Journal of Basrah Researches ((Sciences)) Vol. 35, No.2, 15 April ((2009))

ON GENERALIZATION OF SUMMATION- INTEGRAL SZÃSZ OPERATORS

Amal K. Hassan
Dept of Mathematics, College of Science ,University of Basrah , Basrah, IRAQ.
ISSN –1817–2695
((Received 26/10/2008, Accepted 11/2/2009))

ABSTRACT.

In this paper, we introduce and study some direct results in simultaneous approximation for a generalization of Summation –Integral Szãsz type operator $M_n(f(t);x)$. First, we establish the basic pointwise convergence theorem and then proceed to discuss the Voronovaskaja-type asymptotic formula. Finally, we obtain an error estimate in terms of modulus of continuity of the function being approximated.

Keywords: Linear positive operators, Simultaneous approximation, Voronovaskaja-type asymptotic formula, Degree of approximation, Modulus of continuity.

1. INTRODUCTION

Let $m \in N$ (the set of positive integer), for sufficiently small values of $\eta > 0$, the m-th order modulus of continuity $\omega_m(f,\eta;I)$ for a continuous function f on the interval I is defined as:

$$\omega_m(f,\eta;I) = \sup\left\{\left|\Delta_h^m f(x)\right| : |h| \le \eta,\ x, x+mh \in I\right\},$$

where $\Delta_h^m f(x)$ is the m-th order forward difference with step length h and is given by:

$$\Delta_h^m f(t) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(t+ih).$$

For $m=1$, $\omega_m(f,\eta;I)$ is written simply as $\omega_f(\eta;I)$ or $\omega(f,\eta;I)$.

Voronovaskaja in [8] found the following relation for Bernstein operators $B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}$, $x \in [0,1]$:

$$\lim_{n\to 0} n\left(B_n(f(t);x) - f(x)\right) = \frac{x(1-x)}{2} f''(x) .$$

This relation showed that the degree of approximation Bernstein operators is $O(1)$. Many researchers were found similar relations for another sequences of linear positive operators these relations are called Voronovaskaja-type asymptotic formula.

Many papers introduced new sequences of linear positive operators and discussed the Voronovaskaja-type asymptotic formula for these sequences. In this direction we refer to [1,5,7].

independently Mazher and Totic [6] and Kasana et.al [3] proposed the following summation –Integral Szasz operators to approximate unbounded functions on $[0,\infty)$ as:

$$L_n(f(t);x) = n\sum_{k=0}^{\infty} q_{n,k}(x) \int_0^{\infty} q_{n,k}(t)\, f(t)\, dt,$$

where, $f \in C_\alpha[0,\infty) = \{ f \in C[0,\infty) : |f(t)| \le Ce^{\alpha t} \text{ for some } C > 0, \alpha > 0\}$, $q_{n,k}(x) = \frac{e^{-nx}(nx)^k}{k!}$.

In this paper, we introduce a generalization for the operator $L_n(f(t);x)$ denoted by $M_n(f(t);x)$ as follows:
For $\nu \in N^0 = \{0,1,2,\dots\}$,we define

(1.1) $M_n(f(t);x) = n\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty} q_{n,k+\nu}(t) f(t)\, dt$,

Note that $M_n(f(t);x) = L_n(f(t);x)$ whenever $\nu = 0$.

We may also write the operator (1.1) as $M_n(f(t);x) = \int_0^{\infty} W_n(t,x) f(t)\, dt$ where

$$W_n(t,x) = n\sum_{k=0}^{\infty} q_{n,k}(x) q_{n,k+\nu}(t).$$

The space $C_\alpha[0,\infty)$ is normed by

$$\|f\|_{C_\alpha} = \sup_{0\le t<\infty} |f(t)| e^{-\alpha t}.$$

Throughout this paper, we assume that C denotes a positive constant not necessarily the same at all occurrence and $[\beta]$ denotes the integer part of β.

