A linear Programming Formulation of Assignment Problems

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Abstract

In this work, the problem of job-machine assignment was formulated as a linear programming (LP) models and then solved by the simplex method. Three case studies were involved in this study to cover all kinds of problems may be faced. To verify the results of the LP models, these problems also solved using transportation algorithm and has been found that the LP model is more efficient for solving the assignment problems.

key words: Linear programming, Integer programming

1. Introduction

Linear programming (LP) has been successfully applied to a wide range of problems, such as capital budgeting, maintenance, production scheduling and traveling salesman problems. LP has in the last decade been shown to be a flexible, efficient and commercially successful technique for scheduling, planning and allocation. These problems usually involve permutations, discretization or symmetries that may result in large and intractable LP models [1]. Yazarel and Pappas [2] present a novel method that exploits the structure of linear dynamical systems and the monotonicity of the exponential function in order to obtain safety certificates of continuous linear systems. Juan Alonso and Kevin Fall [3] present an algorithm to solve a deterministic form of a routing problem in delay tolerant networking, in which contact possibilities are known in advance. J´erˆome Galtier[4] present Semi-Definite Programming (SDP) as an extension of linear programming and the basics of the duality theory. He shows that SDP allows to express naturally a large set of flow problems in the telecommunication context with specific notions of fairness. Finally operations research has a rich history of sophisticated mathematical techniques, many of which built on linear programming for generating a global view of large, complex optimization problems [5].

2. Mathemtical LP Model for assignment problem

Some linear programming models for the assignment problem is presented .It is assumed that the cost (or time) for every machine is known denoting that:

C_{ij}=is the cost of machining job(i)on machine(j).
X_{ij}=is the element position in the job-machine matrix.

X_{ij}= {0 if the ith job is not assigned to the jth machine.}
X_{ij}= {1 if the ith job is assigned

to the jth machine.}

Now consider the situation of assigning m jobs to n machines.
A job i(=1,2,……,m)when assigned to machine j(=1,2,……,n)incurs accost C_{ij}.
The objective is to assigne the jobs to the machines (one job per machine at the least total cost.
Before the model can be solved by simplex algorithm, it is necessary to balance the problem by adding function (imaginary) jobs or machines, depending on the m<n

	M_1	M_2	M_3	M_4
J_1	32	40	29	38
J_2	31	37	40	36
J_3	43	39	41	35
J_4	42	38	39	44

or m>n without the loss of generality.
As maintained above three models can be written depending on the number of jobs and machines as follows.

2.1 Model (I) (when m=n)

For this case the LP model becomes as..

$$\text{Min . } Z=\sum_{i=1}^{m}\sum_{j=1}^{n} C_{ij}X_{ij}$$

sub to:

$$\sum_{i=1}^{n} X_{ij}=1 \quad j=1,2,\ldots\ldots,m$$

$$\sum_{j=1}^{m} X_{ij}=1 \quad i=1,2,\ldots\ldots,m$$

$$X_{ij}=0 \text{ or } 1$$

2.2 Model (II) (when m<n)

In this case the number of jobs is less than the number of machine, and to balance the problem imaginary jobs (n-m) should be added, so the LP model becomes.

$$\text{Min. } Z=\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}X_{ij}$$

sub to:

$$\sum_{i=1}^{n} X_{ij}=1 \quad j=1,2,\ldots\ldots,n$$

$$\sum_{j=1}^{n} X_{ij}=1 \quad i=1,2,\ldots\ldots,n$$

$$X_{ij}=0 \text{ or } 1$$

2.3 Model (III) (when n<m)

In this case, the number of machine is less than the number of jobs, and to balance the problem imaginary machine (m-n) should be added, so the LP model becomes.

$$\text{Min. } Z=\sum_{i=1}^{m}\sum_{j=1}^{m} C_{ij}X_{ij}$$

sub to:

$$\sum_{i=1}^{m} X_{ij}=1 \quad j=1,2,\ldots\ldots,m$$

$$\sum_{j=1}^{m} X_{ij}=1 \quad i=1,2,\ldots\ldots,m$$

$$X_{ij}=0 \text{ or } 1$$

Results and Discussions

Case (I): M=n

Consider a job-machine matrix as shown below. The element in the matrix refer to the cost of assigning job (i) to the machine (j) .

(i) Primal LP Model

$$\text{Min. } Z=32X_{11}+40X_{12}+29X_{13}+38X_{14}+$$

$$31X_{21}+37X_{22}+40X_{23}+36X_{24}+$$
$$43X_{31}+39X_{32}+41X_{33}+35X_{34}+$$
$$42X_{41}+38X_{42}+39X_{43}+44X_{44}$$

Sub. to:

$$X_{11}+X_{12}+X_{13}+X_{14}=1$$
$$X_{21}+X_{22}+X_{23}+X_{24}=1$$
$$X_{31}+X_{32}+X_{33}+X_{34}=1$$
$$X_{41}+X_{42}+X_{43}+X_{44}=1$$
$$X_{11}+X_{21}+X_{31}+X_{41}=1$$
$$X_{12}+X_{22}+X_{32}+X_{42}=1$$
$$X_{13}+X_{23}+X_{33}+X_{43}=1$$
$$X_{14}+X_{24}+X_{34}+X_{44}=1$$

With all variables nonnegative and integral

Solving the above model using MAT LAB package give the following results;

$X_{11}=0$ $X_{12}=0$ $X_{13}=1$ $X_{14}=0$
$X_{21}=1$ $X_{22}=0$ $X_{23}=0$ $X_{24}=0$
$X_{31}=0$ $X_{32}=0$ $X_{33}=0$ $X_{34}=1$
$X_{41}=0$ $X_{42}=1$ $X_{43}=0$ $X_{44}=0$

This means that optimum assignment is:

J_1 For M_3 J_2 For M_1 J_3 For M_4 and J_4 For M_2 with a minimum cost of objective function Z=133.
To verify the solution of the primal LP model, another technique called Transportation method [6] is used to solve the job-machine matrix. Applying this method leads to the final-transportation tableau.

