Performance of First-order Loops Incorporating Time Delay

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Abstract

The influence of time delay on the statistical behavior of the first-order phase-locked-loop is investigated in VHF and UHF synchronous communication systems. The Fokker-Plank equation has been proposed to estimate the probability density function (pdf) of phase fluctuations as well as the average time to loss lock in the presence of noise. The result reveal that the degradation in the loop performance occurs under various conditions of detuning when the inherent time delay is present

أداء منظومات قفل الطور من الدرجة الأولى المدمجة بالتأخير الزمني

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الخلاصة

تم تقصي تأثير التأخير الزمني المسلط على أداء منظومات قفل الطور من الدرجة الأولى في نظم الاتصالات المتزامنة ذات ترددات VHF و UHF. اعتمد البحث على معادلة فوكر-بلانك لغاية

1- *Introduction*

First-order PLLs have been extensively studied and their behavior have been well analyzed [1-5]. In such loops, the equivalent noise bandwidth is directly proportional to the loop gain. On the other hand, the tracking performance and acquisition range are directly improved when the loop gain is increased. Such conflicting properties indicate that these loops have no adequate immunity against the effect of the presence of additive noise. This is an essential reason for not using first-order PLLs extensively in various synchronous systems.

However, in modern synchronous communication systems operating at high frequencies, as well as in microwave frequency bands, first – order loops are implemented to achieve synchronous such as in radar systems [3-8]. In this type of applications, the inherent response of the loop elements can be neglected. This is due to the fact that the response of such devices when operating at higher frequencies becomes comparable with the cycle duration of the operating frequency. Thus at these frequencies when analyzing the PLL performance, one should take into account the presence of the inherent time delay as it cause significant deterioration in the loop properties.

In this paper, an exact analysis is employed to describe the exact behavior of the noise fluctuations in the phase of reconstructed carriers at the output of the loop. The analysis is based on Fokker-Planck (FP) equation to estimate the steady-

state probability density function (pdf) of the phase fluctuations as well as the average time to lose lock under various conditions of detuning in the presence of the dominant time delay.

2- *Analytical Model*

The model under consideration shown in Fig. 1 with loop filter F(s)=1, is analyzed by one-dimensional Fokker-Planck equation [1,2]. When the input signal of power A^2 is accompanied by an additive noise, then

$$V_i(t)=\sqrt{2}\,A\sin\phi_i(t)+n(t) \qquad \text{.........(1)}$$

where $\phi_i(t)$ is the phase of the input and n(t) is the narrow band and stationary Gaussian noise, $n(t)=\sqrt{2}\,n_1(t)\sin\omega_o t+\sqrt{2}\,n_1(t)\cos\omega_o t$, with $n_1(t)$ and $n_2(t)$ of zero mean value and are statistically independent of one sided spectral density N_o watt/Hz and taking into account that the bandwidth of the input noise is much larger than the loop noise bandwidth.

The output of the voltage-controlled-oscillator (VCO) is assumed as

$$V_o(t)=\sqrt{2}\cos\phi_o(t) \qquad \text{.......(2-a)}$$

where $\phi_o(t)$ is the overall phase of the VCO output. The phase detector is assumed to be of the multiplying type, then

$$V_d(t)=V_i(t).V_o(t) \qquad \text{.......(2-b)}$$

$V_d(t)$ is the output of the phase detector that depends on the total phase error $\phi(t)=\phi_i(t)-\phi_o(t)$ and on the noise process. After neglecting the double frequency terms from the product and after amplification by factor G, then

$$V_c(t)=K_d\,G\,F(p).\{\,A\,g[\phi(t)]+n(t)\,\} \qquad \text{....(3)}$$

Where p=d/dt is the heaviside operator and K_d is the phase detector sensitivity and $\phi(t)$ accounts for the total instantaneous phase error in the loop input and output phases given respectively by,

$$\phi_i(t)=\omega_i t+\theta_i(t) \qquad \text{..................(4)}$$

and,

$$\phi_o(t)=\omega_o t-\theta o(t) \qquad \text{..................(5)}$$

ω_i is the frequency of the received signal and ω_o is the quiescent of the VCO. However, the equivalent noise process n(t) can be defined as,

