Solution of Fractional Differential Equations by Using Variational Approach

حل المعادلات التفاضلية الكسريه بأستخدام الصياغة التغايرية

Dr.Fadhel S. Fadhel
Department of Mathematics and Computer Applications, College of Science, Baghdad, Iraq.
Basim K. AL-Sultani
Department of Computer, College of Science, Kerbala, Iraq.

Abstract

In this paper, we modify a new approach based on variational techniques for solving fractional differential equations of the form:

$$y^{(\alpha)} = F(x, y)$$

$$y^{(\alpha-1)}(x_0) = y_0,$$

where $0 < \alpha < 1$

This approach has its bases on using Magri's approach (see [8])for every linear operator, the results are established using direct Ritz method as well as optimization method to solve these fractional differential equations numerically.

الخلاصة

الهدف من هذا البحث هو لاستخدام طريقة جديده لحل المعادلات التفاضلية الكسرية (Fractional Differential Equations) والمعرفة بالشكل الاتي :-

$$y^{(\alpha)} = F(x, y)$$

$$y^{(\alpha-1)}(x_0) = y_0,$$

where $0 < \alpha < 1$

وهذه الطريقة تستخدم الاسلوب التغايري (Variational Approach) لحل المعادلات التفاضلية الكسرية وكيفية صياغة مثل هذا النوع من المعادلات التفاضلية الكسرية والتي يمكن كتابتها بصيغة المؤثر (Operator Form) على شكل صياغة التغايريه .

تم ايجاد الصياغة التغايريه لحل المعادلات التفاضلية الكسرية مع مراعات الاصالة في طريقة الحل بدون تحويل المعادلات التفاضلية الكسرية الى معادلات تكاملية كسريه .

1.Introduction

In opposite to differential equations of integer order in which derivatives depends only on the local behavior of the function, an important type of differential equations is the so-called **fractional differential equations** in which the differentiation is of non-integer order, hence fractional differential equations accumulate the whole information of the function in a weighted form, this is so called memory effect and has many applications in physics [1], chemistry [2], engineering [3], etc. for that reason, we need a method for solving such equation, which is effective, easy to use and applied for fractional differential equations.

However, will known method used for solving such type of equations have more disadvantages. Analytical methods [6], which uses the multivariate Mittag-Leffer function and generalizes the previous result, can be used only for linear type of equations.

The initial value problem of the fractional differential equation, considered in this paper is of the form:

$y^{(\alpha)} = F(x, y)$ .... (1)

with the initial condition:

$y^{(\alpha-1)}(x_0) = y_0$

where $0 < \alpha < 1$, $y_o \in R$, F(x, y) is given linear function in y on the interval [0, x], y(x) is known function which is to be determined as the solution of the problem.

The fractional derivative operator D^{α} is defined using the Riemann-Liouville fractional derivative of order α [5]. Other definitions related to fractional differential equations could be found in Caputo[6], Grunwald-Letnikov[6], and weyl-Mochand[5], Riemann-Liouville derivative of function y(x) at x_0 is defined by order α:

$$D^{\alpha} y(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x y(t)(x-t)^{-\alpha} dt \quad \text{..............................} (2)$$

where $\Gamma(1-\alpha)$ is the gamma function.

2. FRACTIONAL DIFFERENTIAL EQUATION

2.1 Basic Concepts

This paper gives at some of the primary problems with fractional calculus, as it is now embodied, and attempts some of these with a modified embodiment. The basic approach that will bee taken in this work is to make the defined mathematics as maximally applicable to the problems of engineering and science as possible. To this end, basic distributed, dynamic systems have been developed for use as reference fractional system [7].

The problem that the researches perceive to bar widespread of applications of fractional calculus in the engineering sciences. The operative basic definitions for this section will be the contemporary, Rirmann-Liouville definition, which is that integration of arbitrary order, given by:

$$D^{-\alpha} y(x) = \frac{d^{-\alpha}}{dx^{-\alpha}} y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x y(t)(x-t)^{\alpha-1} dt \quad \text{....................} (3)$$

where $0 < \alpha < 1$ and $\Gamma(\alpha)$ is the gamma function.

For simplicity, in this paper α is constrained to be real number between 0 and 1. The contemporary differentiation is defined as:

$$D^{\alpha} y(x) = \frac{d^{\alpha}}{dx^{\alpha}} y(x)$$

where $0 < \alpha < 1$.

