Sensitivity Analysis of Nonlinear Optimization Problem Via Differential Equations Approach

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Abstract

In this paper, a new method for obtaining sensitivity information for the parameters of the nonlinear optimization problems $(OP)_v$ is presented. These parameters appear in the objective function as well as the constraints. This method depends on using differential equations approach for solving nonlinear programming problem with equality and inequality constraints, the behavior for the local solution for slight perturbation of the parameters in the neighborhood of their chosen initial values is presented by using the technique of trajectory continuation. Finally, an example is given to show the efficiency of the proposed method.

Keywords: Sensitivity analysis; Nonlinear programming; Differential equation approach.

1. Introduction

A new method for solving the equality constrained nonlinear programming problem by using the differential equation approach whose critical points are the solution to the optimization problem is discussed in [1]. The extension of [1] to include a large class of optimization problem including equality and inequality constraints and differential equation approach was developed in [2]. This method guarantees finding a local optimal solution as well as a global one of nonlinear programming problem with equality and inequality constraints via critical (asymptotic) point of a suitable nonlinear autonomous differential system. Recently this method is regarded preferable to solve the nonlinear programming problem with equality and inequality constraints compared with some well-known optimization methods [3].

In the present work the sensitivity information by using the autonomous system of differential equations approach corresponding to the optimization problem is discussed. This information coincides with explicit representation of the first order partial derivatives of the local solution point and associated Lagrange multipliers to the parametric problem [4]. A development of sensitivity analysis of some class of nonlinear optimization problem based on differential equation approach and the work of [5] is presented. The fundamental problem is described in section 2. By using the technique of trajectory continuation [1, 6], the behavior of the local solution (sensitivity) analysis of the parameters in the neighborhood of the chosen initial values is discussed in section 3. Illustrative example is given in section 4. The conclusions are drawn in Section 5.

2. Problem Formulation

A mathematical nonlinear programming problem with equality and inequality constraints with general perturbation in the objective function and anywhere in the constraints has the form

$$\left.\begin{array}{lll} \text{Minimize} & f(x,v) & \\ \text{subject to} & h_i(x,v)=0 & i=1,2,\dots,m \\ & h_i(x,v)\le 0 & i=m+1,\dots,p \end{array}\right\} \quad 2.1$$

$$x\in R^n \quad , \quad f,h\in C^2 \quad , \quad p\le n$$

By using the technique of differential equation approach [2] the problem (2.1) becomes

$$\left.\begin{array}{ll}\text{Minimize} & F(z,v)\\ \text{subject to} & G(z,v)=0\end{array}\right\} \quad 2.2$$

where

$$F(z,v)=F(x,y,v)=f(x,v)$$

$$G(z,v)=\begin{bmatrix} h_i(x,y,v)=h_i(x,v) & i=1,2,...,m\\ h_i(x,y,v)=h_i(x,v)+y^2 & i=m+1,...p\end{bmatrix}$$

$z\in R^w$, $F,G\in C^2$, $R^w=R^n\times R^{p-m}$, $w=p-m+n$

$z=[z_1\equiv x_1,...,\ z_n\equiv x_n,\ z_{n+1}\equiv y_m,...,\ z_w\equiv y_p]$

Assume that $A=\nabla_z G(z)$ are of full rank, where

$$\left.\begin{array}{l} B\dot{z}+A^T\lambda(z)=-\nabla F(z)\\ A\dot{z}=-G(z)\end{array}\right\} \quad 2.3$$

And B is a symmetric nonsingular matrix of order $w\times w$ from the above autonomous system (2.3) we obtain the unique solution

$$\dot{z}\equiv\varphi(z)=-PB^{-1}\nabla_z F(z)^T-\tilde{P}G(z) \qquad 2.4$$

$$\lambda(z)=-(AB^{-1}A^T)^{-1}AB^{-1}\nabla_z F(z)+(AB^{-1}A^T)^{-1}G$$

2.5

Where

$P=I-q\,,\ q=B^{-1}A^T(AB^{-1}A^T)^{-1}A$,

$\tilde{P}=B^{-1}A^T(AB^{-1}A^T)^{-1}$

$D(z)=\begin{bmatrix} B^{-1} & A^T\\ A & 0\end{bmatrix}$ is nonsingular

In [2] it was proved that the matrix p(z) is a projection operator which projects any vector in R^w into $M(\hat{z})$, where $M(\hat{z})$ is the tangent of the system of constraints at $\hat{z}$

$M(\hat{z})=\{K:A^TK=0\}$.

