Unconditionally Stable Fourth Order compact Finite Difference Scheme for 3D Microscale Heat Equation.

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Abstract

A fourth order compact difference scheme with a Crank-Nicolson technique is employed to discretize three dimensions unsteady state microscale heat transport equation. By introducing an intermediate function for the heat transport equation, we use the fourth order compact scheme. The general form of the solution is solved using the Gauss-Seidel method .The stability of this new scheme is proved unconditionally stable with respect to initial values. We use the test problem to compare the accuracy of this new scheme. The results show that the compact fourth order finite difference scheme is more accurate than the second order finite difference schemes.

Key words: finite difference, fourth order compact, three dimension heat transport equation, Crank-Nicolson.

اسلوب الفروقات المحددة المضغوطة من الرتبة الرابعة المستقر بدون شروط لحل معادلة التوصيل الحراري ثلاثية البعد

عقيل جاسم حرفش

قسم الرياضيات - كلية العلوم - جامعة البصرة

الخلاصة

معادلة التوصيل ا راري ثلاثية البعد حلت هذا البحث باستخدام اسلوب الفروقات ا دة ا ضغوطة من الرتبة الرابعة مع اسلوب كرانك نيكولسن. لذا فقد كنا من ا صول على دقة من الرتبة الرابعة للحيز وثنائية بالنسبة للزمن. النظام ا طي الناتج من صيغة الفروقات ا دة قمنا له باستخدام طريقة كاوس سايدل التكرارية. وباستخدام طريقة الطاقة ا تقطعة قمنا باثبات ان هذه الطريقة غ مشروطة للاستقرارية وبالنسبة لاي شروط حدودية. وقد قمنا بتطبيق هذا الاسلوب على مثال اختباري لقياس الدقة. وقد وجدنا ان دقة هذا الاسلوب عالية جدا مقارنة باسلوب ثنائي الدقة ل هذه ا عادلة. حيث اننا نستطيع رسم البيانات بسهولة وذلك للتباين الكب ب البيانات, حيث ان ا طأ الأسلوب ا ديد صغ جداً مقارنتاً بالاسلوب الثنائي الامر الذي اضطرنا ا عدم تقسيم ا ور بشكل متساوي.

Introduction

Many applications, including from phonon electron interaction model (Qiu et al. 1993), the single energy equation (Tzou 1995-b, Tzou 1995(c)), the phonon scatting model (Joseph et al. 1989), the phonon radiative transfer model (Joshi et al. 1993) and the lagging behavior model (Ozisik et al. 1994, Tzou 1995(a), Tzou 1995(b)), can be modeled by the microscale heat transport equation.

The three dimension Microscale heat transport equation for describing the thermal behavior of thin films and other microstructure can be written as (Zhang et al. 2001(c))

$$\frac{1}{\alpha}\left(\frac{\partial T}{\partial t}+T_q\frac{\partial^2 T}{\partial t^2}\right)=T_q\frac{\partial^3 T}{\partial t\,\partial x^2}+T_q\frac{\partial^3 T}{\partial t\,\partial y^2}+T_T\frac{\partial^3 T}{\partial t\,\partial z^2}+\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+S, \tag{1}$$

The initial and boundary conditions are

$$\begin{aligned} &T(x,y,z,0)=T_0(x,y,z), && \frac{\partial T(x,y,z,0)}{\partial t}=T_1(x,y,z),\\ &T(0,y,z,t)=T_2(y,z,t), && T(L_x,y,z,t)=T_5(y,z,t),\\ &T(x,0,z,t)=T_3(x,z,t), && T(x,L_y,z,t)=T_6(x,z,t),\\ &T(x,y,0,t)=T_4(x,y,t), && T(x,y,L_z,t)=T_7(x,y,t), \end{aligned} \tag{2}$$

where T is the temperature, α, T_T and T_q are positive constants. Here α is the thermal diffusivity. T_T and T_q represent the time lags of heat flux and temperature gradient, respectively. S is the source (Zhang et al. 2001(c)).

Few authors deal with numerical solution of one dimension microscale heat transport equation. By using Crank-Nicolson technique (Qiu et al. 1992) solved the phonon electron interaction model. (Joshi et al. 1993) used the explicit upstream difference method to solve the phonon radiative transfer model in one dimensional medium. (Zhang et al. 2001(a)) solve the one dimension microscale heat transport equation using fourth order compact scheme and prove this new scheme is unconditionally stable with initial value.

(Zhang et al. 2001(b)) and (Zhang et al. 2001(c)) solve the two and three dimension microscale heat transport equation, respectively, using second order approxmations for time and space.