$$n'(t)=-n_1(t)\sin\theta o(t)+n_2((t)\cos\theta o(t) \quad \text{...(6)}$$

$\theta_i(t)$ and $\theta_o(t)$ are arbitrary and random processes. Furthermore, n'(t) is stationary process with exactly the same statistics of $n_1(t)$ and $n_2(t)$. In the case of the first-order loop, F(s)=1 then the equivalent expression is, F'(s)=T(s) with T(s) being an arbitrary transfer function. It indicates the incorporation of the time delay within the loop. Thereby

$$T(s)=e^{-sT_d} \qquad \text{.....................(7)}$$

T_d is assumed to be the overall time delay incorporating within the loop. The operation of the VCO with constant K_v is described by

$$\dot{\theta}_o(t)=K_v\,V_c(t) \qquad \text{.....................(8)}$$

and inconsequence,

$$\dot{\phi}(t)=\dot{\theta}_i(t)-K\,T(p)\,[A\,g(\phi)+n'(t)] \quad \text{....(9)}$$

with $K=K_vK_dG$ is the overall loop gain. The initial radian frequency detuning (frequency offset) can be defined by [1,7].

$$\dot{\theta}_i(t)=\Omega_o=\omega_i-\omega_o \qquad \text{..................(10)}$$

rewriting equation 9 using eqn. 10, the differential equation that describes the loop operation becomes as,

$$\dot{\phi}(t)=\Omega_o-K\,T(p)\,[A\,g(\phi)+n'(t)] \quad \text{....(11)}$$

3- Fokker-Planck Equation

If the filter is omitted in Fig.1, then $F'(s)=\exp(-sT_d)$. Further, since $n(t)$ is a White- Gaussian (WG) process, the instantaneous change in ϕ represented by its derivative which depends only on the present value of $\phi(t)$ and the present value of noise [5,7]. Hence, $\phi(t)$ is a continuous Markov process and a random walk technique is used to determine its probability density function. This property of this process is employed to satisfy the one-dimensional Fokker-Planck equation. Then the following time dependent differential equation becomes,

$$\frac{\partial P(\phi,t)}{\partial t} = -\frac{\partial}{\partial \phi}\{\Omega_o - K A\, T(p) \sin\phi\}\, P(\phi,t) + \frac{1}{2}\,\frac{K^2 N_o}{2}\,\frac{\partial^2 P(\phi,t)}{\partial \phi^2} \quad \text{....(12)}$$

where $\phi(t)$ is taken reduced modulo-2π, and by solving eqn. 12 based on initial condition,

$$P(\phi,0) = \sum_{n=-\infty}^{\infty} \delta(\phi-\phi_o-2n\pi) \quad \text{........(13)}$$

for one period of the phase error (with n=0) over the interval $-\pi\leq\phi\leq\pi$, the initial condition becomes

$$P(\phi,0) = \delta(\phi-\phi_o) \quad \text{.......(14-a)}$$

with boundary condition,

$$P(\pi,0) = P(-\pi,0) = 0 \ \text{ for all t} \quad \text{........(14-b)}$$

and the normalized condition,

$$\int_{-\pi}^{\pi} P(\phi,t)\, d\phi = 1 \qquad \text{for all t} \quad \text{.......(14-c)}$$

using $\mathcal{G}=T(p).g(\phi)$, then for sinusoidal phase detector characteristic $g(\phi)=\sin\phi$, thereby FP-equation becomes,

$$\frac{\partial P(\phi,t)}{\partial t} = -\frac{\partial}{\partial \phi}\{K A\, \mathcal{G}(\phi)-\Omega_o\}P(\phi,t) + \frac{1}{2}\,\frac{K^2 N_o}{2}\,\frac{\partial^2 P(\phi,t)}{\partial \phi^2} \quad \text{....(15)}$$

