Variances Of Maximum Likelihood Estimators Of Variance Components In The 5-Way Nested Classification Random Model

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ABSTRACT

In this study the variances of the maximum likelihood estimators of variance components in the 5-way nested classification random model will be found by using Searle's method .

Key word : Nested random model , Covariance matrix, Maximum likelihood estimators of variance components.

Introduction

The variance of the maximum likelihood estimators of variance components and estimation of these components for balanced or unbalanced models have been discussed by many researchers. The sampling variance of the least squares estimates of the components of variance in an unbalanced one way classification and the large sample variances of maximum likelihood estimates of these quantities are a discussed by Crump (1951) [5] . Searle (1956) [10] using matrix method re-worked Crump's results and extended them to components of variance and he used the same method to derive sampling variance of variance components estimators for both the 2-way crossed and the 2-way nested classification. Blischke (1966) [3] used them on 3-way crossed classification. He (1968) [4] developed this method for r-way classification. In the same way, Henderson (1953) [7] developed three methods of estimating variance components for unbalanced data of any crossed or nested classification, the techniques of which have been further discussed in Searle (1968) [11] . Mahamunulu (1963) [8] used matrix method to find the explicit expression for elements of information matrix of variance components under normality condition and used this matrix in finding the variance of maximum likelihood estimators of variance components and he displayed these results for 2-way nested classification model. Rudan and Searle (1971) [9] used these results for 3-way nested classification model .Abdullah (1997) [1] used these results for 2-way random effect with unbalanced data, and he (2006) [2] used them for random effect models which the measurements have the following form of covariance

$$\operatorname{cov}(y_{ijk}, y_{i'j'k'}) = \begin{cases} \sigma^2 & ; i = i' \ , \ j = j' \ , \ k = k' \\ \sigma^2 \rho_3 & ; i = i' \ , \ j = j' \ , \ k \neq k' \\ \sigma^2 \rho_2 & ; i = i' \ , \ j \neq j' \ , \ k = k' \\ \sigma^2 \rho_1 & ; i = i' \ , \ j \neq j' \ , \ k \neq k' \\ 0 & ; i \neq i' \end{cases}$$

And used these results for 2-way random effect model which has the form

$$y_{ijk} = \theta + a_i + b_{ij} + (ab)_{ik} + e_{ijk}$$

With $i = 1,2,\cdots,n$, $j = 1,2,\cdots,d$, $k = 1,2,\cdots,r$ and θ unknown parameter, and

$a_i \overset{iid}{\sim} n(0,\sigma_a^2)$, $b_{ij} \overset{iid}{\sim} n(0,\sigma_b^2)$, $(ab)_{ik} \overset{iid}{\sim} n(0,\sigma_{ab}^2)$, $e_{ijk} \overset{iid}{\sim} n(0,\sigma_e^2)$

And for 3-way random effect model which has the form

$$y_{ijk} = \theta + a_i + b_{ij} + c_{ik} + e_{ijk}$$

with $i = 1,2,\cdots,n$, $j = 1,2,\cdots,d$ and $k = 1,2,\cdots,r$ and θ unknown parameter. The a_i , b_{ij} , c_{ik} *and* e_{ijk} are independent random variables with zero mean and variance σ_a^2 , σ_b^2 , σ_c^2 *and* σ_e^2 respectively .

Also for 4-way mixed effect model which has the form

$$y_{ijk} = \theta + a_i + b_{ij} + c_{ik} + d_{jk} + e_{ijk}$$

with $i = 1,2,\cdots,n$, $j = 1,2,\cdots,d$ and $k = 1,2,\cdots,r$ and θ, d unknown parameters such that

$$\sum_j d_{jk} = \sum_k d_{jk} = 0 \text{ and } a_i \overset{iid}{\sim} n(0,\sigma_a^2) \ , \ b_{ij} \overset{iid}{\sim} n(0,\sigma_b^2) \ , (c)_{ik} \overset{iid}{\sim} n(0,\sigma_c^2) \ , e_{ijk} \overset{iid}{\sim} n(0,\sigma_e^2)$$

The aim of this study is to use Searle's method to find information matrix for 5-way nested classification model and use this matrix to find the variance of maximum likelihood estimators of variance components of this model.