2. PRELIMINARY RESULTS

For $f \in C[0,\infty)$ the classic Szãsz operators is defined as [2] $S_n(f;x) = \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right)$, $x \in [0,\infty)$ and for $m \in N^0$ (the set of nonnegative integers), the $m-$th order moment of the Szãsz operators is defined as

$$\mu_{n,m}(x) = \sum_{k=0}^{\infty} q_{n,k}(x)\left(\frac{k}{n} - x\right)^m.$$

LEMMA 2.1. [2] For $m \in N^0$, the function $\mu_{n,m}(x)$ defined above, has the following properties:

(i) $\mu_{n,0}(x) = 1$, $\mu_{n,1}(x) = 0$, and the recurrence relation is

$$n\mu_{n,m+1}(x) = x\left(\mu'_{n,m}(x) + m\mu_{n,m-1}(x)\right) \quad , m \ge 1;$$

(ii) $\mu_{n,m}(x)$ is a polynomial in x of degree at most $[m/2]$;

(iii) For every $x \in [0,\infty)$, $\mu_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

From above lemma, we get

$$(2.1) \qquad \sum_{k=0}^{\infty} q_{n,k}(x)(k-nx)^{2j} = n^{2j}\left(\mu_{n,2j}(x)\right) = n^{2j}\{O(n^{-j})\} = O(n^j).$$

For $m \in N^0$, the m-th order moment $T_{n,m}(x)$ for the modified Szãsz operators is defined as:

$$B_{n,m}(x) = L_n((t-x)^m;x) = n\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty} q_{n,k}(t)(t-x)^m\, dt$$

LEMMA 2.2.[3] For the function $B_{n,m}(x)$,we have $B_{n,0}(x) = 1$, $B_{n,1}(x) = \frac{1}{n}$ $\quad B_{n,2}(x) = \frac{2}{n}\left(x + \frac{1}{n}\right)$

and there holds the recurrence relation:

$$nB_{n,m+1}(x) = xB'_{n,m}(x) + (m+1)B_{n,m}(x) + 2mxB_{n,m-1}(x)$$

Further, we have the following consequences of $B_{n,m}(x)$:

(i) $B_{n,m}(x)$ is a polynomial in x of degree exactly m;

(ii) For every $x \in [0,\infty)$, $B_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

For $m \in N^0$, the $m-$th order moment $T_{n,m}(x)$ for the operators (1.1) is defined as:

$$T_{n,m}(x) = M_n\left((t-x)^m;x\right) = n\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty} q_{n,k+\nu}(t)(t-x)^m\, dt.$$

LEMMA 2.3. For the function $T_{n,m}(x)$, we have $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{\nu+1}{n}$, $T_{n,2}(x) = \frac{(\nu+1)(\nu+2)}{n^2} + \frac{2x}{n}$ and there holds the recurrence relation:

(2.2) $nT_{n,m+1}(x) = xT'_{n,m}(x) + (m+\nu+1)T_{n,m}(x) + 2mxT_{n,m-1}(x)$.

Further, we have the following consequences of $T_{n,m}(x)$:

(i) $T_{n,m}(x)$ is a polynomial in x of degree exactly m;

(ii) For every $x \in [0,\infty), T_{n,m}(x) = O\left(n^{-[(m+1)/2]}\right)$.

Proof: By direct computations, we have $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{\nu+1}{n}$ and $T_{n,2}(x) = \frac{(\nu+1)(\nu+2)}{n^2} + \frac{2}{n}x$. Next, we prove (2.2). For $x = 0$ it clearly holds. For $x \in (0,\infty)$, we have

$$T'_{n,m}(x) = n\sum_{k=0}^{\infty} q'_{n,k}(x)\int_0^{\infty} q_{n,k+\nu}(t)(t-x)^m\,dt - mT_{n,m-1}(x)$$

