On the non-existence of Complete (k,n)-arcs in PG(2,q)

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Abstract

In this paper we discuss the non-existence of complete (k,n)-arcs in the projective plane of order q, and we find the largest value of k produces according to theorem (3.1), for which a (k,n)-arc dose not complete in the projective plane of order q "PG(2,q)" for $k \geq n$, $n \neq q+1$, which is denoted by $t_q^*(n)$.

Introduction

A (k,n)-arc K in PG(2,q) is a set of k points such that there is some n but no n+1 of them are collinear. A (k,n)-arc K is complete if there is no (k+1,n)-arc containing it. A line ℓ of PG(2,q) is an i-secant of a (k,n)-arc K , if | ℓ $\cap$ K |= i .The maximum value for which a (k,n)-arc K exist in PG(2,q) will be denoted by $m(n)_{2,q}$. A (k ,2)-arc generally called a k-arc.

Some results on the (k,n)-arcs

Lemma 2.1. [4]

For a (k,n)-arc K in PG(2,q), the following equations are true :

$$\sum_{i=1}^{n} R_i = q+1 \quad .... \quad (2.1)$$

$$\sum_{i=2}^{n} (i-1)R_i = k-1 \quad .... \quad (2.2)$$

where R_i =R_i (P) denote the number of i-secants to K through a point P of K.

Assume the equations (2.1) and (2.2) in lemma (2.1) have k distinct solutions B_j = (R_{1j} ,…, R_{nj}) ; j=1,…,k .Suppose there are b_j points on the (k,n)-arc K , with solution B_j then:

Lemma 2.2 [6]

For a (k,n)-arc K in PG(2,q), the following equations are true :

$$\sum_{j=1}^{k} b_j R_{ij} = it_i \quad ..... \quad (2.3)$$

$$\sum_{j=1}^{k} b_j = k \quad ..... \quad (2.4)$$

where t_i is the total number of i-secants to K .

Lemma 2.3 [10]

If K is a complete (k , n)-arc in PG(2,q), then :

q^2+q+1-k , with equality iff S_n = 1 $\geq$ (q+1-n) t_n for all Q in PG(2,q)\K

where S_i =S_i (Q) denote the number of i-secants to K through a point Q of PG(2,q)\K .

3. Non-existence of coplete (k,n)-arcs in PG(2,q)

Suppose $[x]$ denote the smallest positive integer less than or equal to x,

and $\theta(m)=\frac{(q^{m+1}-1)}{(q-1)}, \alpha=\theta(1)-n, \beta=\theta(1)-n^2, \gamma=(n^2-n)\theta(2)$. then we have the following theorem:

Theorem 3.1

In PG(2,q), a complete (k,n)-arc, with $n \le k \le \bar{n}$, $n \ne q+1$ dose not exist, where $\bar{n}=[(\beta+\sqrt{\beta^2+4\alpha\gamma})/2\alpha]$.

Proof:

when $n \le k \le \bar{n}$ equations (2.1) and (2.2) of lemma (2.1) become

$$R_1+R_2+R_3 + R_4 \ldots + R_n = \theta(1) \ldots\ldots\ldots\ldots\ldots\ldots (3.1)$$
$$R_2+2R_3+3R_4 \ldots+(n-1)R_n = k-1 \ldots\ldots\ldots\ldots\ldots\ldots (3.2)$$

Let $m=\left[{(k-1)}/{(n-1)}\right]$, so the largest value of R_n can accure is m.

Assume that $r_{n-i}(s_1,s_2,\ldots,s_i)=[(k-1)-\sum_{l=1}^{i}(n-l)s_l]/[n-(i+1)]$ where

$i=1,2,\ldots,n-2$, $s_1=0,1,\ldots,m$ and $s_j=0,1,2,\ldots,r_{(n+1)-j}(s_1,s_2,\ldots,s_{j-1})\big|_{s_1=0,s_2=0,\ldots,s_{j-1}=0}$
$j=2,3,\ldots,n-2$.

Suppose that

$r_{(n+1)-j}(s_1,s_2,\ldots,s_{j-1})\big|_{s_1=0,s_2=0,\ldots,s_{j-1}=0}$ denoted by $r_{(n+1)-j}(\tilde{0})$ where $j=2,3,\ldots,n-2$.

It is clearly that m is positive for $k \geq n$. Now, suppose that $\psi(j, r_{n-1}(s_1), r_{n-2}(s_1,s_2),\dots,r_2(s_1,s_2,\dots,s_{n-2}))$ denote the number of points of PG(2,q) of type $(j, r_{n-1}(s_1), r_{n-2}(s_2),\dots,r_2(s_{n-2}))$ denote the number of points of PG(2,q) of type $(j, r_{n-1}(s_1), r_{n-2}(s_1,s_2),\dots,r_2(s_1,s_2,\dots,s_{n-2}))$.

According to equations (2.3) of lemma (2.2) we have

$$\sum_{\xi=0}^{m}\xi\left[\sum_{s_1=0}^{m}\sum_{s_2=0}^{r_{n-1}(\tilde{0})}\cdots\sum_{s_{n-2}=0}^{r_2(\tilde{0})}\psi(j, r_{n-1}(s_1), r_{n-2}(s_1,s_2),\dots,r_2(s_1,s_2,\dots,s_{n-2}))\right] = nt_n \quad \text{.......(3.3)}$$

where t_n is the total number of *n-secants* of (k,n)-arc in PG(2,q), with $n \leq k \leq \overline{n}$.