(1-2) Notation

Let I_n be the $n \times n$ identity matrix, let $J_{n\times m}$ be the matrix of one in every position. If A $n \times m$ matrix and $B = (b_{ij})$ is $p \times q$ matrix then the Kronecker product of A and B written as $A \otimes B$ is the $np \times mq$ matrix (Ab_{ij}). If A $n \times m$ matrix the tr(A) is the sum of elements of leading diagonal of A .The element r^i is the inverse of r_i .

2. The model

Consider the following model

$$y_{ijklmz} = \theta + \alpha_i + \beta_{ij} + \gamma_{ijk} + \delta_{jikl} + \xi_{ijklm} + e_{ijklmz} \quad (1)$$

Where y_{ijklmr} is the $z-th$ response within the $m-th$ level of the ξ factor within $l-th$ level of the δ factor within $k-th$ level of the γ factor within $j-th$ level of the β factor within $i-th$ level of the α factor. This nesting in indicated in the observation identifiers which are taken to be

$i = 1,2,...,a$; $j = 1,2,...,b_i$; $k = 1,2,...,c_{ij}$; $l = 1,2,...,d_{ijk}$, $m = 1,2,...,r_{ijkl}$ *and* $z = 1,2,...,n_{ijklm}$.

The $\alpha's$; $\beta's$; $\gamma's$; $\delta's$, $\xi's$ *and* $e's$ are independent have normal distribution with zero means and variances σ_α^2 , σ_β^2 , σ_γ^2 , σ_δ^2 *and* σ_e^2 respectively.

The variance covariance matrix of Y can be written as:

$$V = \sum_{i=1}^{a}{}^{+} R_i$$

Where $\sum^+$ denote the operating of a direct sum of matrices and each R_i can be written in partition matrix form as :

$R_i = \left\{ R_{ij\times ij'} \right\}$; $j, j' = 1,2,...,b_i$,where

(2)

$$R_{i,jj'} = \begin{cases} \sigma^2[(1-\rho_5) I_{n_{ij...}} + (\rho_5 - \rho_4) \underset{k,l,m}{diag}(J_{n_{ijklm}}) + (\rho_4 - \rho) \underset{k,l}{diag}(J_{n_{ijkl.}}) + (\rho_3 - \rho_2) \underset{k}{diag}(J_{n_{ijk..}}) \\ + \rho_2 J_{n_{ij...}}] & j = j' \\ \sigma^2 \rho_1 J_{n_{ij...} \times n_{ij'...}} & j \neq j' \end{cases}$$

(3)

so, we get

$$R_i = \sigma^2[(1-\rho_5)I_{n_{i....}} + (\rho_5-\rho_4)\underset{j,k,l,m}{diag}(J_{n_{ijklm}}) + (\rho_4-\rho)\underset{j,k,l}{diag}(J_{n_{ijkl.}}) + (\rho_3-\rho_2)\underset{j,k}{diag}(J_{n_{ijk..}})$$
$$+(\rho_2-\rho_1)\underset{j}{diag}(J_{n_{ij...}}) + \rho_1 J_{n_{i....}}] \quad (4)$$

Now without loss of generality, we shall discuss the variance covariance matrix, and the information matrix for the model (1) in balanced case when

$$b_i = b \ , \ c_{ij} = c \ , \ d_{ijk} = d \ , \ r_{ijkl} = r \ , \ n_{ijklm} = n \ \ and \ \ R_i = R$$

So, we get

$$V = R \otimes I_a \quad (5)$$

Where

$$R = \sigma^2[(1-\rho_5)I_{bcdrn} + (\rho_5-\rho_4)J_n \otimes I_{bcdr} + (\rho_4-\rho_3)J_{rn} \otimes I_{bcd}$$
$$+(\rho_3-\rho_2)J_{drn} \otimes I_{bc} + (\rho_2-\rho_1)J_{cdrn} \otimes I_b + \rho_1 J_{bcdrn}] \quad (6)$$