Using the relations $xq'_{n,k}(x) = (k-nx)\,q_{n,k}(x)$, we get:

$$xT'_{n,m}(x) = n\sum_{k=0}^{\infty}(k-nx)\,q_{n,k}(x)\int_0^{\infty} q_{n,k+\nu}(t)(t-x)^m\,dt - mxT_{n,m-1}(x)$$

$$= n\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty}(k+r-nt)q_{n,k+\nu}(t)(t-x)^m\,dt + nT_{n,m+1}(x) - mT_{n,m-1}(x) - \nu xT_{n,m}(x)$$

$$= n\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty} t\,q'_{n,k+\nu}(t)(t-x)^m\,dt + nT_{n,m+1}(x) - mxT_{n,m-1}(x) - \nu T_{n,m}(x)$$

$$= n\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty} q'_{n,k+\nu}(t)(t-x)^{m+1}\,dt + nx\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty} q'_{n,k+\nu}(t)(t-x)^m\,dt$$

$$+ nT_{n,m+1}(x) - \nu T_{n,m}(x) - mxT_{n,m-1}(x).$$

Integrating by parts, we get

$$xT'_{n,m}(x) = n\,T_{n,m+1}(x) - (m+\nu+1)\,T_{n,m}(x) - 2mx\,T_{n,m-1}(x)$$

from which (2.2) is immediate.

From the values of $T_{n,0}(x)$ and $T_{n,1}(x)$, it is clear that the consequences (i) and (ii) holds for $m = 0$ and $m = 1$. By using (2.2) and the induction on m the proof of consequences (i) and (ii) follows, hence the details are omitted. ■

From the above lemma, we have

(2.3) $$n\sum_{k=0}^{\infty} q_{n,k}(x)\int_0^{\infty} q_{n,k+\nu}(t)(t-x)^{2r}\,dt = T_{n,2r}(x) = O(n^{-r})$$

LEMMA 2.4. [4] Let δ and γ be any two positive real numbers and $[a,b] \subset (0,\infty)$. Then, for any $s > 0$, we have

$$\left\| \int_{|t-x|\ge\delta} W_n(t,x)\,e^{\gamma t}\,dt \right\|_{C[a,b]} = O(n^{-s}).$$

Making use of Taylor's expansion, Schwarz inequality for integration and then for summation and (2.3), the lemma easily follows, hence the details are omitted.

LEMMA 2.5. [3] There exist polynomials $Q_{i,j,r}(x)$ independent of n and k such that

$$x^r D^r (q_{n,k}(x)) = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i (k-nx)^j Q_{i,j,r}(x) q_{n,k}(x), \text{ where } D = \frac{d}{dx}.$$

3. MAIN RES

Firstly, we show that the derivative $M_n^{(r)}(f(t);x)$ is an approximation process for $f^{(r)}(x)$, $r = 0,1,2,\ldots$.

Theorem 3.1. If $r \in N$, $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$ and $f^{(r)}$ exists at a point $x \in (0,\infty)$, then

(3.1) $\lim_{n\to\infty} M_n^{(r)}(f(t);x) = f^{(r)}(x)$.

Further, if $f^{(r)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty)$, $\eta > 0$, then (3.1) holds uniformly in $[a,b]$.

Proof: By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where, $\varepsilon(t,x) \to 0$ *as* $t \to x$. Hence

$$M_n^{(r)}(f(t);x) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i\,dt + \int_0^\infty W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r\,dt$$

$$:= I_1 + I_2.$$

Now, using Lemma 2.3 and induction we get that $M_n(t^m;x)$ is a polynomial in x of degree exactly m, for all $m \in N^0$. Further , we can write it as:

(3.2) $M_n(t^m;x) = x^m + \frac{m(m+\nu)}{n} x^{m-1} + O(n^{-2})$.

