J. Edu. & Sci., Vol. (22), No. (2) 2009

Solution of Higher Order Linear Freadholm Integro-differential Equation by Elementary Approximation

Aml Jassim Mohammed
Department of Mathematics / College of Education
University of Mosul

Received 08 / ٠4 / 200٨ **Accepted** 22 / 07 / 2008

الملخص:-

في هذا البحث، استطعنا إيجاد الحل التقريبي لمعادلة فريدهولم التكاملية التفاضلية الخطية، باستخدام طريقة اتكن وذلك بالاستعانة بطريقة التعاقب التكرارية . بعض الأمثلة العددية قدمت لبيان الدقة لهذه الطريقة.

Abstract:-

In this paper, we found an approximate solution for solving linear Fredholm integro-differential equation, by using Aitken's method with the help of the successive iteration method. Some numerical example were presented to show the accuracy of this method.

1.1 Introduction:-

The objection of this study is the numerical treatment of equations of the form:

$$\frac{d^n u(x)}{d x^n} + \sum_{m=0}^{n-1} p_m(x) \frac{d^m u(x)}{d x^m} = f(x) + \lambda \int_a^b \sum_{r=0}^{L} k_r(x,y)\, u^{(r)}(y) dy \qquad (1)$$

where

$k_r(x,y),(r=0,\ldots,L), p_m(x),(m=0,\ldots n-1)$ and $f(x)$ are given functions, and $u(x)$ is an unknown function, $u^{(r)}(x)$ denote the r^{th} derivative of $u(x)$, and λ is a scalar parameter.

In our work we will consider the following form of higher order linear Fredholm integro-differential equation (FIE's)of order, n, integer number, $n>0$, the kernel is degenerate, $\lambda=1$ and $L=0$.

$$\frac{d^n u(x)}{d\,x^n} = f(x) + \int_a^b k(x,y)\,u(y)dy \tag{2}$$

with boundary condition

$$\sum_{m=0}^{n-1}\left[r_{j,m}D^m u(a) + r_{j,n+m}D^m u(b)\right] = \alpha_j \qquad , j=0,1,\ldots n-1$$

In [2] Delves &Mohamed deals with non linear integro-diffrential equation. In [3] Mustafa Khiralla used simplex and dual simplex methods to find the optimum solution for first order FIDE's2nd kind. In [4] Fatima Al-Hammeed used spline function to find the numerical solution for higher order FIDE's2nd kind. In [5] Roger Alexander applied Aitken extrapolation to certain sequences. In [6] Sepandar &Taher &Christopher &Gene present novel algorithm (Aitken extrapolation) for the fast computation of PageRank.

1.2 Aitken Elementary Approximation :-

This method deals with the linear Fredholm integral equation (FIE's) of the second kind having approximate solutions $u_1(x), u_2(x), u_3(x)$, we can extrapolate (elementary) to an improve and estimate, this can be done by considering the following formula [1]:

$$S(x) = \frac{u_3(x)u_1(x) - u^2{}_2(x)}{u_3(x) - 2u_2(x) + u_1(x)} \tag{3}$$

1.3 The Basic Technique:-

Our intention is using Aitken elementary approximation method on successive approximations for solving FIE's. Published codes of algorithms for the treatment of various types of integro-differential equation including equation (2)

$$\frac{d^n u(x)}{d\,x^n} = f(x) + \int_a^b k(x,y)\,u(y)dy$$

subject to boundary condition

$$\sum_{m=0}^{n-1}\left[r_{j,m}D^{m}u(a)+r_{j,n+m}D^{m}u(b)\right]=\alpha_{j} \qquad , j=0,1,\ldots n-1$$

successive approximation method with the first iteration gives:

$$\frac{d^{n}u_{1}(x)}{d\,x^{n}}=f(x)+\int_{a}^{b}k(x,y)\,u_{0}(y)dy \tag{4}$$

substituted the primary initial condition $u_0(y)=c$ into (4)

$$\frac{d^{n}u_{1}(x)}{d\,x^{n}}=f(x)+\int_{a}^{b}k(x,y)\,c\,dy$$

$\because$ c is constant then

$$\frac{d^{n}u_{1}(x)}{d\,x^{n}}=f(x)+c\int_{a}^{b}k(x,y)\,dy \tag{5}$$

assume that $g(x)=f(x)+c\int_{a}^{b}k(x,y)\,dy$

then equation (5) become $\dfrac{d^{n}u_{1}(x)}{d\,x^{n}}=g(x)$ (6)

after integrating equation (6) n-times with respect to x ,from a to b, yield

$$u_{1}(x)=G(x)+\sum_{i=0}^{n}a_{i}x^{n-i} \tag{7}$$

by using the boundary conditions in equation (7) to obtain the constants a_i's and substituting in equation (7), we get $u_1(x)$, then substituting the first approximation $u_1(x)$ again in equation (2) to obtain second approximate $u_2(x)$.

$$\frac{d^{n}u_{2}(x)}{d\,x^{n}}=f(x)+\int_{a}^{b}k(x,y)\,u_{1}(y)dy$$

as before and after using the boundary conditions, we have $u_1(x)$, $u_2(x)$ and $u_3(x)$ substituted them in Aitken elementary formula.