In this paper we develop the work of (Zhang et al. 2001(c)), to solve the microscale heat transport equation in three dimensions for $T_T = T_q$. (Zhang et al. 2001(c)) drive finite difference scheme with second order accuracy for space and time. We have a fourth order accuracy for space (using compact scheme) and second order for time(using Crank-Nicolson method). In section 3, we prove this new scheme is unconditionally stable with initial value using the discrete energy method. In section 4, our results are compared with the results of (Zhang et al. 2001(c)).

Fourth order compact discretization

For convenience, let us consider a cubic domain $\Omega = [0, L_x]\times[0, L_y]\times[0, L_z]$. Here subscripts are obviously not derivatives. We discretize Ω with uniform mesh sizes Δx, Δy and Δz respectively in the x , y and z coordinate directions. Define $Nx = L_x/\Delta x$, $Ny = L_y/\Delta y$ and $Nz = L_z/\Delta z$ the numbers of uniform subintervals along the x , y and z coordinate directions, respectively. The mesh points are (x_i, y_j, z_k) where $x_i = i\,\Delta x$, $y_j = j\,\Delta y$, $z_k = k\,\Delta z$, $0 \le i \le Nx$, $0 \le j \le Ny$, $0 \le k \le Nz$. In the sequel, we may also use the index pair (i, j, k) to represent the mesh point (x_i, y_j, z_k). Also, we discretize the time interval with uniform mesh size Δt.

If $T_T = T_q$ equation (1) can be written as:

$$\frac{1}{\alpha}\left(\frac{\partial T}{\partial t} + T_q\frac{\partial^2 T}{\partial t^2}\right) = T_q\frac{\partial^3 T}{\partial t\,\partial x^2} + T_q\frac{\partial^3 T}{\partial t\,\partial y^2}T_q\frac{\partial^3 T}{\partial t\,\partial z^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + S. \tag{3}$$

suppose that the following function

$$\theta = T + T_q\frac{\partial T}{\partial t}. \tag{4}$$

Substituting (4) into (3), and after simplification we have

$$'f = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} \tag{5}$$

$$f = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} - S$$

where .

The modified initial and boundary conditions

$$\theta(x, y, 0) = T_0(x, y, z) + T_q T_1(x, y, z) \qquad (6)$$

$$\theta(0, y, z, t) = T_2(y, z, t) + T_q \frac{\partial T_2(y, z, t)}{\partial t}, \qquad \theta(L_x, y, z, t) = T_5(y, z, t) + T_q \frac{\partial T_5(y, z, t)}{\partial t},$$

$$\theta(x, 0, z, t) = T_3(x, z, t) + T_q \frac{\partial T_3(x, z, t)}{\partial t}, \qquad \theta(x, L_y, z, t) = T_6(x, z, t) + T_q \frac{\partial T_6(x, z, t)}{\partial t},$$

$$\theta(x, y, 0, t) = T_4(x, y, t) + T_q \frac{\partial T_4(x, y, t)}{\partial t}, \qquad \theta(x, y, L_z, t) = T_7(x, y, t) + T_q \frac{\partial T_7(x, y, t)}{\partial t}. \qquad (7)$$

The derivatives of Eq.(5) can be approximated by a second order accuracy as

$$\left.\begin{aligned} \left.\frac{\partial^2 \theta}{\partial x^2}\right|_{ijk} &= \delta_x^2 \theta_{ijk} - \frac{\Delta x^2}{12}\frac{\partial^4 \theta}{\partial x^4} + O(\Delta x^4), \\ \left.\frac{\partial^2 \theta}{\partial y^2}\right|_{ijk} &= \delta_y^2 \theta_{ijk} - \frac{\Delta y^2}{12}\frac{\partial^4 \theta}{\partial y^4} + O(\Delta y^4), \\ \left.\frac{\partial^2 \theta}{\partial z^2}\right|_{ijk} &= \delta_z^2 \theta_{ijk} - \frac{\Delta z^2}{12}\frac{\partial^4 \theta}{\partial z^4} + O(\Delta z^4). \end{aligned}\right\} \qquad (8)$$

By using these finite difference approximations, Eq.(5) can be discretized at a given grid point (i, j, k) as

$$\delta_x^2 \theta_{ijk} + \delta_y^2 \theta_{ijk} + \delta_z^2 \theta_{ijk} - \tau_{ijk} = f_{ijk}, \qquad (9)$$

where

$$\tau_{ijk} = \left[\frac{\Delta x^2}{12}\frac{\partial^4 \theta}{\partial x^4} + \frac{\Delta y^2}{12}\frac{\partial^4 \theta}{\partial y^4} + \frac{\Delta z^2}{12}\frac{\partial^4 \theta}{\partial z^4}\right]_{ijk} + O(\Delta x^4, \Delta y^4, \Delta z^4). \qquad (10)$$