$$I_1 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \int_0^\infty W_n^{(r)}(t,x) t^j\,dt$$

$$= \frac{f^{(r)}(x)}{r!}(r!) = f^{(r)}(x) \text{ as } n \to \infty.$$

Therefore,

Next, making use of Lemma 2.5 we have

$$|I_2| \le \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i \frac{|Q_{i,j,r}(x)|}{x^r} n \sum_{k=0}^{\infty} q_{n,k}(x)|k-nx|^j \int_0^\infty q_{n,k+\nu}(t)|\varepsilon(t,x)||t-x|^r\,dt$$

$$:= I_3.$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, then for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$, whenever $0 < |t-x| < \delta$. For $|t-x| \ge \delta$, there exists a constant $C > 0$ such that $|\varepsilon(t,x)(t-x)^r| \le Ce^{\alpha t}$.

Now, since

$$\sup_{\substack{2i+j\le r\\ i,j\ge 0}} \frac{|Q_{i,j,r}(x)|}{x^r} := M(x) = C \ \forall x \in (0,\infty).$$

Hence,

$$I_3 \le C \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i n \sum_{k=0}^{\infty} q_{n,k}(x)|k-nx|^j \left(\int_{|t-x|<\delta} q_{n,k+\nu}(t)\varepsilon|t-x|^r\,dt + \int_{|t-x|\ge\delta} q_{n,k+\nu}(t)|t-x|^\gamma\,dt \right)$$

$$:= I_4 + I_5.$$

Now, applying Schwarz inequality for integration and then for summation, (2.1) and (2.3) we are led to

$$I_4 \le \varepsilon C \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i n \sum_{k=0}^{\infty} q_{n,k}(x)|k-nx|^j \left(\int_0^\infty q_{n,k+\nu}(t)\,dt\right)^{1/2} \left(\int_0^\infty q_{n,k+\nu}(t)(t-x)^{2r}\,dt\right)^{1/2}$$

(in view of $\int_0^\infty q_{n,k+\nu}(t)\,dt=\frac{1}{n}$)

$$\le \varepsilon C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\left(\sum_{k=0}^{\infty}q_{n,k}(x)(k-nx)^{2j}\right)^{1/2}\left(n\sum_{k=0}^{\infty}q_{n,k}(x)\int_0^\infty q_{n,k+\nu}(t)(t-x)^{2r}\,dt\right)^{1/2}$$

$$\le \varepsilon C\,O(n^{-r/2})\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O(n^{j/2})=\varepsilon O(1)=o(1)$$.(since ε arbitrary)

Again using Schwarz inequality for integration and then for summation, in view of (2.1) and Lemma 2.3, we have

$$I_5\le C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i n\sum_{k=0}^{\infty}q_{n,k}(x)\left|k-nx\right|^j\int_{|t-x|\ge\delta}q_{n,k+\nu}(t)e^{\alpha t}dt$$

$$\le C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i n\sum_{k=0}^{\infty}q_{n,k}(x)\left|k-nx\right|^j\left(\int_0^\infty q_{n,k+\nu}(t)dt\right)^{1/2}\left(\int_0^\infty q_{n,k+\nu}(t)e^{2\alpha t}dt\right)^{1/2}$$

$$\le C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\left(\sum_{k=0}^{\infty}q_{n,k}(x)(k-nx)^{2j}\right)^{1/2}\left(n\sum_{k=0}^{\infty}q_{n,k}(x)\int_0^\infty q_{n,k+\nu}(t)e^{2\alpha t}dt\right)^{1/2}$$

$$\le O(n^{-s})\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O(n^{j/2})$$ (for any $s>0$)

$$=O\left(n^{r/2-s}\right)=o(1)$$ (for $s>r/2$).

Hence $I_2=o(1)$, combining the estimates of I_1 and I_2, we obtain (3.1).

To prove the uniformity assertion, it is sufficient to remark that $\delta(\varepsilon)$ in the above proof can be chosen to be independent of $x\in[a,b]$, and also that the other estimates hold uniformly in $[a,b]$.

Next theorem is a Voronovaskaja-type asymptotic formula for the operators $M_n^{(r)}(f(t);x)$, $r=0,1,2,\dots$.