1.4 Numerical Results:-

We test some of the numerical examples performed in solving this linear integro-differential equation. The exact solution is used only to show the accuracy of the numerical solution obtained with our method.

Example (1):

Consider the problem which 2nd order linear FIDE:

$$\frac{d^2u(x)}{dx^2} = f(x) + \int_{-1}^{0} (x+y)u(y)\,dy$$

where

$$f(x) = -\cos(x) - 0.8414709\ x + 0.381773$$

and boundary conditions $u(-1) = \cos(-1)$, $u(0) = 1$

while the exact solution is $u(x) = \cos(x)$.

The successive elementary approximations methods are used to solve this example.

Successive approximations method with the first iteration gives:

$$\frac{d^2u_1(x)}{dx^2} = -\cos(x) - 0.8414709\ x + 0.381773 + \int_{-1}^{0} (x+y)u_0(y)\,dy$$

substituted the primary initial condition $u_0(-1) = \cos(-1)$,

$$\frac{d^2u_1(x)}{dx^2} = -\cos(x) - 0.8414709\ x + 0.381773 + \int_{-1}^{0} (x+y)\cos(-1)\,dy$$

$$\frac{d^2u_1(x)}{dx^2} = -\cos(x) - 0.301169\ x + 0.111622$$

integrating above equation 2-times with respect to x and using the boundary condition we get the first approximation

$$u_1(x) = \cos(x) - 0.050195\ x^3 + 0.055811\ x^2 + 0.106005\ x$$

substituted the first approximation into the above example to obtain the second approximation

$$\frac{d^2u_2(x)}{dx^2} = -\cos(x) - 0.8414709\ x + 0.381773 + \int_{-1}^{0} (x+y)u_1(y)\,dy$$

integrating this equation with respect to x, and using boundary condition, yields

$$u_2(x) = \cos(x) - 0.003642\,x^3 + 0.0056716\ x^2 + 0.009313\ x$$

again substituting second approximation to get the third approximation

$$\frac{d^2u_3(x)}{dx^2} = -\cos(x) - 0.8414709\ x + 0.381773 + \int_{-1}^{0} (x+y)\ u_2(y)\ dy$$

integrate this equation, we get

$$u_3(x) = \cos(x) - 0.000309\, x^3 + 0.000479\, x^2 + 0.000788\, x$$

now use $u_1(x), u_2(x), and\ u_3(x)$ and substituting them in Aitken formula (3).

$$S(x) = \frac{u_3(x)u_1(x) - u^2{}_2(x)}{u_3(x) - 2u_2(x) + u_1(x)}$$

$$S(x) =$$

$$\left[\frac{-0.043221\cos(x)x^2 + 0.449466\ \cos(x)\,x + 0.088167\ \cos(x) + 0.0000023\ x^5 \ldots + 0.0000000068\ x^4 - 0.00000099\, x^3 - 0.0000108\ x^2 - 0.00000318\ x}{-0.043221x^2 + 0.0449467\,x + 0.088167}\right]$$

Example (2):

Consider the 3rd order linear FIDE:

$$\frac{d^3u(x)}{dx^3} = f(x) + \int_0^1 (x-y)\,u(y)\,dy$$

where $\quad f(x) = \frac{103}{15} - \frac{5}{4}x$

with boundary conditions

$$u(0) = 0,\ u(1) = 3,\ u'(0) = 2$$

and the exact solution is $u(x) = 2x + x^3$.

By using the primary initial condition $u_0(0) = 0$ to get first iteration

$$\frac{d^3u_1(x)}{dx^3} = \frac{103}{15} - \frac{5}{4}x + \int_0^1 (x-y)\,u_0(y)\,dy$$

$$\frac{d^3u_1(x)}{dx^3} = \frac{103}{15} - \frac{5}{4}x$$

integrating above equation 3-times with respect to x and using the boundary condition we get the first approximation

$$u_1(x) = 1.14444\, x^3 - 0.05208\ x^4 - 0.09236\, x^2 + 2x$$

the same way we obtain

$$u_2(x) = 1.00048\ x^3 - 0.0000212\ x^4 - 0.000268\,x^2 + 2x$$

and

$$u_3(x) = 1.000001\ x^3 - 0.00000048\ x^4 - 0.00000056\,x^2 + 2x$$

then using Aitken formula equation (3)

$$S(x) = \frac{u_3(x)u_1(x) - u^2{}_2(x)}{u_3(x) - 2u_2(x) + u_1(x)}$$

$$S(x) = \left[\frac{\begin{array}{c} 0.143484\,x^4 - 0.051659\,x^5 - 0.195144\,x^3 \ldots \\ + 0.286969\,x^2 - 0.000000019\,x^6 - 0.183651\,x \end{array}}{0.143485\,x - 0.051659\,x^2 - 0.091825} \right]$$

Conclusions:

Some numerical methods were used to find an approximate solution for higher order linear Fredholm integro-differential equation.