We have include both $O(h^2)$ term in Eq.(10) because we wish to approximate all of them in order to construct an $O(h^4)$ scheme. To obtain fourth order compact approximation to the $O(h^2)$ terms in Eq.(10), we simply take the appropriate derivatives of Eq.(5),

$$\begin{aligned} \frac{\partial^4 \theta}{\partial x^4} &= \frac{\partial^2 f}{\partial x^2} - \frac{\partial^4 \theta}{\partial x^2 \partial y^2} - \frac{\partial^4 \theta}{\partial x^2 \partial z^2}, \\ \frac{\partial^4 \theta}{\partial y^4} &= \frac{\partial^2 f}{\partial y^2} - \frac{\partial^4 \theta}{\partial y^2 \partial x^2} - \frac{\partial^4 \theta}{\partial y^2 \partial z^2}, \\ \frac{\partial^4 \theta}{\partial z^4} &= \frac{\partial^2 f}{\partial z^2} - \frac{\partial^4 \theta}{\partial z^2 \partial x^2} - \frac{\partial^4 \theta}{\partial z^2 \partial y^2}. \end{aligned} \qquad (11)$$

Substituting Eqs.(11) into Eq.(10) yields

$$\tau_{ijk} = \begin{bmatrix} \frac{\Delta x^2}{12}\frac{\partial^2 f}{\partial x^2} + \frac{\Delta y^2}{12}\frac{\partial^2 f}{\partial y^2} + \frac{\Delta z^2}{12}\frac{\partial^2 f}{\partial z^2} - (\frac{\Delta x^2}{12} + \frac{\Delta y^2}{12})\frac{\partial^4 \theta}{\partial x^2 \partial y^2} \\ -(\frac{\Delta x^2}{12} + \frac{\Delta z^2}{12})\frac{\partial^4 \theta}{\partial x^2 \partial z^2} - (\frac{\Delta y^2}{12} + \frac{\Delta z^2}{12})\frac{\partial^4 \theta}{\partial y^2 \partial z^2} \end{bmatrix}_{ijk} + O(\Delta x^4, \Delta y^4, \Delta z^4). \quad (12)$$

Substituting the truncation error τ_{ijk} (12) into (9) we get the fourth order compact scheme for (5) as follows:

(13)

$$\delta_x^2\theta_{ijk} + \delta_y^2\theta_{ijk} + \delta_z^2\theta_{ijk} + \frac{1}{12}(\Delta x^2 + \Delta y^2)\delta_x^2\delta_y^2\theta_{ijk} + \frac{1}{12}(\Delta x^2 + \Delta z^2)\delta_x^2\delta_z^2\theta_{ijk}$$
$$+ \frac{1}{12}(\Delta y^2 + \Delta z^2)\delta_y^2\delta_z^2\theta_{ijk} = f_{ijk} + \frac{\Delta x^2}{12}\delta_x^2 f_{ijk} + \frac{\Delta y^2}{12}\delta_y^2 f_{ijk} + \frac{\Delta z^2}{12}\delta_z^2 f_{ijk}$$
$$+ O(\Delta x^4 + \Delta y^4 + \Delta z^4)$$

We use the fourth order compact scheme (13) with Crank-Nicolson technique in (5) by considering $f = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} - S$, we have

$$\frac{1}{2}\left(\delta_x^2(\theta_{ijk}^{n+1} + \theta_{ijk}^n) + \delta_y^2(\theta_{ijk}^{n+1} + \theta_{ijk}^n) + \delta_z^2(\theta_{ijk}^{n+1} + \theta_{ijk}^n)\right) + \frac{1}{24}(\Delta x^2 + \Delta y^2)\delta_x^2\delta_y^2(\theta_{ijk}^{n+1} + \theta_{ijk}^n)$$
$$+ \frac{1}{24}(\Delta x^2 + \Delta z^2)\delta_x^2\delta_z^2(\theta_{ijk}^{n+1} + \theta_{ijk}^n) + \frac{1}{24}(\Delta y^2 + \Delta z^2)\delta_y^2\delta_z^2(\theta_{ijk}^{n+1} + \theta_{ijk}^n) = [\frac{1}{\alpha}\frac{\partial\theta}{\partial t} - S^{n+\frac{1}{2}}$$
$$+ \frac{\Delta x^2}{12}\delta_x^2(\frac{1}{\alpha}\frac{\partial\theta}{\partial t} - S^{n+\frac{1}{2}}) + \frac{\Delta y^2}{12}\delta_y^2(\frac{1}{\alpha}\frac{\partial\theta}{\partial t} - S^{n+\frac{1}{2}}) + \frac{\Delta z^2}{12}\delta_z^2(\frac{1}{\alpha}\frac{\partial\theta}{\partial t} - S^{n+\frac{1}{2}})]_{ijk}, \quad (14)$$