To simplify notation in the writing we use

$$\alpha = \sigma_\alpha^2 \ , \beta = \sigma_\beta^2 \ , \ \gamma = \sigma_\gamma^2 \ , \delta = \sigma_\delta^2 \ , \xi = \sigma_\xi^2 \ \ and \ \ e = \sigma_e^2$$

Thus we shall get that

$$\sigma^2 = \alpha+\beta+\gamma+\delta+\xi+e \ , \sigma^2\rho_5 = \alpha+\beta+\gamma+\delta+\xi \ , \ \sigma^2\rho_4 = \alpha+\beta+\gamma+\delta \ ,$$
$$\sigma^2\rho_3 = \alpha+\beta+\gamma \ , \ \sigma^2\rho_2 = \alpha+\beta \ \ and \ \ \sigma^2\rho_1 = \alpha$$

Now, the matrix R in (2) can be written as:

$$R = eI_{bcdrn} + \xi J_n \otimes I_{bcdr} + \delta \ J_{rn} \otimes I_{bcd} + \gamma \ J_{drn} \otimes I_{bc} + \beta \ J_{cdrn} \otimes I_b + \alpha \ J_{bcdrn}] \quad (7)$$

Therefore

$$V = [eI_{bcdrn} + \xi J_n \otimes I_{bcdr} + \delta \ J_{rn} \otimes I_{bcd} + \gamma \ J_{drn} \otimes I_{bc} + \beta \ J_{cdrn} \otimes I_b + \alpha \ J_{bcdrn}] \otimes I_a \quad (8)$$

By Urquhart's method [13] we get the inverse of the matrix R by the form:

$$R^{-1} = \frac{1}{\sigma^2}[\frac{1}{1-\rho_5} \ I_{bcdrn} + \phi_1 \ J_n \otimes I_{bcdr} + \phi_2 \ J_{rn} \otimes I_{bcd} + \phi_{13} \ J_{drn} \otimes I_{bc} + \phi_4 \ J_{cdrn} \otimes I_b + \phi_5 J_{bcdrn}] \quad (9)$$

and the algebraic details involved in finding R^{-1} from R as given in the appendix

So, the inverse of the variance covariance matrix V is

$$V^{-1} = R^{-1} \otimes I_a \quad (10)$$

3. Sampling variances

By Searle's method the covariance matrix of large sample maximum likelihood estimators of $\alpha\ ,\ \beta\ ,\ \gamma\ ,\ \delta\ ,\ \xi$ *and* e is

$$H_{ML}=\begin{bmatrix} V(\hat{\alpha}) & \text{cov}(\hat{\alpha},\hat{\beta}) & \text{cov}(\hat{\alpha},\hat{\gamma}) & \text{cov}(\hat{\alpha},\hat{\delta}) & \text{cov}(\hat{\alpha},\hat{\xi}) & \text{cov}(\hat{\alpha},\hat{e}) \\ \text{cov}(\hat{\beta},\hat{\alpha}) & V(\hat{\beta}) & \text{cov}(\hat{\beta},\hat{\gamma}) & \text{cov}(\hat{\beta},\hat{\delta}) & \text{cov}(\hat{\beta},\hat{\xi}) & \text{cov}(\hat{\beta},\hat{e}) \\ \text{cov}(\hat{\gamma},\hat{\alpha}) & \text{cov}(\hat{\gamma},\hat{\beta}) & V(\hat{\gamma}) & \text{cov}(\hat{\gamma},\hat{\delta}) & \text{cov}(\hat{\gamma},\hat{\xi}) & \text{cov}(\hat{\gamma},\hat{e}) \\ \text{cov}(\hat{\delta},\hat{\alpha}) & \text{cov}(\hat{\delta},\hat{\beta}) & \text{cov}(\hat{\delta},\hat{\gamma}) & V(\hat{\delta}) & \text{cov}(\hat{\delta},\hat{\xi}) & \text{cov}(\hat{\delta},\hat{e}) \\ \text{cov}(\hat{\xi},\hat{\alpha}) & \text{cov}(\hat{\xi},\hat{\beta}) & \text{cov}(\hat{\xi},\hat{\gamma}) & \text{cov}(\hat{\xi},\hat{\delta}) & V(\hat{\xi}) & \text{cov}(\hat{\xi},\hat{e}) \\ \text{cov}(\hat{e},\hat{\alpha}) & \text{cov}(\hat{e},\hat{\beta}) & \text{cov}(\hat{e},\hat{\gamma}) & \text{cov}(\hat{e},\hat{\delta}) & \text{cov}(\hat{e},\hat{\xi}) & V(\hat{e}) \end{bmatrix}$$