Initial Transportation Tableau

Cost=29+31+35+38=133
Final transportation tableau

Case II m>n
(ii): Primal LP model:

Min.

$$Z=35X_{11}+37X_{12}+31X_{13}+36X_{14}+$$

Jobs \ Machines	M_1	M_2	M_3	M_4
J_1	35	37	31	36
J_2	31	30	34	38
J_3	36	31	37	29

$$31X_{21}+30X_{22}+34X_{23}+38X_{24}+$$
$$36X_{31}+31X_{32}+37X_{33}+29X_{34}+$$

Sub to:

$$X_{11}+X_{12}+X_{13}+X_{14}=1$$
$$X_{21}+X_{22}+X_{23}+X_{24}=1$$
$$X_{31}+X_{32}+X_{33}+X_{34}=1$$
$$X_{11}+X_{21}+X_{31}=1$$
$$X_{12}+X_{22}+X_{23}=1$$
$$X_{13}+X_{23}+X_{33}=1$$

	M_1	M_2	M_3	M_4	
J_1	32	40	29	38	1
J_2	31	37	40	36	1
J_3	43	39	41	35	1
J_4	42	38	39	44	1
	1	1	1	1	

$$X_{14}+X_{24}+X_{34}=1$$

With all variables nonnegative and integral

Jobs \ Machines	M_1	M_2	M_3	M_4
J_1	32	40	29 1	38
J_2	31 1	37	40	36
J_3	43	39	41	35 1
J_4	42	38 1	39	44

The final tubule of the simplex method leads to the following results:

J_1 for M_3 J_2 for M_2 J_3 for M_4 with minimum objective function cost Z=90.
Also applying the transported technique gives:

	M_1	**M_2**	**M_3**	**M_4**	
J_1	35	37	31	36	1
J_2	31	30	34	38	1
J_3	36	31	37	29	1
J_I	∞	∞	∞	∞	1
	1	1	1	1	

J_I=Imaginary job

Initial Transportation Tableau

	M_1	**M_2**	**M_3**	**M_4**
J_1	35	37	31 1	36
J_2	31	30 1	34	38
J_3	36	31	37	29 1
J_I	∞	∞	∞	∞

Cost=90

Final Transportation Tableau

As seen from the final transportation tableau the optimal solution is J1 assigned for M3, J2 assigned for M2, J3 for M4, with

	M_1	**M_2**	**M_3**	**M_I**	
J_1	35	30	33	∞	1
J_2	34	37	36	∞	1
J_3	27	30	32	∞	1
J_4	35	37	31	∞	1
	1	1	1	1	

optimal cost of 90 unit .This solution give the same result of the LP model .

Case III : m < n

(i) Primal LP Model

$$\text{Min. } Z=35X_{11}+30X_{12}+33X_{13}+34X_{21}+37X_{22}+36X_{23}+27X_{31}+30X_{32}+32X_{33}+35X_{41}+37X_{42}+31X_{43}$$

Sub. to :

$$X_{11}+X_{12}+X_{13}=1$$
$$X_{21}+X_{22}+X_{23}=1$$
$$X_{31}+X_{32}+X_{33}=1$$
$$X_{41}+X_{42}+X_{43}=1$$
$$X_{11}+X_{21}+X_{31}+X_{41}=1$$
$$X_{12}+X_{22}+X_{32}+X_{42}=1$$
$$X_{13}+X_{23}+X_{33}+X_{43}=1$$

With all variables nonnegative and integral

The optimal solution of the previous

	M_1	**M_2**	**M_3**	**M_I**
J_1	35	30 1	33	∞
J_2	34	37	36	∞
J_3	27 1	30	32	∞
J_4	35	37	31 1	∞

LP model is ;

$X_{11}=0$ $X_{12}=1$ $X_{13}=0$ $X_{21}=0$ $X_{22}=0$ $X_{23}=0$

$X_{31}=1$ $X_{32}=0$ $X_{33}=0$ $X_{41}=0$ $X_{42}=0$ $X_{43}=1$

Thus, the optimal assignment is

J_1 For M_2 J_3 For M_1 J_4 For M_3

With minimum objective function cost Z=88 Unit .

MI=Imaginary machine

Initial Transportation Tableau

Applying the transportation technique gives the same optimal solution as shown below.

Cost=88

Final Transportation Tableau

Conclusion

In this paper, it has been presented a mathematical LP models to obtain the optimal assignment of tasks to resources. The verification method was done by transportation algorithm, which is equivalent to the linear programming model. The results showed that the LP model is more efficient than the traditional technique in solving the assignment problems.

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نموذج رياضي خطي لحل مشاكل التخصيص

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الخلاصة:

تهتم الدراسة الحالية ببناء نماذج رياضية خطية لحل مشكلات تخصيص الاعمال على المكائن .تم حل النمـــاذج المقترحـــة باستعمال طريقة التبسيط *simplex method*. تم دراسة ثلاث حالات لغرض تغطية جميع حالات عملية تخصيص الاعمال على المكائن .لغرض التاكد من صحة النتائج و دقةها تم حل نفس النماذج باستعمال الطرائق التقليدية ولوحظ ان الحل باستعمال النماذج الخطية اكثر كفاءة" في حل مشكلات التخصيص.