with $\left.\frac{\partial\theta}{\cdot\partial t}\right|_{ijk} = \frac{\theta_{ijk}^{n+1} - \theta_{ijk}^n}{\Delta t}$

where Δt is the time step, (14) becomes

$$\frac{1}{2}\delta_x^2(\theta_{ijk}^{n+1} + \theta_{ijk}^n) + \frac{1}{2}\delta_y^2(\theta_{ijk}^{n+1} + \theta_{ijk}^n) + \frac{1}{2}\delta_z^2(\theta_{ijk}^{n+1} + \theta_{ijk}^n) + \frac{1}{24}(\Delta x^2 + \Delta y^2)\delta_x^2\delta_y^2(\theta_{ijk}^{n+1}$$
$$+ \theta_{ijk}^n) + \frac{1}{24}(\Delta x^2 + \Delta z^2)\delta_x^2\delta_z^2(\theta_{ijk}^{n+1} + \theta_{ijk}^n) + \frac{1}{24}(\Delta y^2 + \Delta z^2)\delta_y^2\delta_z^2(\theta_{ijk}^{n+1} + \theta_{ijk}^n) =$$
$$\frac{1}{\alpha}(\frac{\theta_{ijk}^{n+1} - \theta_{ijk}^n}{\Delta t}) - S_{ijk}^{n+\frac{1}{2}} + \frac{\Delta x^2}{12}\delta_x^2(\frac{1}{\alpha}(\frac{\theta_{ijk}^{n+1} - \theta_{ijk}^n}{\Delta t}) - S_{ijk}^{n+\frac{1}{2}}) + \frac{\Delta y^2}{12}\delta_y^2(\frac{1}{\alpha}(\frac{\theta_{ijk}^{n+1} - \theta_{ijk}^n}{\Delta t})$$
$$- S_{ijk}^{n+\frac{1}{2}}) + \frac{\Delta z^2}{12}\delta_z^2(\frac{1}{\alpha}(\frac{\theta_{ijk}^{n+1} - \theta_{ijk}^n}{\Delta t}) - S_{ijk}^{n+\frac{1}{2}}), \quad (15)$$

where

$$\delta_x^2\delta_y^2\theta_{ijk} = \frac{1}{(\Delta x\Delta y)^2}\begin{bmatrix} 4\theta_{ijk} - 2(\theta_{i-1jk} + \theta_{i+1jk} + \theta_{ij-1k} + \theta_{ij+1k}) \\ + \theta_{i-1j-1k} + \theta_{i+1j-1k} + \theta_{i+1j+1k} + \theta_{i-1j+1}k \end{bmatrix} + O(\Delta x^2, \Delta y^2),$$

$$\delta_x^2\delta_z^2\theta_{ijk} = \frac{1}{(\Delta x\Delta z)^2}\begin{bmatrix} 4\theta_{ijk} - 2(\theta_{i-1jk} + \theta_{i+1jk} + \theta_{ijk-1} + \theta_{ijk+1}) \\ + \theta_{i-1jk-1} + \theta_{i+1jk-1} + \theta_{i+1jk+1} + \theta_{i-1jk+1} \end{bmatrix} + O(\Delta x^2, \Delta z^2),$$

$$\delta_y^2\delta_z^2\theta_{ijk} = \frac{1}{(\Delta y\Delta z)^2}\begin{bmatrix} 4\theta_{ijk} - 2(\theta_{ijk-1} + \theta_{ijk+1} + \theta_{ij-1k} + \theta_{ij+1k}) \\ + \theta_{ij-1k-1} + \theta_{ij-1k+1} + \theta_{ij+1k+1} + \theta_{ij+1k-1} \end{bmatrix} + O(\Delta y^2, \Delta z^2).$$

Equation (15) used to evaluate θ_{ijk}^{n+1}. For computing T_{ijk}^{n+1}, we discretize (4) using Crank-Nicolson method

$$\frac{1}{2}(\theta_{ijk}^{n+1} + \theta_{ijk}^{n}) = \frac{1}{2}(T_{ijk}^{n+1} + T_{ijk}^{n}) + \frac{T_q}{\Delta t}(T_{ijk}^{n+1} - T_{ijk}^{n}). \qquad (16)$$

Simplification equation (16) to get T_{ijk}^{n+1}, we have

$$T_{ijk}^{n+1} = (\frac{\Delta t}{2} + T_q)^{-1}(T_q - \frac{\Delta t}{2})T_{ijk}^{n} + (\frac{\Delta t}{2} + T_q)^{-1}\frac{\Delta t}{2}(\theta_{ijk}^{n+1} + \theta_{ijk}^{n}) \qquad (17)$$

which can be used to evaluate T_{ijk}^{n+1}.