in the steady-state, as $t\to\infty$,

$$\lim_{t\to\infty} \frac{\partial P}{\partial t} = 0 \quad \text{......(16)}$$

then $P_{ss}(\phi)=P(\phi,t)$ is the steady-state pdf of ϕ,

$$-\frac{d}{d\phi}\{K A\,\mathcal{G}_\infty(\phi)-\Omega_o\}P(\phi)+ \frac{1}{2}\,\frac{K^2 N_o}{2}\,\frac{d^2P(\phi)}{d\phi^2} \quad \text{.......(17)}$$

equation 17 is an ordinary differential equation and in which,

$$K_1(\phi) = \Omega_o - K A\,\mathcal{G}_\infty(\phi) \quad \text{..........(18-a)}$$

$$K_2(\phi) = K^2 N_o/2 \quad \text{..........(18-b)}$$

Furthermore as $t\to\infty$, considering

$$\lim_{t\to\infty} \mathcal{G}(\phi) = \mathcal{G}_\infty(\phi) = C\,\mathcal{G}(\phi) \quad \text{...(18-c)}$$

where C is assumed to be an arbitrary constant depending on the bandwidth parameter of the loop in linear model, and

$$\alpha = 4\, A/KN_o = A^2/N_o\, B_o = \rho_L \quad \text{....(19-a)}$$

$$\beta = 4\,\Omega_o / K\, N_o \quad \text{....(19-b)}$$

are parameters to characterize the loop performance, whereby α represents the loop signal-to-noise ratio (S/N) relative to the loop bandwidth B_L and then $\gamma=\beta/\alpha=\Omega_o/N_o$ is the normalized initial detuning in the loop. Thereby, the general solution of FP-equation in the steady-state becomes

$$P(\phi) = C'_o \exp[-U_o(\phi)] \int_{\phi}^{\phi+\pi} \exp[U_o(y)]dy \quad \text{..(20)}$$

with,

$$U_o(\phi) = -\beta\phi + \alpha\int^{\phi} \mathcal{G}_m(x)\ dx \qquad \text{....(21)}$$

and C'_o is the normalized constant found from the normalized condition of eqn. 14-c as

$$1/C'_o = \int_{-\pi}^{\pi}\int_{\phi}^{\phi+2\pi} \exp[U_o(\phi)+U_o(x)]\ d\phi\ dx \ \text{...(22)}$$

However, two significant cases must be investigated to obtain the actual modulo-2π pdf of phase error:

CASE(1): Pdf with non-zero detuning ($\gamma \neq 0$)

When the frequency of the incoming signal differs from the quiescent frequency of the VCO, there may be actual difference between transmitter and receiver (called Doppler shift). As a result. The steady-state pdf of ϕ can be deduced as,

$$P(\phi)=C'_o \exp[\alpha(\cos\phi/R_d - \gamma\phi)].$$

$$\int^{\phi+2\pi} \exp[-\alpha(\cos\phi/R_d+\gamma y)]dy \ \text{....(23)}$$

with $C=1/R_d$ (or $\alpha'=C\alpha=\alpha_d$), where α' is considered as the effective S/N ratio, then the constant C'_o becomes

$$1/C'_o = \int_{-\pi}^{\pi}\int_{\phi}^{\phi+2\pi} \exp[\alpha(\cos\phi-\cos y)]/R_d + \alpha\gamma(\phi-y)]\ d\phi\ dy \ \text{...(24)}$$

Consequently, the results of pdf in the steady-state are plotted in Figures 2-4 for various values of γ, α and KT_d. The curves tend to degrade as the amount of noise increases and as the time delay increases.

CASE(2): Pdf with zero-detuning ($\gamma=0$)

When the incoming signal is known so that the VCO frequency can be set according; the problem is merely one of tracking the phase. Then

$$P(\phi) = C'_o \exp[\alpha' \cos\phi] = \exp[\alpha' \cos\phi]/ 2\pi I_o(\alpha') \qquad \text{.......(25)}$$

where $I_o(\alpha')$ is the Bessel function of zero-order. For large α',

$$I_o(\alpha') = \exp[\alpha'] / \sqrt{2\pi\alpha} \quad \text{for } \alpha' >> 1 \ \text{....(26)}$$

4- The statistics in the steady-state

The statistical properties of the reduced modulo-2π of the error $\phi(t)$ determine the accuracy of the formation of the recovered signal involving the mean-value and variance due to the presence of additive noise. The cycle slipping determines the synchronization reliability by the mean time to first-slip (loss-of-lock), frequency of slipping cycles and the probability of synchronization failure.