Since $m > 0$ for $k \geq n$ we obtain the expression

$$m\left[\sum_{\xi=0}^{m}\sum_{s_1=0}^{m}\sum_{s_2=0}^{r_{n-1}(\tilde{0})}\cdots\sum_{s_{n-2}=0}^{r_2(\tilde{0})}\psi(\xi, r_{n-1}(s_1), r_{n-2}(s_1,s_2),\dots,r_2(s_1,s_2,\dots,s_{n-2}))\right] \quad \text{.......(3.4)}$$

is bigger than the expression

$$\sum_{\xi=0}^{m}\xi\left[\sum_{s_1=0}^{m}\sum_{s_2=0}^{r_{n-1}(\tilde{0})}\cdots\sum_{s_{n-2}=0}^{r_2(\tilde{0})}\psi(j, r_{n-1}(s_1), r_{n-2}(s_1,s_2),\dots,r_2(s_1,s_2,\dots,s_{n-2}))\right] \quad \text{.......(3.5)}$$

but by using the equation (2.4) of lemma (2.2) we have

$$m\left[\sum_{\xi=0}^{m}\sum_{s_1=0}^{m}\sum_{s_2=0}^{r_{n-1}(\tilde{0})}\cdots\sum_{s_{n-2}=0}^{r_2(\tilde{0})}\psi(\xi, r_{n-1}(s_1), r_{n-2}(s_1,s_2),\dots,r_2(s_1,s_2,\dots,s_{n-2}))\right] = mk \quad \text{.......(3.6)}$$

This implies $mk > nt_2$ or $t_n < mk/n$.

Furthermore, $t_n < \frac{k(k-1)}{n(n-1)}$........(3.7) since $m = \left[\frac{k-1}{n-1}\right] \leq \frac{k-1}{n-1}$.

On the other hand, if the (k,n)-arc is complete for $n \le k \le \bar{n}$, then lemma (2.3) predicated that $(q+1-n)t_n \ge \theta(2)-k$ or $t_n \ge \frac{\theta(2)-k}{q+1-n}$......(3.8)
Finally, the solution of the two inequalities (3.7) and (3.8) gives the smallest possible value of k for which a (k,n)-arc can be complete which is

$$\bar{n} = \left[\frac{\beta + \sqrt{\beta^2 + 4\alpha\gamma}}{2\alpha} \right].$$

4. Non-existence of a complete (k,n)-arcs in Small planes

In this subsection we give two examples of non-existence of a complete (k,n)-arcs in small planes and comparable the results with the theorem (3.1) in this paper .

Example 4.1.

In projective plane of order two "PG(2,2)", a smallest complete (k,2)-arc accures when $k=4$,this means there is no complete (k,2)-arc for $2 \le k \le 3$ which is exactly the same interval produced by theorem (3.1) .

Example 4.2.

The smallest value of k for which a (k,4)-arc is complete in
projective plane of order five "PG(2,5)" is $k=13$,which represented by the following set of points {(1,0,0), (0,1,0), (0,0,1), (1,0,1), (1,1,1), (1,2,4), (1,4,4), (1,4,2), (1,3,3), (1,2,0), (0,1,2), (1,0,4), (1,4,1)},so there is no complete (k,4)-arc in PG(2,5) for $4 \le k \le 12$,which is also the same interval produced by theorem (3.1) .

In general, if $t_q^*(n)$ denote the largest value of k produces according to theorem (3.1), for which a (k,n)-arc dose not complete in projective plane of order q "PG(2,q)" and $k \ge n$, then we have the following useful table:

q	2	3	4	5	7	8	9	11	13	16	17	19	23
$t_q^*(2)$	3	3	3	4	4	4	5	5	5	6	6	6	7
$t_q^*(3)$		6	6	7	8	8	8	9	10	11	11	11	12
$t_q^*(4)$			11	11	12	12	12	13	14	15	16	17	18
$t_q^*(5)$				17	16	17	17	18	19	20	21	22	24
$t_q^*(6)$					23	22	23	23	24	26	26	28	30
$t_q^*(7)$					56	30	29	29	30	32	32	33	36
$t_q^*(8)$						72	38	37	37	38	39	40	42
$t_q^*(9)$							90	46	45	45	46	47	49
$t_q^*(10)$								132	54	53	53	54	56

If $t_q(n)$ denote the exact largest value of k for which a (k,n)-arc dose not complete in projective plane of order q "PG(2,q)" ,then the following table give the exact value of $t_q(n)$ for some value of q and n , we can see the references [1], [2], [3],[5] ,[7] , [8], [9] and [10] for these results :

q	2	3	4	5	7	8	9	11	13	16	17	19	23
$t_q(2)$	3	3	5	5	5	5	5	6	7	8	9	9	9
$t_q(3)$	6	6	6	8	8	10	11	13	14				
$t_q(4)$		12	11	12									

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