Stability analysis

We shall prove the unconditionally stability of the compact finite difference (12) and (14) with respect to the initial values. The technique that we will use in our proof is the discrete energy method (Dia, 1992, Less 1961). To this end, we denote D to the set of discrete values

$$\left\{e^n = \left\{e_{ijk}^n\right\}, with\ e_{0jk}^n = e_{Njk}^n = e_{i0k}^n = e_{iNk}^n = e_{ij0}^n = e_{ijN}^n = 0.\ \ 1 \le i, j, k \le N.\right\}$$

We then make the following norm definitions for any e^n, $f^n \in D$,

$$\left(e^n, f^n\right) = \Delta x^2 \sum_{i,j,k=1}^{n} e_{ijk}^n f_{ijk}^n, \qquad \left\|e^n\right\| = \left(e^n, e^n\right).$$

The following results can be verified easily [1, 4].

Lemma 1: For any e^n, $f^n \in D$, the following equalities hold.

$$\left(\delta_x^2 e^n, f^n\right) = -\left(\delta_x e^n, \delta_x f^n\right), \quad \left(\delta_y^2 e^n, f^n\right) = -\left(\delta_y e^n, \delta_y f^n\right), \quad \left(\delta_z^2 e^n, f^n\right) = -\left(\delta_z e^n, \delta_z f^n\right),$$

$$\left(\delta_x^2\delta_y^2 e^n, f^n\right) = -\left(\delta_x\delta_y e^n, \delta_x\delta_y f^n\right), \qquad \left(\delta_x^2\delta_z^2 e^n, f^n\right) = -\left(\delta_x\delta_z e^n, \delta_x\delta_z f^n\right),$$

$$\left(\delta_y^2\delta_z^2 e^n, f^n\right) = -\left(\delta_y\delta_z e^n, \delta_y\delta_z f^n\right),$$

where

$$\delta_x e^n = \frac{e^n_{i+1\,jk} - e^n_{ijk}}{\Delta x}, \qquad \delta_y e^n = \frac{e^n_{ij+1\,k} - e^n_{ijk}}{\Delta y}, \qquad \delta_z e^n = \frac{e^n_{ijk+1} - e^n_{ijk}}{\Delta z}.$$

are the forward difference operators.

Theorem: Suppose that $\{T^n_{ijk}, \theta^n_{ijk}\}$ and $\{V^n_{ijk}, \xi^n_{ijk}\}$ are the solution of the finite difference scheme (15) and (16) which satisfy the initial and boundary conditions (5) and (6), and have different initial values $\{T^0_{ijk}, \theta^0_{ijk}\}$ and $\{V^0_{ijk}, \xi^0_{ijk}\}$, respectively. Let, $w^n_{ijk} = \theta^n_{ijk} - \xi^n_{ijk}$ and $\varepsilon^n_{ijk} = T^n_{ijk} - V^n_{ijk}$ then $\{w^n_{ijk}, \varepsilon^n_{ijk}\}$ satisfy

$$\begin{aligned}
&\frac{1}{\alpha}\left[\|w^n\|^2 + \frac{\Delta x^2}{12}\|\delta_x w^n\|^2 + \frac{\Delta y^2}{12}\|\delta_y w^n\|^2 + \frac{\Delta z^2}{12}\|\delta_z w^n\|^2\right] + 2T_q(\|\delta_x \varepsilon^n\|^2 + \|\delta_y \varepsilon^n\|^2 + \\
&\|\delta_z \varepsilon^n\|^2) + \frac{T_q}{6}\left(\Delta x^2 + \Delta y^2\right)\|\delta_x\delta_y \varepsilon^n\|^2 + \frac{T_q}{6}\left(\Delta x^2 + \Delta z^2\right)\|\delta_x\delta_z \varepsilon^n\|^2 + \frac{T_q}{6}\left(\Delta y^2 + \Delta z^2\right)\|\delta_y\delta_z \varepsilon^n\|^2 \\
&\le \frac{1}{\alpha}\left[\|w^0\|^2 + \frac{\Delta x^2}{12}\|\delta_x w^0\|^2 + \frac{\Delta y^2}{12}\|\delta_y w^0\|^2 + \frac{\Delta z^2}{12}\|\delta_z w^0\|^2\right] + 2T_q(\|\delta_x \varepsilon^0\|^2 + \|\delta_y \varepsilon^0\|^2 + \\
&\|\delta_z \varepsilon^0\|^2) + \frac{T_q}{6}\left(\Delta x^2 + \Delta y^2\right)\|\delta_x\delta_y \varepsilon^0\|^2 + \frac{T_q}{6}\left(\Delta x^2 + \Delta z^2\right)\|\delta_x\delta_z \varepsilon^0\|^2 + \frac{T_q}{6}\left(\Delta y^2 + \Delta z^2\right)\|\delta_y\delta_z \varepsilon^0\|^2 .
\end{aligned} \tag{18}$$

for any $0 \le n\,\Delta t \le t_{stop}$. This implies that the finite difference scheme is unconditionally stable with respect to the initial values.