4-1 Moments of phase error:

The second-moment $\phi(t)$ denotes the mean-square value and the variance of phase error is expressed by,

$$\sigma_\phi^2 = \overline{\phi^2(t)} = \int_{-\pi}^{\pi} \phi^2\ P(\phi)\ d\phi \qquad \text{........(27)}$$

assuming that the mean value of $\phi(t)$ equals zero. Thus the results obtained are illustrated in Fig.5.

4-2 The Mean-Time To First-Slip

When the loop is initially in-lock the phase error must remain within the limits $|\phi| < \phi_L$, where $\pm\phi_L$ are the boundaries of $\phi(t)$, and then the first-passage time becomes,

$$2\tau(\phi_1).B_L = 0.5\ \alpha \int_{-\phi_L}^{\phi_L}\int_{-\phi_L}^{\phi} \exp[\alpha\{\cos\phi-\cos y\} + \beta\{\phi-y\}]\ dy\ d\phi \qquad \text{.........(28)}$$

Letting $\phi_L=\phi_o+2\pi$ and $-\phi_L=\phi_o-2\pi$, while assuming $\phi_o=0$ then eqn. 28 produces the mean-time to first-slip. For the special case of $\beta=0$ (i.e. $\gamma=0$) when the frequency of the incoming signal equals the VCO frequency (i.e. $\omega_i=\omega_o$), the equilibrium phase error position becomes $\phi=0$; otherwise when $\omega_i \neq \omega_o$ the equilibrium position becomes nonzero. However, the results are given in

Figures 6 and 7 under different conditions of detuning and time delay.

4-3 *Number of Cycles gained/lost per unit time:*

The cycle slipping phenomenon is characterized by the average number of cycles per unit time (S). The average number of cycles gained per second (N+) and that of cycles (N-) have been obtained for the first-order loop incorporating time delay. For the special case of $\gamma=0$, the slips either in right (plus) or left (minus) has the same probability and the probability of occurrence of synchronization failure can be described by [1,7],

$$P(t) = 1 - \exp(-S\, t) \qquad \text{.......(29)}$$

Figure 8 illustrates an example of synchronization failure.

5- *Discussion*

The paper has dealt with Fokker-Planck technique in order to investigate the effect of the incorporated time delay in the presence of noise under different conditions of detuning. The results highlight the performance of the first-order loop in the steady-state as shown in Figures 2-8. However, the following points are obtained to summarize the loop performance:

1- For high S/N ratio, it can be concluded that the pdf of the phase error $P(\phi)$ becomes uniformly distributed for high values of additive noise. In the presence of delay error, the degradation in the phase error probability takes place for high values of KT_d and low S/N ratios.
2- The noise bandwidth of the loop steadily increases with time delay until the loop eventually becomes unstable at limit $KT_d=\pi/2$ (i.e. 90^0).
3- The parameter (γ) has important effect on $P(\phi)$. For any increment in γ there is a resulting shift in the maximum value of $P(\phi)$ in the direction of detuning and $P(\phi)$ becomes asymmetric as the detuning approaches the synchronization limit (i.e. when γ=1).
4- An increment in the degradation of $P(\phi)$ will take place as the delay error increases in the loop. This is due to the deterioration in the noise bandwidth in the presence of inherent delay error.
5- Fokker-Planck technique gives unlimited solution for PLLs performance and the threshold can be evaluated by the cycle slip rate. If the additive noise is sufficiently large, the variance of the VCO increases causing cycle slipping phenomenon. The phenomenon increases due to the additive noise as well as the initial frequency detuning.
6- In the case of first-order loop incorporating delay error, the average number of cycles slips per unit time will increase as compared with the case of non-delay error. This means that the delay will deteriorate the loop bandwidth and in consequence the degradation in the effective S/N ratio accompanies the loop operation.