A comparison between these methods depending on least square error(L.S.E), which was calculated from the numerical solution against the exact solution.

Successive method and Aitken elementary method were used to treat the higher order linear Fredholm integro-differential equations and perfect results are presented and comparison is done as follows:

Tables (1-2) show a comparison between the results obtained form solving testing examples (1-2) respectively by using successive method and Aitken elementary method.

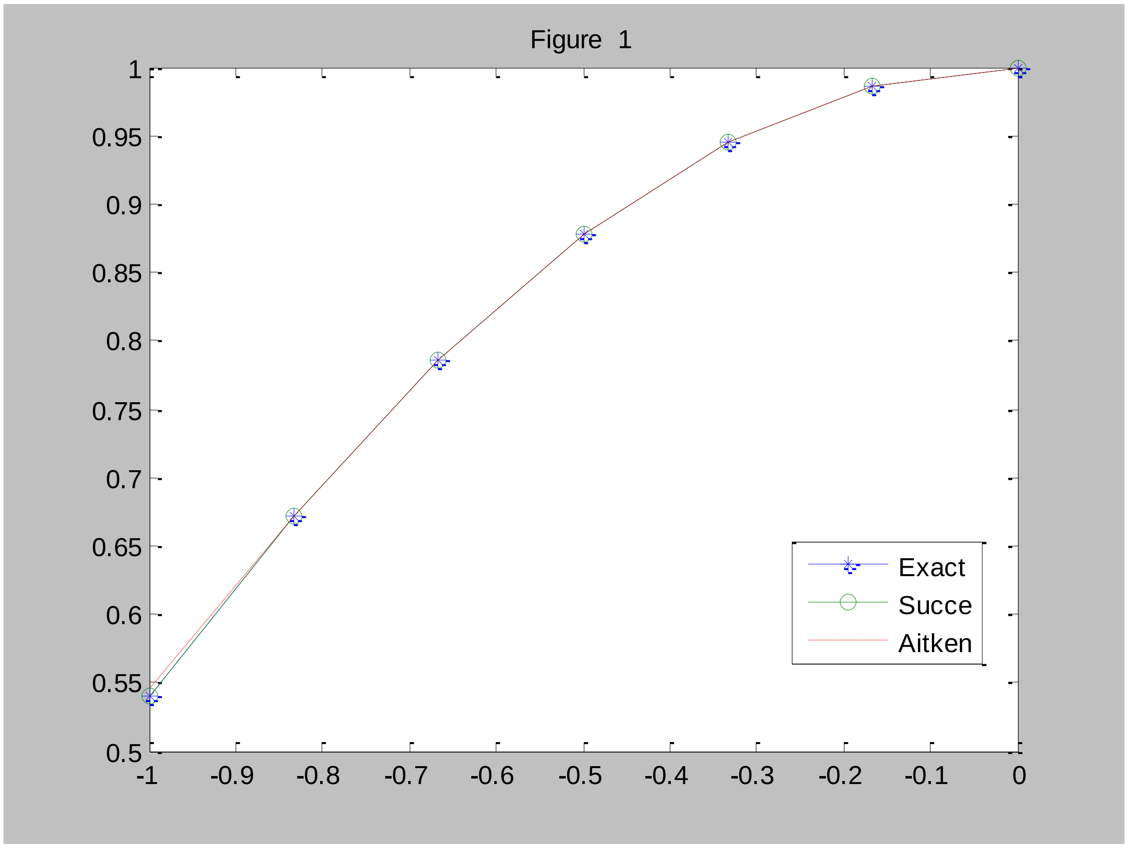
X	Exact	Successive method	Aitken elementary method
-1	0.54030231	0.54030231	0.54545732
-5/6	0.67241224	0.6722697	0.67240983
-4/6	0.78588726	0.7866627	0.78588567
-3/6	0.87758256	0.87734684	0.87758337
-2/6	0.94495695	0.94475887	0.94496003
-1/6	0.98614323	0.98602659	0.98614668
0	1.00000000	1.00000000	1.00000000
L.S.E		1.7834111e-7	2.657419e-005