(11)

$$=2T^{-1}=2\begin{bmatrix} t_{\alpha\alpha} & t_{\alpha\beta} & t_{\alpha\gamma} & t_{\alpha\delta} & t_{\alpha\xi} & t_{\alpha e} \\ t_{\beta\alpha} & t_{\beta\beta} & t_{\beta\gamma} & t_{\beta\delta} & t_{\beta\xi} & t_{\beta e} \\ t_{\gamma\alpha} & t_{\gamma\beta} & t_{\gamma\gamma} & t_{\gamma\delta} & t_{\gamma\xi} & t_{\gamma e} \\ t_{\delta\alpha} & t_{\delta\beta} & t_{\delta\gamma} & t_{\delta\delta} & t_{\delta\xi} & t_{\delta e} \\ t_{\xi\alpha} & t_{\xi\beta} & t_{\xi\gamma} & t_{\xi\delta} & t_{\xi\xi} & t_{\xi e} \\ t_{e\alpha} & t_{e\beta} & t_{e\gamma} & t_{e\delta} & t_{e\xi} & t_{ee} \end{bmatrix}^{-1}$$

Where T is (6×6) symmetric matrix its elements can be find by the following law

$$t_{ij}=tr(V^{-1}V_iV^{-1}V_j) \qquad for\ i,j=\alpha,\beta,\gamma,\delta,\xi,e$$

With $V_i=\dfrac{\partial V}{\partial\sigma_i^2}$ is the partial derivative of V with respect to σ_i^2 .

From (8) we get

$$V_\alpha=\frac{\partial V}{\partial\alpha}=J_{bcdrn}\otimes I_a \quad , \quad V_\beta=\frac{\partial V}{\partial\beta}=J_{cdrn}\otimes I_{ab}$$

$$V_\gamma=\frac{\partial V}{\partial\gamma}=J_{drn}\otimes I_{abc} \quad , \quad V_\delta=\frac{\partial V}{\partial\delta}=J_{rn}\otimes I_{abcd}$$

$$V_\xi=\frac{\partial V}{\partial\xi}=J_n\otimes I_{abcdr} \quad , \quad V_e=\frac{\partial V}{\partial e}=I_{abcdrn}$$

(12)

Now Searle's results are used in finding explicit expression for the elements of the matrix T .
From (9) and (10) we get

$$V^{-1}V_\alpha = (R^{-1}\otimes I_a)(J_{bcdrn}\otimes I_a) = (R^{-1}J_{bcdrn})\otimes I_a$$

$$= [\frac{1}{\sigma^2}(\frac{1}{1-\rho_5} I_{bcdrn} + \phi_1 J_n \otimes I_{bcdr} + \phi_2 J_{rn}\otimes I_{bcd} + \phi_3 J_{drn}\otimes I_{bc} +$$

$$\phi_4 J_{cdrn}\otimes I_b + \phi_5 J_{bcdrn}) J_{bcdrn}]\otimes I_a$$

$$= \frac{1}{\sigma^2}(\frac{1}{1-\rho_5} + n\phi_1 + rn\phi_2 + drn\phi_3 + cdrn\ \phi_4 + bcdrn\phi_5) J_{bcdrn}\otimes I_a$$

Thus

$$t_{\alpha\alpha} = tr(V^{-1}V_\alpha V^{-1}V_\alpha) = tr((V^{-1}V_\alpha)^2)$$

$$= tr(\frac{bcdrn}{\sigma^4}(\frac{1}{1-\rho_5} + n\phi_1 + rn\phi_2 + drn\phi_3 + cdrn\ \phi_4 + bcdrn\phi_5)^2 J_{bcdrn}\otimes I_a)$$

$$= \frac{a(bcdrn)^2}{\sigma^4}(\frac{1}{1-\rho_5} + n\phi_1 + rn\phi_2 + drn\phi_3 + cdrn\ \phi_4 + bcdrn\phi_5)^2 = \frac{a(bcdrn)^2}{\sigma^4}L_1$$