Finally, the FP-equation can be applied for higher-order PLLs in the presence of delay error but it requires more details in the analysis to obtain the suitable approximated solution to n-dimensional loops. In this case, the cycle slips rate can not be evaluated as the inverse of the mean-time to first-slip for higher-order loops. This is due to the fact that the small time t, many cycles have short duration compared to the mean-time between slips and then the slips occur in burst mode [1,6].

6- *Conclusion*

The paper has discussed the exact nonlinear behavior of the first-order PLL incorporating time delay for VHF and UHF systems in the presence of noise. The analysis is based on Fokker-Planck equation in the steady-state. In fact, the study tends to find an exact solution for the first-order loop and introduces a good line to predicate the statistical behavior of the higher-order loops following FP-equation. The exact solution leads to evaluate and compute the threshold phenomenon through the cycle slip rate (S) and the probability of synchronization failure can be obtained for such loop. In fact, the analysis can be employed to describe and investigate optical first-order loops in many recent optical communication systems.

7- *References*

[1] W.C. Lindsey, Synchronization systems in communication and control, 1976.

[2] F.M.Gardener, Phaselock technique, 2nd ed., 1979.

[3] J.C.Ward,"Delay reduction techniques in phase locked loop amplifiers", Elect. Lett., Vol. 17, No. 7, pp. 253-255, 1981.

[4] K.Daikoku,"Analysis of phase-locked acquisition with delay time", Rev Elect. Comm., Vol. 23, No. 7-8, pp. 976-983, Feb. 1975.

[5] G.A.AL-Suheal, "Transient behaviour of First-order phase-locked loop with the aid of RLC ladder networks", AL-Kufa journal, Vol. 1, No. 1&2, pp.10-18, August 1997.

[6] G.Ascheid and H. Meyr,"Cycles slips in phase locked loops: A tutorial survy," IEEE Trans. Comm., Vol. Com-30, No.10, pp. 2228-2241, Oct. 1982.

[7] A.J. Viterbi, "Phase-locked loop dynamics in the presence of noise by Fokker-Planck techniques,"proc. IEEE, Vol. 51, pp. 1137-1153, Dec. 1963.

[8] E. A. Bazzoni, G. Marchetti. U. Mengali and F. Russo, " An extension of Viterbi's analysis of the cycle slipping in a first-order phase locked loop," IEEE Trans. Aerosp. Electron Syst., Vol. AES-6, No. 4, pp. 484-490, July 1970.

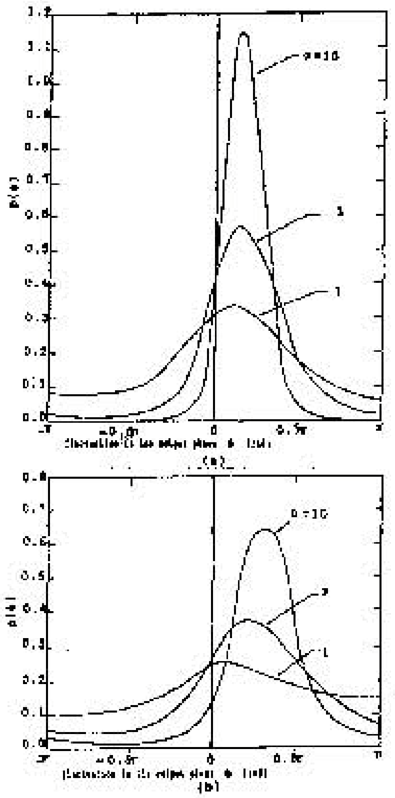

Figure 3 Steady-state probability density function of phase error for various values of α, with detuning $\gamma=0.5$
(a) $KT_d=0.0$ *(b) $KT_d=0.5$ rad*