THEOREM 3.2. Let $f\in C_\alpha[0,\infty)$ for some $\alpha>0$ .If $f^{(r+2)}$ exists at a point $x\in(0,\infty)$, then,

$$\lim_{n\to 0} n\left(M_n^{(r)}(f(t);x)-f^{(r)}(x)\right)=(r+\nu+1)f^{(r+1)}(x)+xf^{(r+2)}(x). \tag{3.3}$$

Further, if $f^{(r+2)}$ exists and is continuous on the interval $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, then (3.3) holds uniformly on $[a,b]$.

Proof: By the Taylor's expansion of $f(t)$, we get

$$M_n^{(r)}(f(t);x)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}M_n^{(r)}\left((t-x)^i;x\right)+M_n^{(r)}\left(\varepsilon(t,x)(t-x)^{r+2};x\right)$$

$$:=I_1+I_2,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$.

By Lemma 2.3 and (3.2), we have

$$I_1=\sum_{i=r}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=r}^{i}\binom{i}{j}(-x)^{i-j}M_n^{(r)}(t^j;x)$$

$$=\frac{f^{(r)}(x)}{r!}M_n^{(r)}(t^r;x)+\frac{f^{(r+1)}(x)}{(r+1)!}\left((r+1)(-x)M_n^{(r)}(t^r;x)+M_n^{(r)}(t^{r+1};x)\right)$$

$$+\frac{f^{(r+2)}(x)}{(r+2)!}\left(\frac{(r+2)(r+1)}{2}x^2M_n^{(r)}(t^r;x)+(r+2)(-x)M_n^{(r)}(t^{r+1};x)+M_n^{(r)}(t^{r+2};x)\right)$$

$$=f^{(r)}(x)+\frac{f^{(r+1)}(x)}{(r+1)!}\left((r+1)(-x)r!+(r+1)!x+\frac{(r+1)(r+\nu+1)}{n}r!\right)$$

$$+\frac{f^{(r+2)}(x)}{(r+2)!}\left\{\frac{(r+1)(r+2)}{2}x^2r!+(r+2)(-x)\left((r+1)!x+\frac{(r+1)(r+2)}{n}r!\right)+\right.$$

$$\left.\left((r+2)!\frac{x^2}{2}+\frac{(r+2)(r+\nu+2)}{n}(r+1)!x\right)\right\}+O(n^{-2}).$$

Hence in order to prove (3.3) it sufficient to show that $nI_2\to 0$ as $n\to\infty$, which follows on proceeding along the lines of proof of $I_2\to 0$ as $n\to\infty$ in Theorem 3.1.

The uniformity assertion follows as in the proof of Theorem 3.1. ■

Finally, we present a theorem which gives an estimate of the degree of approximation by $M_n^{(r)}(.;x)$ for smooth functions in $C_\alpha[0,\infty)$.

THEOREM 3.3. Let $f\in C_\alpha[0,\infty)$ for some $\alpha>0$ and $r\le q\le r+2$. If $f^{(q)}$ exists and it is continuous on $(a-\eta,b+\eta)\subset(0,\infty),\eta>0$, then for sufficiently large n,

$$\left\|M_n^{(r)}(f(t);x)-f^{(r)}(x)\right\|_{C[a,b]}\le C_1n^{-1}\sum_{i=r}^{q}\left\|f^{(i)}\right\|_{C[a,b]}+C_2\,n^{-1/2}\omega_{f^{(q)}}\left(n^{-1/2}\right)+O(n^{-2})$$

where C_1,C_2 are constants independent of f and n, $\omega_f(\delta)$ is the modulus of continuity of f on $(a-\eta,b+\eta)$, and $\|.\|_{C[a,b]}$ denotes the sup-norm on $[a,b]$.