Where

$$L_1 = \frac{1}{1-\rho_5} + n\phi_1 + rn\phi_2 + drn\phi_3 + cdrn\ \phi_4 + bcdrn\phi_5$$

Also

$$V^{-1}V_\beta = (R^{-1}\otimes I_a)(J_{cdrn}\otimes I_{ab}) = (R^{-1}(J_{cdrn}\otimes I_b))\otimes I_a$$

$$= [\frac{1}{\sigma^2}(\frac{1}{1-\rho_5} I_{bcdrn} + \phi_1 J_n \otimes I_{bcdr} + \phi_2 J_{rn}\otimes I_{bcd} + \phi_3 J_{drn}\otimes I_{bc} +$$

$$\phi_4 J_{cdrn}\otimes I_b + \phi_5 J_{bcdrn})(J_{cdrn}\otimes I_b)]\otimes I_a$$

$$= \frac{1}{\sigma^2}[(\frac{1}{1-\rho_5} + n\phi_1 + rn\phi_2 + drn\phi_3 + cdrn\ \phi_4) J_{cdrn}\otimes I_b + cdrn\ \phi_5\ J_{bcdrn}]\otimes I_a$$

Thus

$$t_{\beta\beta} = tr(V^{-1}V_\alpha V^{-1}V_\beta) = tr((V^{-1}V_\beta)^2)$$

$$= \frac{1}{\sigma^4}tr[cdrn(\frac{1}{1-\rho_5} + n\phi_1 + rn\phi_2 + drn\phi_3 + cdrn\ \phi_4)^2 J_{cdrn}\otimes I_b + (bc^3d^3r^3n^3\ \phi_5^2$$

$$+ 2c^2d^2r^2n^2\ \phi_5(\frac{1}{1-\rho_5} + n\phi_1 + rn\phi_2 + drn\phi_3 + cdrn\ \phi_4)) J_{bcdrn}]\otimes I_b$$

$$= \frac{a(bcdrn)^2}{\sigma^4}(L_2 + cdrn\phi_5(2L_2 + cdrn\phi_5))$$

Where

$$L_2 = \frac{1}{1-\rho_5} + n\phi_1 + rn\phi_2 + drn\phi_3 + cdrn\,\phi_4$$

Now, let $A_{pq} = \frac{abcdrn}{\sigma^4}[L_p^2 + qcdrn\,\phi_5(L_1 + L_2)]$

Then

$t_{\alpha\alpha} = bcdrn A_{10}$ *and* $t_{\beta\beta} = cdrn A_{21}$

Similarly the other elements of the matrix T are

$t_{\alpha\beta} = cdrn\,A_{10}$, $t_{\alpha\gamma} = drn\,A_{10}$, $t_{\alpha\delta} = rn\,A_{10}$, $t_{\alpha\xi} = n\,A_{10}$, $t_{\alpha e} = n\,A_{10}$

$t_{\beta\gamma} = drn\,A_{21}$, $t_{\beta\delta} = rn\,A_{21}$, $t_{\beta\xi} = n\,A_{21}$, $t_{\beta e} = A_{21}$, $t_{\gamma\gamma} = drn\,B_{31100}$

$t_{\gamma\delta} = rn\,B_{31100}$, $t_{\gamma\xi} = n\,B_{31100}$, $t_{\gamma e} = B_{31100}$, $t_{\delta\delta} = rn\,B_{40101}$, $t_{\delta\xi} = n\,B_{40101}$

$t_{\delta e} = B_{40101}$, $t_{\xi\xi} = n\,B_{50011}$, $t_{\xi e} = B_{50011}$ *and* $t_{ee} = C$