Proof: Firstly, we substitute (16) into (15) we get,

$$\begin{aligned}
&\frac{1}{\alpha\Delta t}\left[\theta^{n+1}_{ijk} - \theta^n_{ijk} + \frac{\Delta x^2}{12}\delta_x^2(\theta^{n+1}_{ijk} - \theta^n_{ijk}) + \frac{\Delta y^2}{12}\delta_y^2(\theta^{n+1}_{ijk} - \theta^n_{ijk}) + \frac{\Delta z^2}{12}\delta_z^2(\theta^{n+1}_{ijk} - \theta^n_{ijk})\right] \\
&= \frac{1}{2}\delta_x^2(T^{n+1}_{ijk} + T^n_{ijk}) + \frac{T_q}{\Delta t}\delta_x^2(T^{n+1}_{ijk} - T^n_{ijk}) + \frac{1}{2}\delta_y^2(T^{n+1}_{ijk} + T^n_{ijk}) + \frac{T_q}{\Delta t}\delta_y^2(T^{n+1}_{ijk} - T^n_{ijk}) \\
&+ \frac{1}{2}\delta_z^2(T^{n+1}_{ijk} + T^n_{ijk}) + \frac{T_q}{\Delta t}\delta_z^2(T^{n+1}_{ijk} - T^n_{ijk}) \\
&+ \frac{1}{24}(\Delta x^2 + \Delta y^2)\delta_x^2\delta_y^2(T^{n+1}_{ijk} + T^n_{ijk}) + \frac{T_q}{12\Delta t}(\Delta x^2 + \Delta y^2)\delta_x^2\delta_y^2(T^{n+1}_{ijk} - T^n_{ijk}) \\
&+ \frac{1}{24}(\Delta x^2 + \Delta z^2)\delta_x^2\delta_z^2(T^{n+1}_{ijk} + T^n_{ijk}) + \frac{T_q}{12\Delta t}(\Delta x^2 + \Delta z^2)\delta_x^2\delta_z^2(T^{n+1}_{ijk} - T^n_{ijk}) \\
&+ \frac{1}{24}(\Delta y^2 + \Delta z^2)\delta_y^2\delta_z^2(T^{n+1}_{ijk} + T^n_{ijk}) + \frac{T_q}{12\Delta t}(\Delta y^2 + \Delta z^2)\delta_y^2\delta_z^2(T^{n+1}_{ijk} - T^n_{ijk}).
\end{aligned} \tag{19}$$

Since $\{T_{ijk}^{n}, \theta_{ijk}^{n}\}$ and $\{V_{ijk}^{n}, \xi_{ijk}^{n}\}$ are both the solutions of (19) with the same boundary conditions, so $\{w_{ijk}^{n}, \varepsilon_{ijk}^{n}\} \in D$, and they also satisfy

$$\frac{1}{\alpha \Delta t}\left[w_{ijk}^{n+1} - w_{ijk}^{n} + \frac{\Delta x^2}{12}\delta_x^2 (w_{ijk}^{n+1} - w_{ijk}^{n}) + \frac{\Delta y^2}{12}\delta_y^2 (w_{ijk}^{n+1} - w_{ijk}^{n}) + \frac{\Delta y^2}{12}\delta_z^2 (w_{ijk}^{n+1} - w_{ijk}^{n}) \right]$$

$$= \frac{1}{2}\left(\delta_x^2\left(\varepsilon_{ijk}^{n+1} + \varepsilon_{ijk}^{n}\right)\right) + \frac{1}{2}\left(\delta_y^2\left(\varepsilon_{ijk}^{n+1} + \varepsilon_{ijk}^{n}\right)\right) + \frac{1}{2}\left(\delta_z^2\left(\varepsilon_{ijk}^{n+1} + \varepsilon_{ijk}^{n}\right)\right) + \frac{T_q}{\Delta t}\left(\delta_x^2\left(\varepsilon_{ijk}^{n+1} - \varepsilon_{ijk}^{n}\right)\right) +$$

$$\frac{T_q}{\Delta t}\left(\delta_y^2\left(\varepsilon_{ijk}^{n+1} - \varepsilon_{ijk}^{n}\right)\right) + \frac{T_q}{\Delta t}\left(\delta_z^2\left(\varepsilon_{ijk}^{n+1} - \varepsilon_{ijk}^{n}\right)\right) + \frac{1}{24}\left(\Delta x^2 + \Delta y^2\right)\delta_x^2\delta_y^2\left(\varepsilon_{ijk}^{n+1} + \varepsilon_{ijk}^{n}\right) +$$