Also it was proved that

1- z^* is the regular point of the constraints.

2- The second order sufficient condition are satisfied at z^*.

3- There are no degenerate constraints at z^*.

Then, any trajectory starting from a point with some neighborhood of the local minimal point z^* converges to z^* as $t\to\infty$.

In the following section for simplicity we omit the variable z and write F or G instead of F(z) or G(z).

3. Sensitivity Information

A methodology for conducting a local perturbation (sensitivity) analysis and finite perturbation (stability) analysis of solution behavior with respect to problem changes is a well established requirement of any scientific discipline. A sensitivity and stability analysis should be an integral part of any solution methodology. The status of a solution cannot be understood without such information. This has been well recognized since the inception of scientific inquiry and has been explicitly addressed from the beginning of mathematics. In mathematical programming the sensitivity and stability techniques have been used to obtain optimality conditions, duality results, solution algorithms, convergence and rate of convergence proofs, and acceleration of convergence of algorithms, in addition to their more obvious and immediate applications in estimating near solution with different data. By using the technique of trajectory continuation [1, 6], we will discuss the behavior of the local solution z^* for slight perturbation of the parameters in the neighborhood of their chosen initial values.

The following existence theorem, which is based on the implicit function theorem [4], holds

Theorem 1

Let z^* be a unique local solution of $(OP)_{\bar{v}}$ satisfying the assumptions (1-3). Then there exists a continuously differentiable vector valued function $z(\cdot)$ defined in some neighborhood $N(\bar{v})$, so that $z(v)=z^*$ where $z(v)$ is a unique local solution for the problem $(OP)_v$, for any $v\in N(\bar{v})$ satisfying the assumptions (1-3).

Proof

For any $v\in N(\bar{v})$ the fundamental equations corresponding to $(OP)_v$ have the following form.

$$B\dot{z}(v)+A^T\lambda=-\nabla_z F$$

$$A\dot{z}(v)=-G$$

And consequently

$\varphi(z(v))=\dot{z}(v)=-B^{-1}(A^T\lambda+\nabla_z F)$ from equation (2.4) near z^* one can write [1, 6]

$$\phi(z(v))=\frac{dz(v)}{dt}=\frac{\partial\phi(z^*)}{\partial z}(z(v)-z^*)$$

Proposition 1

If $B=\nabla_z^2F^T+\lambda^T\nabla_z^2G$, then $\frac{\partial\phi(z^*)}{\partial z}=-I$, i.e. the local minimal point z^* is asymptotically stable.

Proof

$$\frac{\partial\varphi\ (z^*)}{\partial z}=\frac{\partial}{\partial z}\left\{-B^{-1}A\lambda-B^{-1}\nabla F\right\}_{Z=Z^*}$$

see [1, 6]

$$\frac{\partial\varphi\ (z^*)}{\partial z}=-B^{-1}\left\{\nabla_z^2F+\lambda^T\nabla_z^2G-A^T(AB^{-1}A^T)^{-1}AB^{-1}(\nabla_z^2F+\lambda^T\nabla_z^2G-B)\right\}_{Z=Z^*}$$

Since $B(z)=\nabla_z^2F^T+\lambda^T\nabla_z^2G(z)$

$$\frac{\partial\varphi\ (z^*)}{\partial z}=-I$$

That is $\lim_{t\to\infty} z(t)=z^*$

The solution near z^* becomes $z(t)\approx z^*+(z(0)-z^*)\,e^{-t}$ asymptotically stable, see [2].

For obtaining sensitivity information for the first order estimation of solution of the nonlinear optimization problem, we introduce the following system of the differential equations.

$$B\frac{dz}{dv}+\nabla_v(A^T\lambda)=-\nabla_v(\nabla_zF)$$

$$A\frac{dz}{dv}=-\nabla_vG$$

Then one can obtain

$$\frac{dz(v)}{dv}=-PB^{-1}(\nabla_v(\nabla_zF))-\tilde{P}\nabla_vG \quad 3.1$$

$$\nabla_v\lambda=-(AB^{-1}A^T)^{-1}AB^{-1}\nabla_v(\nabla_zF)-(AB^{-1}A^T)^{-1}\nabla_vG$$

3.2

where P, $\tilde{P}$ and B defined above in (2.4)

After solving $(OP)_{\bar v}$, we may wish to answer the following question: if the problem $(OP)_{\bar v}$ replaced by $(OP)_v$, $v\in N(\bar v)$ what is the new efficient solution of the problem $(OP)_v$ $v\in N(\bar v)$ without solving it again.