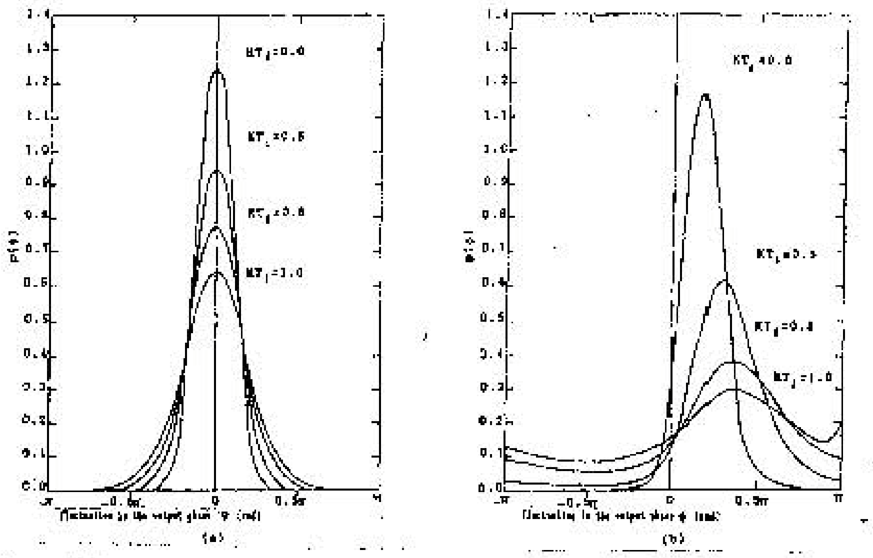

Figure 4 Steady-state probability density function of phase error for various values of KT_d
(a) with zero detuning *(b) with non-zero detuning $\gamma=0.5$*

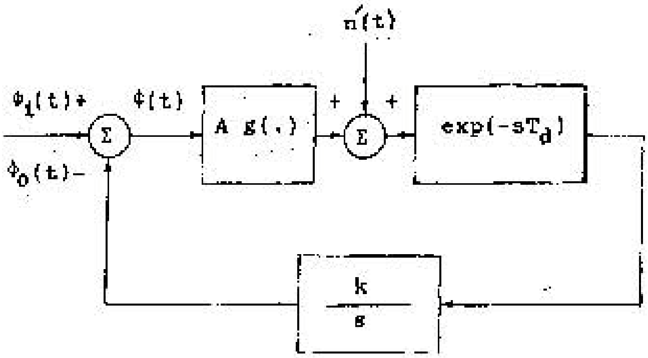

Figure 1 Analytical model of the first-order PLL incorporating time delay.

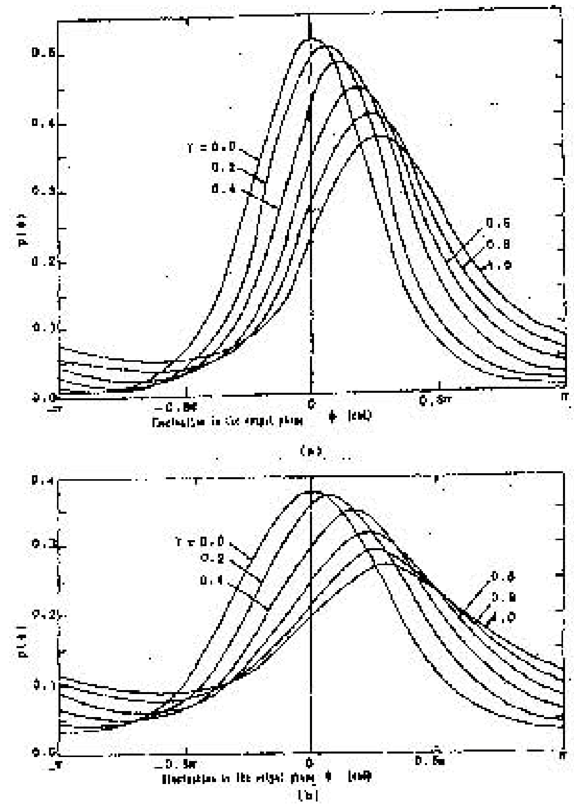

Figure 2 Steady-state probability density function of the phase-error for various values of detuning with α=2

(a) KT_d=0.0 (b) KT_d=0.5 rad

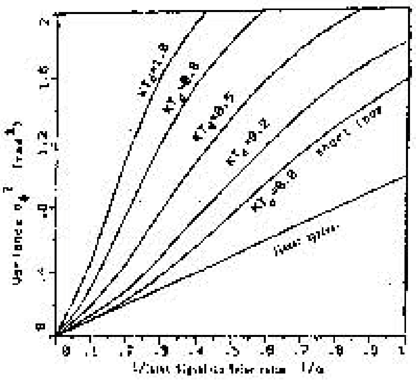

Figure 5 Variance of phase error versus the inverse of signal-to-noise ratio for γ=0.0