Proof. By Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)+h(t,x)(1-\chi(t)),$$

where ξ lies between t and x, and $\chi(t)$ is the characteristic function of the interval $(a-\eta,b+\eta)$. Now,

$$M_n^{(r)}(f(t);x)-f^{(r)}(x)=\left(\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i\,dt-f^{(r)}(x)\right)$$

$$+\int_0^\infty W_n^{(r)}(t,x)\left\{\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)\right\}dt+\int_0^\infty W_n^{(r)}(t,x)h(t,x)(1-\chi(t))\,dt$$

$$:=I_1+I_2+I_3.$$

By using Lemma 2.2 and (3.2), we get

$$I_1=\sum_{i=r}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=r}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^r}{dx^r}\left(x^j+\frac{r(r+\nu)}{n}x^{j-1}+O(n^{-2})\right)-f^{(r)}(x)\;.$$

Consequently,

$$\|I_1\|_{C[a,b]} \le C_1 n^{-1}\left(\sum_{i=r}^{q}\left\|f^{(i)}\right\|_{C[a,b]}\right)+O(n^{-2}), \text{ uniformly on } [a,b].$$

To estimate I_2 we proceed as follows:

$$\left|I_2\right| \le \int_0^\infty \left|W_n^{(r)}(t,x)\right|\left\{\frac{\left|f^{(q)}(\xi)-f^{(q)}(x)\right|}{q!}\left|t-x\right|^q \chi(t)\right\}dt$$

$$\le \frac{\omega_{f^{(q)}}(\delta)}{q!}\int_0^\infty \left|W_n^{(r)}(t,x)\right|\left(1+\frac{|t-x|}{\delta}\right)|t-x|^q\,dt$$

$$\le \frac{\omega_{f^{(q)}}(\delta)}{q!}\left[n\sum_{k=0}^{\infty}\left|q_{n,k}^{(r)}(x)\right|\int_0^\infty q_{n,k+\nu}(t)\left(|t-x|^q+\delta^{-1}|t-x|^{q+1}\right)dt\right], \qquad \delta>0 .$$

Now, for $s=0,1,2,\ldots$, using Schwarz inequality for integration and then for summation, (2.1) and (2.3) we have

$$(3.4)\qquad n\sum_{k=0}^{\infty}q_{n,k}(x)|k-nx|^j\int_0^\infty q_{n,k+\nu}(t)|t-x|^s dt \le n\sum_{k=0}^{\infty}q_{n,k}(x)|k-nx|^j\left\{\left(\int_0^\infty q_{n,k+\nu}(t)dt\right)^{1/2}\times\left(\int_0^\infty q_{n,k+\nu}(t)(t-x)^{2s}dt\right)^{1/2}\right\}$$

$$\le\left(\sum_{k=0}^{\infty}q_{n,k}(x)(k-nx)^{2j}\right)^{1/2}\left(n\sum_{k=0}^{\infty}q_{n,k}(x)\int_0^\infty q_{n,k+\nu}(t)(t-x)^{2s}dt\right)^{1/2}$$

$$=O(n^{(j-s)/2}), \text{ uniformly on } [a,b].$$

Therefore, by Lemma 2.4 and (3.4), we get

$$(3.5)\qquad n\sum_{k=0}^{\infty}\left|q_{n,k}^{(r)}(x)\right|\int_0^\infty q_{n,k+\nu}(t)|t-x|^s dt \le n\sum_{k=0}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i|k-nx|^j\frac{\left|Q_{i,j,r}(x)\right|}{x^r}q_{n,k}(x)\times\int_0^\infty q_{n,k+\nu}(t)\,|t-x|^s dt$$

$$\le\left(\sup_{\substack{2i+j\le r\\ i,j\ge 0}}\sup_{x\in[a,b]}\frac{\left|Q_{i,j,r}(x)\right|}{x^r}\right)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\left(n\sum_{k=0}^{\infty}q_{n,k}(x)|k-nx|^j\int_0^\infty q_{n,k+\nu}(t)|t-x|^s dt\right)$$

$$=C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O(n^{(j-s)/2})=O(n^{(r-s)/2}), \text{uniformly on } [a,b] .$$

