd} = 0.$

$H_{08}: (\beta\tau)_{k1} = (\beta\tau)_{k2} = \cdots = (\beta\tau)_{kt} = 0.$

$H_{09}: (\beta\gamma)_{k1} = (\beta\gamma)_{k2} = \cdots = (\beta\gamma)_{kd} = 0.$

$H_{010}: (\alpha\beta\tau)_{jk1} = (\alpha\beta\tau)_{jk2} = \cdots = (\alpha\beta\tau)_{jkt} = 0.$

$H_{011}: (\alpha\beta\gamma)_{jk1} = (\alpha\beta\gamma)_{jk2} = \cdots = (\alpha\beta\gamma)_{jkd} = 0.$

$H_{012}: (\tau\gamma)_{11} = (\tau\gamma)_{12} = \cdots = (\tau\gamma)_{1d} = (\tau\gamma)_{21} = (\tau\gamma)_{22} = \cdots = (\tau\gamma)_{2d}$

$= \cdots = (\tau\gamma)_{t1} = (\tau\gamma)_{t2} = \cdots = (\tau\gamma)_{td} = 0.$

$H_{013}: (\alpha\tau\gamma)_{j11} = (\alpha\tau\gamma)_{j12} = \cdots = (\alpha\tau\gamma)_{j1d} = \cdots = (\alpha\tau\gamma)_{jtd} = 0.$

$H_{014}: (\beta\tau\gamma)_{k11} = (\beta\tau\gamma)_{k12} = \cdots = (\beta\tau\gamma)_{k1d} = \cdots = (\beta\tau\gamma)_{ktd} = 0.$

$H_{015}: (\alpha\beta\tau\gamma)_{jk11} = (\alpha\beta\tau\gamma)_{jk12} = \cdots = (\alpha\beta\tau\gamma)_{jktd} = 0.$

The ANOVA based on the set of transformed observations above , the $[Y^*_{ijk2}, \ldots, Y^*_{ijkp}]$ provides the ANOVA for within-units effects. This leads to the following forms for the sum square terms :

$$S_T = \sum_{l=2}^{t} n\,(\overline{Y^*_l}(\overline{Y^*_l})') \qquad , \qquad (3.28)$$

where

$$\overline{Y_l^*} = \sum_{j=1}^{a} \sum_{k=1}^{b} n_{jk} \overline{Y_{jklm}^*} / n \quad \text{for each } l = 2, \ldots, t, \qquad (3.29)$$

and

$$\overline{Y_{jklm}^*} = \sum_{i=1}^{n_{jk}} Y_{jklm}^* / n_{jk} \quad \text{for each } j = 1, \ldots, a, \text{ and } k = 1, \ldots, b\,. \qquad (3.30)$$

$$S_D = \sum_{m=2}^{d} n\,(\overline{Y_m^*}(\overline{Y_m^*})') \quad , \qquad (3.31)$$

where

$$\overline{Y_m^*} = \sum_{j=1}^{a} \sum_{k=1}^{b} n_{jk} \overline{Y_m^*} / n \quad \text{for } m = 2, \ldots, d \qquad (3.32)$$

$$S_{AxT} = \sum_{m=2}^{d} \sum_{j=1}^{a} n_{jk} \left(\overline{Y_{jklm}^*} - \overline{Y_l^*}\right)\left(\overline{Y_{jklm}^*} - \overline{Y_l^*}\right)', \quad \text{for each } k = 1, \ldots, b \qquad (3.33)$$

$$S_{AxD} = \sum_{m=2}^{d} \sum_{j=1}^{a} n_{jk} \left(\overline{Y_{jklm}^*} - \overline{Y_m^*}\right)\left(\overline{Y_{jklm}^*} - \overline{Y_m^*}\right)', \text{ for each } k = 1, \ldots, b \qquad (3.34)$$