With $(z,\lambda)=(z(v),\lambda(v))$ is identically for v near zero under the assumption of theorem 1 and proposition 1 a first order approximation of $(z(v),\lambda(v))$ in the neighborhood of $v=0$ is given by [4].

$$\begin{pmatrix} z(v)\\ \lambda(v)\end{pmatrix}=\begin{pmatrix} z^*\\ \lambda^*\end{pmatrix}+\begin{pmatrix} \frac{dz(v)}{dv}\\ \nabla_v\lambda\end{pmatrix}v+0(\|v\|)$$

Sensitivity information (3.1) and (3.2) minimize the computation efforts for finding many efficient solutions for the problem $(OP)_v$, $v\in N(\bar v)$

4. Illustrative Example

In this section, we provide numerical example to clarify the theory developed in this paper

Example *[4].*

Minimize $f(x,v)=x_1+v_2x_2$

subject to $g(x,v)=x_1^2+x_2^2-v_1^2\le 0$

To illustrate the application of the general equations (3.1) and (3.2) for obtaining sensitivity information for the first order estimation of the solution of the optimization problem, we write the following: Reformulate the problem with equality constraints in the form

Minimize $F(z,v)=x_1+v_2x_2$

subject to $G(z,v)=x_1^2+x_2^2-v_1^2+y^2=0$

Formulate (3.1) and(3.2)

with $B(z)=\nabla_z^2F(z)^T+\lambda(z)^T\nabla^2G(z)$, then

$$B^{-1} = \frac{1}{\lambda}\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \quad , \quad D = \begin{bmatrix} 2\lambda & 0 & 0 & 2x_1 \\ 0 & 2\lambda & 0 & 2x_2 \\ 0 & 0 & 2\lambda & 2y \\ 2x_1 & 2x_2 & 2y & 0 \end{bmatrix}.$$

$$-\tilde{P}\nabla_v G(z) = \frac{1}{2(x_1^2 + x_2^2 + y^2)}\begin{bmatrix} 2x_1 v_1 & 0 \\ 2x_2 v_1 & 0 \\ 2y v_1 & 0 \end{bmatrix}$$

$$-PB^{-1}(\nabla_v(\nabla_z F(z)^T)) = \begin{bmatrix} 0 & \frac{x_1 x_2}{2\lambda v_1^2} \\ 0 & -\frac{x_1^2 + y^2}{2\lambda v_1^2} \\ 0 & \frac{y x_2}{2\lambda v_1^2} \end{bmatrix}.$$

$$\frac{dz}{dv} = \begin{bmatrix} \frac{x_1}{v_1} & \frac{x_1 x_2}{2\lambda v_1^2} \\ \frac{x_2}{v_1} & -\frac{x_1^2 + y^2}{2\lambda v_1^2} \\ \frac{y}{v_1} & \frac{y x_2}{2\lambda v_1^2} \end{bmatrix}. \qquad 4.1$$

$$\frac{d\lambda}{dv} = \begin{bmatrix} -\frac{y}{v_1} & -\frac{x_2}{2v_1^2} \end{bmatrix}. \qquad 4.2$$

Sensitivity information which we obtained in (4.1) and (4.2) coincide completely with Fiacco's results in [4 page 106].

5. Conclusions

In this work, the sensitivity information for parametric nonlinear optimization problem is analyzed successfully. This information coincides with the explicit representation of the first order partial derivative of the local solution point and associated Lagrange multipliers to the parametric problem [4]. Moreover, a new modified approach with sensitivity information in equations (3.1) and (3.2) is represented by minimizing the computation efforts compared with [4].

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تحليل حساسية مسألة الأمثلية الغير الخطية
باستعمال طريقة المعادلات التفاضلية

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قدمنا في هذا البحث طريقة جديدة للحصول على معلومات حول حساسية مؤثرات مسائل الأمثلية غير الخطية المقيدة، هذه المؤثرات تكون في دالة الهدف أو في القيود.

هذه الطريقة اعتمدت على طريقة المعادلات التفاضلية لحل مسائل الأمثلية غير الخطية المقيدة وان سلوك الحل تمت دراسته باستخدام (trajectory continuation). تم تقديم مثال لإثبات كفاءة الطريقة.