$$\frac{T_q}{12\Delta t}\left(\Delta x^2 + \Delta y^2\right)\delta_x^2\delta_y^2\left(\varepsilon_{ijk}^{n+1} - \varepsilon_{ijk}^{n}\right) + \frac{1}{24}\left(\Delta x^2 + \Delta z^2\right)\delta_x^2\delta_z^2\left(\varepsilon_{ijk}^{n+1} + \varepsilon_{ijk}^{n}\right) +$$

$$\frac{T_q}{12\Delta t}\left(\Delta x^2 + \Delta z^2\right)\delta_x^2\delta_z^2\left(\varepsilon_{ijk}^{n+1} - \varepsilon_{ijk}^{n}\right) + \frac{1}{24}\left(\Delta y^2 + \Delta z^2\right)\delta_y^2\delta_z^2\left(\varepsilon_{ijk}^{n+1} + \varepsilon_{ijk}^{n}\right) +$$

$$\frac{T_q}{12\Delta t}\left(\Delta y^2 + \Delta z^2\right)\delta_y^2\delta_z^2\left(\varepsilon_{ijk}^{n+1} - \varepsilon_{ijk}^{n}\right). \tag{20}$$

From (16), we can see that

$$w_{ijk}^{n+1} + w_{ijk}^{n} = \left(\varepsilon_{ijk}^{n+1} + \varepsilon_{ijk}^{n}\right) + \frac{2T_q}{\Delta t}\left(\varepsilon_{ijk}^{n+1} - \varepsilon_{ijk}^{n}\right). \tag{21}$$

Using (21) and Lemma 1, we can obtain the following equalities:

$$\left(\delta_x\left(\varepsilon^{n+1} + \varepsilon^{n}\right), \delta_x\left(w^{n+1} + w^{n}\right)\right) = \left\|\delta_x\left(\varepsilon^{n+1} + \varepsilon^{n}\right)\right\|^2 + \frac{2T_q}{\Delta t}\left(\left\|\delta_x\varepsilon^{n+1}\right\|^2 - \left\|\delta_x\varepsilon^{n}\right\|^2\right),$$

$$\left(\delta_x\left(\varepsilon^{n+1} - \varepsilon^{n}\right), \delta_x\left(w^{n+1} + w^{n}\right)\right) = \left\|\delta_x\varepsilon^{n+1}\right\|^2 - \left\|\delta_x\varepsilon^{n}\right\|^2 + \frac{2T_q}{\Delta t}\left(\left\|\delta_x\left(\varepsilon^{n+1} - \varepsilon^{n}\right)\right\|^2\right),$$

$$\left(\delta_y\left(\varepsilon^{n+1} + \varepsilon^{n}\right), \delta_y\left(w^{n+1} + w^{n}\right)\right) = \left\|\delta_y\left(\varepsilon^{n+1} + \varepsilon^{n}\right)\right\|^2 + \frac{2T_q}{\Delta t}\left(\left\|\delta_y\varepsilon^{n+1}\right\|^2 - \left\|\delta_y\varepsilon^{n}\right\|^2\right),$$

$$\left(\delta_y\left(\varepsilon^{n+1} - \varepsilon^{n}\right), \delta_y\left(w^{n+1} + w^{n}\right)\right) = \left\|\delta_y\varepsilon^{n+1}\right\|^2 - \left\|\delta_y\varepsilon^{n}\right\|^2 + \frac{2T_q}{\Delta t}\left(\left\|\delta_y\left(\varepsilon^{n+1} - \varepsilon^{n}\right)\right\|^2\right),$$

$$\left(\delta_z\left(\varepsilon^{n+1} + \varepsilon^{n}\right), \delta_z\left(w^{n+1} + w^{n}\right)\right) = \left\|\delta_z\left(\varepsilon^{n+1} + \varepsilon^{n}\right)\right\|^2 + \frac{2T_q}{\Delta t}\left(\left\|\delta_z\varepsilon^{n+1}\right\|^2 - \left\|\delta_z\varepsilon^{n}\right\|^2\right),$$

$$\left(\delta_z\left(\varepsilon^{n+1} - \varepsilon^{n}\right), \delta_z\left(w^{n+1} + w^{n}\right)\right) = \left\|\delta_z\varepsilon^{n+1}\right\|^2 - \left\|\delta_z\varepsilon^{n}\right\|^2 + \frac{2T_q}{\Delta t}\left(\left\|\delta_z\left(\varepsilon^{n+1} - \varepsilon^{n}\right)\right\|^2\right),$$