Where

$$B_{pqwst} = \frac{abcdrn}{\sigma^4}[L_p^2 + sn(r)^w(d)^q\,\phi_2(L_5 + L_4) + t\,\phi_3(L_4 + L_3) + \phi_4(L_3 + L_2) + \phi_5(L_2 + L_1)]$$

$$C = \frac{abcdrn}{\sigma^4}[\frac{1}{(1-\rho_5)^2} + \phi_1(\frac{1}{1-\rho_5} + L_5) + \phi_2(L_5 + L_4) + \phi_3(L_4 + L_3) + \phi_4(L_3 + L_2) + \phi_5(L_2 + L_1)]$$

and

$$L_3 = \frac{1}{1-\rho_5} + n\phi_1 + rn\,\phi_2 + drn\phi_3$$

$$L_4 = \frac{1}{1-\rho_5} + n\phi_1 + rn\,\phi_2$$

$$L_5 = \frac{1}{1-\rho_5} + n\phi_1$$

Therefore, the matrix T is

$$\begin{bmatrix} bcdrnA_{10} & cdrnA_{10} & drnA_{10} & rnA_{10} & nA_{10} & A_{10} \\ cdrnA_{10} & cdrnA_{21} & drnA_{21} & rnA_{21} & nA_{21} & A_{21} \\ drnA_{10} & drnA_{21} & drnB_{31100} & rnB_{31100} & nB_{31100} & B_{31100} \\ rnA_{10} & rnA_{21} & rnB_{31100} & rnB_{40101} & nB_{40101} & B_{40101} \\ nA_{10} & nA_{21} & nB_{31100} & nB_{40101} & nB_{50011} & B_{50011} \\ A_{10} & A_{21} & B_{31100} & B_{40101} & B_{50011} & C \end{bmatrix}$$

Using this matrix leads to explicit expression for the elements of $H_{ML} = 2T^{-1}$ of (11) and find the variance of maximum likelihood estimators of variance components $\sigma_\alpha^2, \sigma_\beta^2, \sigma_\gamma^2, \sigma_\delta^2, \sigma_\xi^2$ *and* σ_e^2 .

4. Conclusions

The model of this study is extension of the model which is studied by Searle [12] and Rudan and Searle [9] and we can get Searle's results[12]of the 2-way nested classification random model by ignoring the $\gamma's$, $\delta's$ *and* $\xi's$ in (1) and putting $\sigma_\gamma^2 = 0$, $\sigma_\delta^2 = 0$ *and* $\sigma_\xi^2 = 0$,also we can get the results of Rudan and Searle [9] by ignoring the $\delta's$ *and* $\xi's$ in (1) and putting $\sigma_\delta^2 = 0$ *and* $\sigma_\xi^2 = 0$ which is confirms the validity of results of this paper.

The Searle's method is suitable for any type of models, so this study can be extended to higher models which are more than 5-way nested classification model where the covariance matrix and its inverse can be obtained.

The covariance matrix of large sample maximum likelihood estimators of variance components in this work has different order than the one considered by Searle[12] and Rudan and Searle [9] because the order of this matrix depend by the number of variance components.

The maximum likelihood estimation of variance components for unbalanced data has received somewhat less attention than other method of estimation due to its underlying complexities.

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Appendix

Urquhart lemma

Concerning a matrix A partition as:

$$A = \{A_{pq} \ of \ order \ n_p \times n_q\} \text{ For } p, q = 1,2,\dots, N \quad \text{(A1)}$$

Where, $A_{pp} = b_p I_{n_p} + g_{pp} J_{n_p}$ and, $A_{pq} = g_{pq} J_{n_p \times n_q}$ for all $p \neq q$, with, $G = \{g_{pq}\}$. (A2)

The inverse of A is given by,

$$A^{-1} = \left\{ \left(A^{-1}\right)_{pq} \;\; for\;\; order\;\; n_p \times n_q \right\}.$$

(A3)

With, $\left(A^{-1}\right)_{pq} = \frac{1}{b_p} I_{n_p} + h_{pp}\, J_{n_p}$ and $\left(A^{-1}\right)_{pq} = h_{pq}\, J_{n_p \times n_q}$ for $p \neq q$