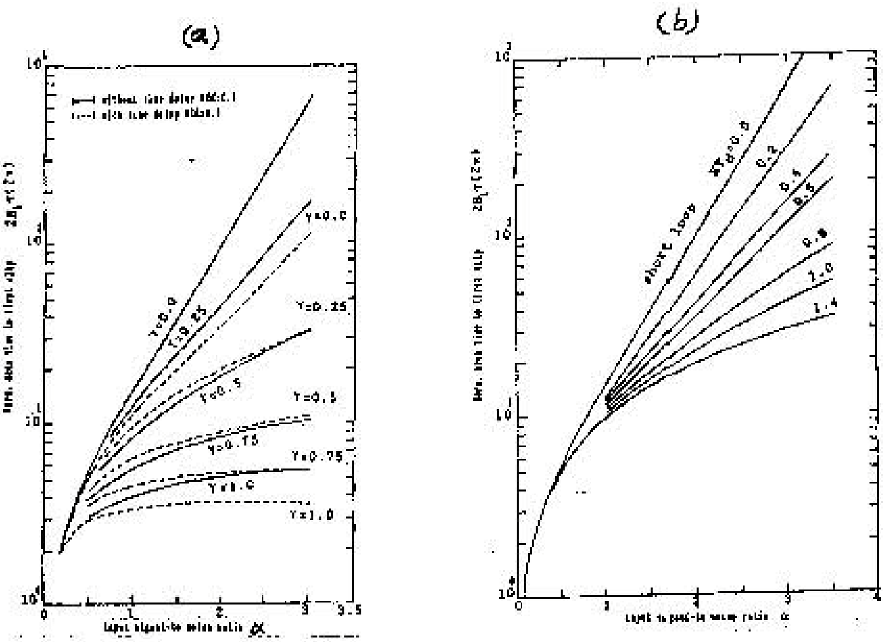

Figure 6 Mean time to first slip versus signal-to-noise ratio
(b)for varoius values of KT_4 (a)for various values of detuning γ

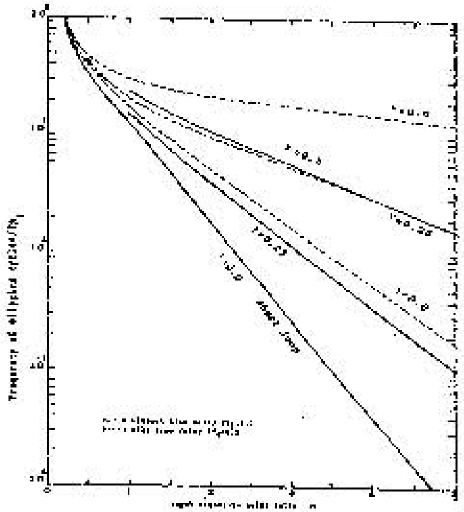

Figure 7 Frequency of slipping cycles (cycle slip rate) versus signal-to-noise ratio

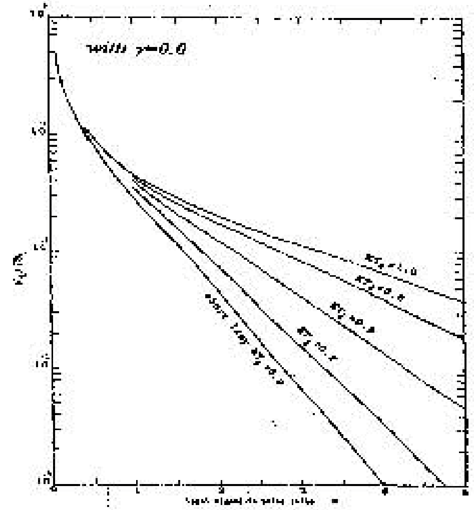

Figure 8 Average number of cycle slips N⊥ versus signal-to-noise ratio