$$S_{BxT} = \sum_{l=2}^{t} \sum_{k=1}^{b} n_{jk} \left(\overline{Y_{jklm}^*} - \overline{Y_l^*}\right)\left(\overline{Y_{jklm}^*} - \overline{Y_l^*}\right)', \quad \text{for each } j = 1, \ldots, a \qquad (3.35)$$

$$S_{BxD} = \sum_{m=2}^{d} \sum_{k=1}^{b} n_{jk} \left(\overline{Y_{jklm}^*} - \overline{Y_m^*}\right)\left(\overline{Y_{jklm}^*} - \overline{Y_m^*}\right)', \text{ for each } j = 1, \ldots, a \qquad (3.36)$$

$$S_{AxBxT} = \sum_{l=2}^{t} \sum_{k=1}^{b} \sum_{j=1}^{a} n_{jk} \left(\overline{Y_{jklm}^*} - \overline{Y_l^*}\right)\left(\overline{Y_{jklm}^*} - \overline{Y_l^*}\right)' \qquad (3.37)$$

$$S_{AxBxD} = \sum_{m=2}^{d} \sum_{k=1}^{b} \sum_{j=1}^{a} n_{jk} \left(\overline{Y_{jklm}^*} - \overline{Y_m^*}\right)\left(\overline{Y_{jklm}^*} - \overline{Y_m^*}\right)' \qquad (3.38)$$

$$S_{TxD} = \sum_{m=2}^{d} \sum_{l=2}^{b} n \left(\overline{Y_{jklm}^*} - \overline{Y_{lm}^*}\right)\left(\overline{Y_{jklm}^*} - \overline{Y_{lm}^*}\right)' \qquad (3.39)$$

where

$$\overline{Y_{lm}^*} = \sum_{j=1}^{a} \sum_{k=1}^{b} n_{jk} \overline{Y_{jklm}^*} / n \quad \text{for each } l = 2, \ldots, t, \text{ and } m = 2, \ldots, d \qquad (3.40)$$

$$S_{AxTxD} = \sum_{m=2}^{d} \sum_{l=2}^{t} \sum_{j=1}^{a} n_{jk} \left(\overline{Y^*_{jklm}} - \overline{Y^*_{lm}}\right)\left(\overline{Y^*_{jklm}} - \overline{Y^*_{lm}}\right)' \quad (3.41)$$

$$S_{BxTxD} = \sum_{m=2}^{d} \sum_{l=2}^{t} \sum_{k=1}^{b} n_{jk} \left(\overline{Y^*_{jklm}} - \overline{Y^*_{lm}}\right)\left(\overline{Y^*_{jklm}} - \overline{Y^*_{lm}}\right)' \quad (3.42)$$

$$S_{AxBxTxD} = \sum_{m=2}^{d} \sum_{l=2}^{t} \sum_{k=1}^{b} \sum_{j=1}^{d} n_{jk} \left(\overline{Y^*_{jklm}} - \overline{Y^*_{lm}}\right)\left(\overline{Y^*_{jklm}} - \overline{Y^*_{lm}}\right)' \quad (3.43)$$

$$S_{E} = \sum_{m=2}^{d} \sum_{l=2}^{t} \sum_{k=1}^{b} \sum_{j=1}^{a} \sum_{i=1}^{n\,jk} n_{jk} \left(\overline{Y^*_{ijklm}} - \overline{Y^*_{lm}}\right)\left(\overline{Y^*_{jklm}} - \overline{Y^*_{lm}}\right)' \quad (3.44)$$

Then from above sum square terms, we have:

(1) $S_T \sim W_r(t-1, \Sigma_\in)$.

(2) $S_D \sim W_r(d-1, \Sigma_\in)$.

(3) $S_{A\times T} \sim W_r((t-1)(a-1), \Sigma_\in)$.

(4) $S_{A\times D} \sim W_r((d-1)(a-1), \Sigma_\in)$.

(5) $S_{B\times T} \sim W_r((t-1)(b-1), \Sigma_\in)$.