(A4)

where, $H = (h)_{pq} = \left[(GD + B)^{-1} \right] D^{-1}$, $D = diag(n_1, \ldots, n_N)$ and, $B = diag(b_1, \ldots, b_N)$ (A5)

In applying the Urquhart lemma to R_i of (2),(3) and (4) we get

$$p, q = jklm,\, j'k'l'm' \quad with \quad j = 1,2,...,b\;;\; k = 1,2,...,c\;; l = 1,2,...,d \;\; and \;\; m = 1,2,...,r$$

And

$$b_p = b_{jklm} = (1-\rho_5) \quad for\;\; all\;\; j, k, l\;\; and\;\; m$$

$$g_{pq} = g_{jklm,\, j'k'l'm'} = \begin{cases} \rho_5 & j = j', k = k', l = l'. m = m' \\ \rho_4 & j = j', k = k', l = l', m \neq m' \\ \rho_3 & j = j', k = k', l \neq l' \\ \rho_2 & j = j', k \neq k' \\ \rho_1 & j \neq j' \end{cases}$$

Hence

$$G = (\rho_5 - \rho_4) I_{bcdr} + (\rho_4 - \rho_3) J_r \otimes I_{bcd} + (\rho_3 - \rho_2) J_{rd} \otimes I_{bc} + (\rho_2 - \rho_1) J_{cdr} \otimes I_b + \rho_1 J_{bcdr}$$

Also

$$B = (1-\rho_5) I_{bcdr} \quad \text{with} \quad B^{-1} = \frac{1}{(1-\rho_5)} I_{bcdr}$$

And

$$D = \underset{j,k,l,m}{diag}(n_{jklm}) \;\; \text{with} \;\; D^{-1} = \underset{j,k,l,m}{diag}(n^{jklm})$$

From (A3) , (A4) and (A5) we get

$$R^{-1} = ((R^{-1}) \;\; for\;\; order\;\; n_{jklm} \times n_{j'k'l'm'})$$

With

$$(R^{-1})_{jklm\times j'k'l'm'} = \delta_{jklm\times j'k'l'm'} (\frac{1}{1-\rho_5}) I_{jklm\times j'k'l'm'} + h_{jklm\times j'k'l'm'} J_{jklm\times j'k'l'm'}$$

Where

$$H = (h)_{jklm\times j'k'l'm'} = \underset{j,k,l,m}{diag}(n^{jklm})[F]^{-1} \underset{j,k,l,m}{diag}(n^{jklm}) - \frac{1}{1-\rho_5} I_{bcdr} \underset{j,k,l,m}{diag}(n^{jklm})$$

where

$$F = (\rho_5 - \rho_4) I_{bcdr} + (\rho_4 - \rho_3) J_r \otimes I_{bcd} + (\rho_3 - \rho_2) J_{rd} \otimes I_{bc} + (\rho_2 - \rho_1) J_{cdr} \otimes I_b + \rho_1 J_{bcdr} + (1-\rho_5) I_{bcdr} \underset{j,k,l,m}{diag}(n^{jklm})$$

Now, if $n_{jklm} = n$ we get

Let $F = (A-B)\otimes I_b + B \otimes I_b$

Where $A = (\frac{1-\rho_5}{n} + \rho_5 - \rho_4) I_{cdr} + (\rho_4 - \rho_3) J_r \otimes I_{cd} + (\rho_3 - \rho_2) J_{rd} \otimes I_c + \rho_2 J_{cdr}$

And $B = \rho_1 J_{cdr}$

Then the inverse of F will be as follows

$$F^{-1} = (A-B)^{-1} \otimes [I_b - \frac{1}{b} J_b] + \frac{1}{b}[(A-B)+bB]^{-1} \otimes J_b \quad \text{(Gabbara)[6]}$$

Now, let $L = A - B = O \otimes I_c + E \otimes J_c$

Where

$O = y_1 I_{dr} + y_2 J_r \otimes I_d + y_3 J_{rd}$, $E = y_4 J_{dr}$, $y_1 = \frac{1-\rho_5}{n} + \rho_5 - \rho_4$, $y_2 = \rho_4 - \rho_3$