(since $\sup_{\substack{2i+j\le r\\ i,j\ge 0}}\sup_{x\in[a,b]}\frac{\left|Q_{i,j,r}(x)\right|}{x^r}:=M(x)$ but fixed)

Choosing $\delta=n^{-1/2}$ and applying (3.5), we obtain

$$\|I_2\|_{C[a,b]}\le\frac{\omega_{f^{(q)}}\left(n^{-1/2}\right)}{q!}\left[O(n^{(r-q)/2})+n^{1/2}O(n^{(r-q-1)/2})+O(n^{-m})\right], \text{ (for any } m>0 \text{)}$$

$\le C_2\, n^{-(r-q)/2}\omega_{f^{(q)}}\left(n^{-1/2}\right).$

Since $t \in [0,\infty)\setminus(a-\eta, b+\eta)$, we can choose $\delta > 0$ in such a way that $|t-x| \ge \delta$ for all $x \in [a,b]$. Thus, by Lemmas 2.3 and 2.4 , we obtain

$$|I_3| \le \sum_{k=0}^{\infty} \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i |k-nx|^j \frac{|Q_{i,j,r}(x)|}{x^r} q_{n,k}(x) \int_{|t-x|\ge\delta} q_{n,k+\nu}(t)|h(t,x)|\,dt .$$

For $|t-x| \ge \delta$, we can find a constant C such that $|h(t,x)| \le Ce^{\alpha t}$. Hence, using Schwarz inequality for integration and then for summation ,(2.1), (2.3), it easily follows that $I_3 = O(n^{-s})$ for any $s>0$, uniformly on $[a,b]$.

Combining the estimates of $I_1, I_2\, I_3$, the required result is immediate. ■

REFERENCES

[1] Z. Balczk : On approximation by modified Szãsz – mirakyan operators, Glasnik Matematicki 303-319, 37(57) (2003) .

[2] Vijay Gupta, G.S. Servastava and A. Shahai: On simultaneous approximation by Szãsz -Beta operators, Soochow J. Math. **21**,1-11, (1995).

[3] H.S. Kasana, G. Prasad, P.N. Agrawal and A. Sahai, On modified Szãsz operators, Proc. Int. Conf. Math. Anal. And its Appl.Kuwait 29-41,(1985),Perogram Press, Oxford(1988).

[4] H.S. Kasan, P.N. Agrawal and Vijay Gupta, Inverse and Saturtion theorems for linear combination of modified Baskakov operators, Approx. Theory Appl.7,65-82,(1991).

[5] G.Beylkin, L.Monzon : On approximation of functions by exponential sums, AppComput . Harmon . Anal. 17-48, 19(2005).

[6] S.M. Mazhar and V. Totik, Approximation by modified Szãsz operators, Act. Sci. Math. 49,257-269,(1985).

[7] S.Guo, Q.Qi and S.Yue: approximation for Baskakov - Kndtorovich – Bezier operators in the space $L_p[0,\infty)$,Taiwanese J.Math. , 161-177, 11(1) (2007) .

[8] E. Voronovskaja : Determination de la forme asymptotique d approximation des functions parles polynomes de S.N. Bernstein , C.R. Adad. Sci USSR 79-85 (1932)

حول تعميم مؤثر Szãsz من النمط مجموع – تكامل

المستخلص

في بحثنا هذا نقدم وندرس بعض النتائج المباشرة في التقريب المتعدد لتعميم مؤثر Szãsz من النمط مجموع – تكامل $M_n(f(t);x)$. بدايةً , نثبت مبرهنة التقارب النقطي الأساسية ومن ثم نناقش صيغة فورونوفسكي Voronovaskaja-type (asymptotic formula) . وأخيرا نجد درجة التقريب بدلالة معيار الاستمرارية للدالة المستخدمة في التقريب .