One of the fundamental problems of contemporary fractional calculus is the requirement that the function f(x) and its derivatives to be identically equal to zero for $x = x_0$ [4], or lacking this, to limit the functions handled to special classes. This is needed to assure that the composition rule (the index law) holds, that is, to assure that:

$$D_x^{\alpha} D_x^{\beta} y(x) = D_x^{\beta} D_x^{\alpha} y(x) = D_x^{\alpha+\beta} y(x)$$

It is difficult, in engineering sciences, to always require that the functions and its derivatives to be equals zero at initialization.

This fundamental rule says that; there can be either no initialization or the composition is lost. Thus, it is not in general true that:

$$y - \frac{d^{-\alpha}}{dx^{-\alpha}} \frac{d^{\alpha} y}{dx^{\alpha}} = 0 \quad \text{...} (4)$$

(See [4] for more details).

Thus, when solving a fractional differential equation of the form:

$$\frac{d^{\alpha} y}{dx^{\alpha}} = F, \; 0 < \alpha < 1$$

Additional terms must be added to equation (4), after applying D^{α} to both sides gives:

$$y - \frac{d^{-\alpha}}{dx^{-\alpha}} \frac{d^{\alpha} y}{dx^{\alpha}} = c_1 x^{\alpha-1}$$

where c_1 is arbitrary constant. to achieve the most general solution of eq(1), given by:

$$y = \frac{d^{-\alpha}}{dx^{-\alpha}} F + c_1 x^{\alpha-1}$$

The reader is referred to [4] for detailed exposition in this area. The added constant of integration in the integer order calculus.

Following, some examples to illustrate the solution of fractional differential equations.

Example (1):

Consider the fractional differential equation:

$$\frac{d^{1/2}}{dx^{1/2}} y = x^{1/2} \quad \text{..........(5)}$$

with the initial condition:

$y^{(-1/2)}(x_0) = y_0$

Applying $\frac{d^{-1/2}y}{dx^{-1/2}}$ to both sides of equation (5), we get:

$$y = \frac{d^{-1/2} x^{1/2}}{dx^{-1/2}} + c_1 x^{-1/2}$$

and from the initial condition, we have $c_1 = \frac{y_0}{\Gamma(1/2)}$, therefore:

$$y = x\Gamma(3/2) + \frac{y_0 x^{-1/2}}{\Gamma(1/2)}.$$

Example (2):

Consider the fractional differential equation:

$y^{(2/3)}(x) = x^5$

with initial condition

$y^{(-1/3)}(x_0) = 0$

and upon applying $\frac{d^{-2/3}y}{dx^{-2/3}}$ to both sides, then similarly as in example (1), we get that:

$$y(x) = \frac{\Gamma(6)x^{17/3}}{\Gamma(26/3)}.$$

3. Variational Approach for Solving Fractional Differential Equations

As it is pointed previously, the most important difficulty of the subject of calculus of variation is how to find the variational formulation which corresponds to the linear fractional differential

$$Lu=F \quad \text{..........(6)}$$

One of the most important approaches for evaluating this functional F [u] is the non-classical variational approach which is proposed by Magri in 1974 [8], which is of evaluating the functional F[u] defined on the domain of the linear operator L (satisfying certain conditions) whose critical points are the solution of the given equation (6).

The problem may be called the inverse problem of calculus of variation. If the given linear operator L is symmetric with respect to the chosen bilinear form <u, v> which is non degenerate, then u is the solution of eq. (6) if and only if u is a critical point of the functional:

$$F[u] = \frac{1}{2}\int_0^T u(t)Lu(t)dt - \int_0^T f(t)u(t)dt \quad \text{...}(7)$$

Hence, if a given linear operator L is symmetric with respect to a non degenerate bilinear form <u, v>, then there is a variational formulation of the given linear equation (6), (this is called the classical variational formulation). The main difficulty, which may arise frequently in this subject, is when the given linear operator L is not symmetric with respect to the chosen bilinear form, and the problem is to find the variational formulation in such cases, we can proceed into two approaches.