$y_3 = \rho_3 - \rho_2$ *and* $y_4 = \rho_2 - \rho_1$

So, we get $L^{-1} = O^{-1} \otimes [I_c - \frac{1}{c} J_c] + \frac{1}{c}[O + cE]^{-1} \otimes J_c$

Where

$O^{-1} = y^1 I_{dr} - y^1 y_5 J_r \otimes I_d + y_8 J_{rd}$ *and* $(O + cE)^{-1} = y^1 I_{dr} - y^1 y_5 J_r \otimes I_d + y_{12} J_{rd}$

With

$$y_5 = \frac{y_2}{y_1 + r\,y_2} \;,\; y_6 = y_2 + d\,y_3 \;,\; y_7 = \frac{y_6}{y_1 + r\,y_6} \;,.\; y_8 = \frac{y_5}{d\,y_1} - \frac{y_1 y_7}{d} \;,$$

$$y_9 = y_3 + c\,y_4 \;,\; y_{10} = y_2 + d\,y_9 \;,\; y_{11} = \frac{y_{10}}{y_1 + r\,y_{10}} \;\; and \;\; y_{12} = \frac{y_5}{d\,y_1} - \frac{y_1 y_{11}}{d}$$

Thus $L^{-1} = y^1 I_{cdr} - y^1 y_5 J_r \otimes I_{cd} + y_8 J_{rd} \otimes I_c + y_{13} J_{cdr}$

Where $y_{13} = \frac{1}{c}(y_{12} - y_8)$

Similarly we get that

$$(A-(b-1)B)^{-1} = y^1 I_{cdr} - y^1 y_5 J_r \otimes I_{cd} + y_8 J_{rd} \otimes I_c + y_{14} J_{cdr}$$

With $y_{14} = \frac{1}{c}(y_{15} - y_8)$, $y_{15} = \frac{y_5}{d\,y_1} - \frac{y_1 y_{16}}{d}$, $y_{16} = \frac{y_{17}}{y_1 + r\,y_{17}}$, $y_{17} = y_2 + d\,y_{18}$
and $y_{18} = y_3 + c(y_4 + b\,y_1)$

Therefore

$$F^{-1} = y^1 I_{bcdr} - y^1 y_5 J_r \otimes I_{bcd} + y_8 J_{rd} \otimes I_{bc} + y_{13} J_{cdr} \otimes I_b + y_{19} J_{bcdr}$$

Where $y_{19} = \frac{1}{b}(y_{14} - y_{13})$

Therefore

$$H = \left(\frac{y^1}{n^2} - \frac{1}{n(1-\rho_5)}\right) I_{bcdr} - \frac{y^1 y_5}{n^2} J_r \otimes I_{bcd} + \frac{y_8}{n^2} J_{rd} \otimes I_{bc} + \frac{y_{13}}{n^2} J_{cdr} \otimes I_b + y_{19} J_{bcdr}$$

Hence from (A4) we get

$$R^{-1} = \frac{1}{\sigma^2}\left[\frac{1}{1-\rho_5} I_{bcdrn} + \phi_1 J_n \otimes I_{bcdr} + \phi_2 J_{rn} \otimes I_{bcd} + \phi_{13} J_{drn} \otimes I_{bc} + \phi_4 J_{cdrn} \otimes I_b + \phi_5 J_{bcdrn}\right]$$

where

$$\phi_1 = \frac{(1-\rho_5) - n\,y_1}{n^2 y_1 (1-\rho_5)} \;,\; \phi_2 = \frac{-y_5}{n^2 y_1} \;,\; \phi_3 = \frac{y_8}{n^2} \;,\; \phi_4 = \frac{y_{13}}{n^2} \;\; and \;\; \phi_5 = y_{19}$$

تباينات مقدرات الإمكان الأعظم لمركبات التباين لنموذج عشوائي متداخل ذي خمسة اتجاهات

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الخلاصة

في هذه الدراسة تم إيجاد تباينات مقدرات الإمكان الأعظم لمركبات التباين لنموذج عشوائي متداخل ذي خمسة اتجاهات باستخدام طريقة سيرل.