(6) $S_{B\times D} \sim W_r((d-1)(b-1), \Sigma_\in)$.

(7) $S_{A\times B\times T} \sim W_r((t-1)(b-1)(a-1), \Sigma_\in)$.

(8) $S_{A\times B\times D} \sim W_r((d-1)(b-1)(a-1), \Sigma_\in)$.

(9) $S_{T\times D} \sim W_r((d-1)(t-1), \Sigma_\in)$.

(10) $S_{A\times T\times D} \sim W_r((d-1)(t-1)(a-1), \Sigma_\in)$.

(11) $S_{B\times T\times D} \sim W_r((d-1)(t-1)(b-1), \Sigma_\in)$.

(12) $S_{A\times B\times T\times D} \sim W_r((d-1)(t-1)(b-1)(a-1), \Sigma_\in)$.

(13) $S_E \sim W_r((d-1)(t-1)(n-ab), \Sigma_\in)$,

Let $S_4 = S_T$, $S_5 = S_D$, $S_6 = S_{A\times T}$, $S_7 = S_{A\times D}$, $S_8 = S_{B\times T}$, $S_9 = S_{B\times D}$,

$S_{10} = S_{A\times B\times T}$, $S_{11} = S_{A\times B\times D}$, $S_{12} = S_{T\times D}$, $S_{13} = S_{A\times T\times D}$

$S_{14} = S_{B\times T\times D}$, $S_{15} = S_{A\times B\times T\times D}$.

Then the test statistics are summarized as follows :
First : The multivariate Wilks test (Wilks, 1932) [10]:

$$T_W = \frac{|S_E|}{|S_E + S_k|}, \text{ when } \mathrm{H}_{0k} \text{ is true, for } k = 4,\dots,15. \quad (3.45)$$

Second : The Lawley – Hotelling trace (Lawley, 1938; Hotelling, 1947) [5,4]:

$$T_{LH} = trace(S_k\ S_E^{-1}), \text{ when } \mathrm{H}_{0k} \text{ is true, for } k = 4,\dots,15. \quad (3.46)$$

Third : The Bartlett-Nanda-Pillai trace (Bartlett, 1939; Nanda, 1950; Pillai, 1955) [2,6,7]:

$$T_{BNP} = trace\{S_k\ S_E^{-1}\ (\mathrm{I} + S_k\ S_E^{-1})^{-1}\}, \text{ when } \mathrm{H}_{0k} \text{ is true, for } k = 4,\dots,15. \quad (3.47)$$

Forth: Roy's Union-Intersection (UI) test or largest root (Ray, 1953):[8]:

$$T_R = \text{largest characteristics root of } (S_k\ S_E^{-1}), \text{ when } \mathrm{H}_{0k} \text{ is true, for } k = 4,\dots,15. \quad (3.48)$$

4. Conclusions

The ANOVA based on the first set of transformed observations provides the ANOVA for the between-units effects, while the ANOVA based on the set of transformed observations the Y^*_{ijk1m} 's for each $l = 2,\dots,t$, and $m = 2,\dots,d$ provides the ANOVA for within-units effects. The hypotheses for the between-units effects, within-units effects, and the interaction between them are tested. The multivariate Wilks test, Lawley – Hotelling trace test, Bartlett-Nanda-Pillai trace test, and Roy's Union-Intersection (UI) test are obtained.

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نموذج تحليل التباين ذي الطرفين للقياسات المتكررة المتعدد المتغيرات

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المستخلص

تمت دراسة نموذج تحليل التباين ذي الطرفين للقياسات المتكررة المتعدد المتغيرات للبيانات الكاملة في حالة متغيرات الاستجابة المتعددة المتغيرات. لقد أعطيت الاختبارات الإحصائية لمختلف الفرضيات المتعلقة بالعوامل بين الوحدات و العوامل داخل الوحدات و التفاعل بينهما.