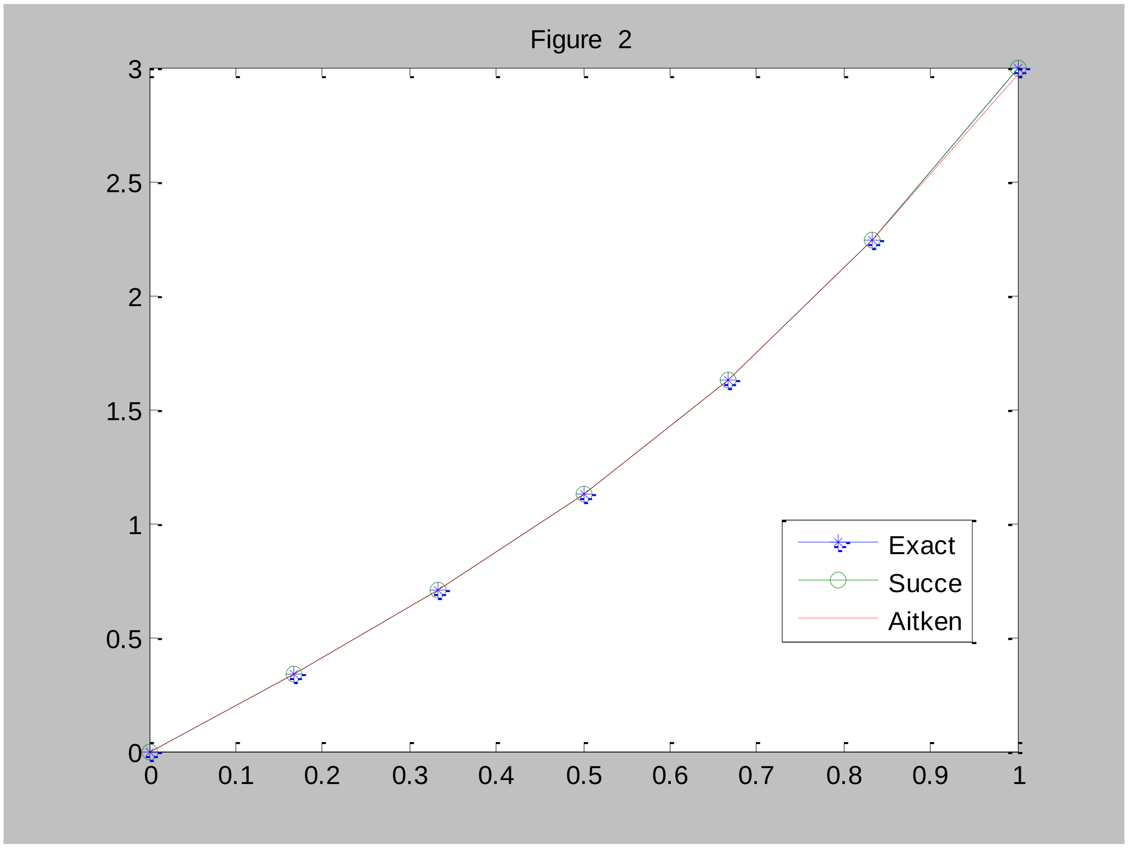
Table (1)
Comparison results between exact solution and numerical solutions (Successive and Aitken)

X	Exact	Successive Method	Aitken elementary method
0	0.00000000	0.00000000	0.00000000
1/6	0.33796296	0.33796295	0.33796297
2/6	0.70370370	0.70370367	0.70370371
3/6	1.12500000	1.24999959	1.25000009
4/6	1.62962963	1.62962959	1.62962964
5/6	2.24537037	2.24537035	2.24537037
1	3.00000000	3.00000000	2.999973
L.S.E		4.252821e-015	7.452902e-010

Table(2)
Comparison results between exact solution and numerical solutions (Successive and Aitken)

The comparison between the solution by successive methods and Aitken elementary method with the exact solution for some test examples have been illustrated in tables (1-2). Figures (1-2) show a comparison between the analytic and numerical solution of higher order linear Fredholm integro-differential equations which was presented in test examples (1-2) respectively.





Reference:

1) Gerald, C. F, and Wheathley, P. O; (1979), "*Applied Numerical Analysis* " Adssdison Wesley printed in the United State of American.

2) Delves, L. M and Mohamed, J. L. (1985), "*Computational Methods for Integral Equations*", Cambridge University press.

3) Mustafa Khiralla, (2002), "*Numerical Treatmentment of Fredholm Integro-differential equations",* University of Mustansiriyah, unpublished thesis.

4) Fatima Al-Hammeed, (2002), "*Numerical Solution of Fredholm Integral Equations Using Spline Functions*", University of Technology, unpublished thesis.

5) Roger Alexander, (2003), "*Index-Doubling in Sequences by Aitken Extrapolation*", American Mathematical Society.

6) Sepandar, Taher, Christopher and Gene,(2003), "*Extrapolation Methods for Accelerating PageRank Computations*", University of Stanford.