The first approach is to retain the bilinear form <u,v>and look for methods of modifying the given equation so as to a new equation which is symmetric with respect to the chosen bilinear form, [8]. This is usual procedure when one make use of the Cartesian bilinear form:

$$(u, v) = \int_0^T u(x)\,v(x)\,dx,$$

where, u, v: C[0,T] $\longrightarrow$ R. such that C[0,T] continuous function.

The second approach is to retain the given linear operator and attempt to change the bilinear form so that the given operator is symmetric with respect to the new bilinear form, [8]. In order to construct a bilinear form that makes the given linear operator L symmetric, let us consider, as a preliminary tool, an arbitrary symmetric bilinear form:

$$(u,v)=\langle u,Lu\rangle \quad \text{...}(8)$$

which is defined for every pair of elements $v \in V$ and $u \in D(L)$. Now, it is so simple to show that the bilinear form equation (8) makes the given linear operator symmetric whatever the choice of the first symmetric bilinear form.

Therefore, we can make use of the bilinear form equation (6) to given a variational formulation corresponding to the linear equation (6). Because of the symmetric of L therefore by using the above in connection with equation (7). The solution of equation (6) is the critical points of the functional:

$$F[u] = \frac{1}{2} \langle Lu, u\rangle - \langle f, u\rangle$$

$$= \frac{1}{2}\ (Lu, Lu) - (f, Lu)$$

Now, to our problem under consideration, consider fractional differential equation: $y^{(\alpha)} = f(x, y),\ 0 < \alpha < 1$

And $f(x, y) = g(x)$, then applying the operator $\frac{d^{1-\alpha}}{dx^{1-\alpha}}$ to both sides of the last equation by techniques like those discussed in section fractional differential equations, and by using the initial conditions, we get:

$$\frac{dy}{dx} = \frac{d^{1-\alpha}}{dx^{1-\alpha}} g(x) + \frac{y_0}{\Gamma(\alpha-1)} x^{\alpha-2}$$

$$= D^{1-\alpha} g(x) + \frac{y_0}{\Gamma(\alpha-1)} x^{\alpha-2}$$

$$= F(x),$$

where D is the differential operator.

Therefore, the related equation in operator form is given by:

$Ly = F(x)$,

Note: $L = D = \frac{d}{dx}$ and $F = D^{1-\alpha} g(x) + \frac{y_0}{\Gamma(\alpha-1)} x^{\alpha-2}$

While, the related variational formulation is given by:

$$J(y) = \frac{1}{2}\ (Ly, Ly) - (F, Ly)$$

$$= \frac{1}{2}\int_0^1 \left[(Ly)^2 - 2(D^{1-\alpha} g(x) + \frac{y_0}{\Gamma(\alpha-1)} x^{\alpha-2})(Ly) \right] dx \quad \text{.......................} (9)$$

This is the related functional corresponding to the linear equation $Ly = F$.

Therefore, one can find the critical of (9), which the desired solution of fractional differential equations.

Consider the fractional differential equation given in example (1):

$y^{(1/2)}(x) = x^{1/2}$,

with initial condition

$y^{(-1/2)}(0) = 0.1$,

By applying the operator $\frac{d^{1/2}}{dx^{1/2}}$ to both sides of the fractional differential equation and carrying out the initial conditions, we get:

$$\frac{dy}{dx} = \Gamma(3/2) - \frac{0.1}{2\Gamma(1/2)} x^{-3/2}$$

and by letting

$$L = \frac{d}{dx} \quad \text{and} \quad F = \Gamma(3/2) - \frac{0.1}{2\Gamma(1/2)} x^{-3/2}$$

Then the related variational formulation takes the form:

$$J(y) = 0.5\int_0^1 \left[\left(\frac{dy}{dx}\right)^2 - 2\left(\Gamma(3/2) - \frac{0.1}{2\Gamma(1/2)} x^{-3/2} \right)\left(\frac{dy}{dx}\right) \right] dx$$

Carrying out and the computer program with approximate solution by using Ritz method:

$y(x) = 0.1x^{-1/2} / \Gamma(1/2) + (a_1 x + a_2 x^2)$

We get the results presented in table (3) with its comparison with the exact solution:

$y(x)= 0.1x^{-1/2} / \Gamma(1/2) + (0.8880001x - 0.001999791x^2)$.

x	***Approximate solution***	***Exact solution***
0.1	0.2671024	0.267
0.2	0.3035267	0.303
0.3	0.3690165	0.369

(Table1)

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