$$\left(\delta_x\delta_y\left(\varepsilon^{n+1} + \varepsilon^{n}\right), \delta_x\delta_y\left(w^{n+1} + w^{n}\right)\right) = \left\|\delta_x\delta_y\left(\varepsilon^{n+1} + \varepsilon^{n}\right)\right\|^2 + \frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2 - \left\|\delta_x\delta_y\varepsilon^{n}\right\|^2\right),$$

$$\left(\delta_x\delta_y\left(\varepsilon^{n+1} - \varepsilon^{n}\right), \delta_x\delta_y\left(w^{n+1} + w^{n}\right)\right) = \left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2 - \left\|\delta_x\delta_y\varepsilon^{n}\right\|^2 + \frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_y\left(\varepsilon^{n+1} - \varepsilon^{n}\right)\right\|^2\right),$$

$$\left(\delta_x\delta_z\left(\varepsilon^{n+1}+\varepsilon^n\right),\ \delta_x\delta_z\left(w^{n+1}+w^n\right)\right)=\left\|\delta_x\delta_z\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_z\varepsilon^n\right\|^2\right),$$

$$\left(\delta_x\delta_z\left(\varepsilon^{n+1}-\varepsilon^n\right),\ \delta_x\delta_z\left(w^{n+1}+w^n\right)\right)=\left\|\delta_x\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_z\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_z\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right),$$

$$\left(\delta_y\delta_z\left(\varepsilon^{n+1}+\varepsilon^n\right),\ \delta_y\delta_z\left(w^{n+1}+w^n\right)\right)=\left\|\delta_y\delta_z\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_y\delta_z\varepsilon^n\right\|^2\right),$$

$$\left(\delta_y\delta_z\left(\varepsilon^{n+1}-\varepsilon^n\right),\ \delta_y\delta_z\left(w^{n+1}+w^n\right)\right)=\left\|\delta_y\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_y\delta_z\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\delta_z\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right).$$

(22)

By multiplying both sides of (20) by $\left(w_{ij}^{n+1}+w_{ij}^n\right)\Delta x\Delta y\Delta z$ and sum over i, j and k we can get

$$\frac{1}{\alpha\,\Delta t}\left[\left(\left\|w^{n+1}\right\|^2-\left\|w^n\right\|^2\right)+\frac{\Delta x^2}{12}\left(\left\|\delta_x w^{n+1}\right\|^2-\left\|\delta_x w^n\right\|^2\right)+\frac{\Delta y^2}{12}\left(\left\|\delta_y w^{n+1}\right\|^2-\left\|\delta_y w^n\right\|^2\right)\right.$$
$$\left.+\frac{\Delta z^2}{12}\left(\left\|\delta_z w^{n+1}\right\|^2-\left\|\delta_z w^n\right\|^2\right)\right]=\frac{1}{2}\left(\delta_x^2\left(\varepsilon^{n+1}+\varepsilon^n\right),\ w^{n+1}+w^n\right)$$
$$+\frac{1}{2}\left(\delta_y^2\left(\varepsilon^{n+1}+\varepsilon^n\right),\ w^{n+1}+w^n\right)+\frac{1}{2}\left(\delta_z^2\left(\varepsilon^{n+1}+\varepsilon^n\right),\ w^{n+1}+w^n\right)$$
$$+\frac{T_q}{\Delta t}\left(\delta_x^2\left(\varepsilon^{n+1}-\varepsilon^n\right),\ w^{n+1}+w^n\right)+\frac{T_q}{\Delta t}\left(\delta_y^2\left(\varepsilon^{n+1}-\varepsilon^n\right),\ w^{n+1}+w^n\right)$$
$$+\frac{T_q}{\Delta t}\left(\delta_z^2\left(\varepsilon^{n+1}-\varepsilon^n\right),\ w^{n+1}+w^n\right)$$
$$+\left(\Delta x^2+\Delta y^2\right)\left[\frac{1}{24}\left(\delta_x^2\delta_y^2\left(\varepsilon^{n+1}+\varepsilon^n\right),w^{n+1}+w^n\right)+\frac{T_q}{12\Delta t}\left(\delta_x^2\delta_y^2\left(\varepsilon^{n+1}-\varepsilon^n\right),w^{n+1}+w^n\right)\right]$$
$$+\left(\Delta x^2+\Delta z^2\right)\left[\frac{1}{24}\left(\delta_x^2\delta_z^2\left(\varepsilon^{n+1}+\varepsilon^n\right),w^{n+1}+w^n\right)+\frac{T_q}{12\Delta t}\left(\delta_x^2\delta_z^2\left(\varepsilon^{n+1}-\varepsilon^n\right),w^{n+1}+w^n\right)\right]$$
$$+\left(\Delta y^2+\Delta z^2\right)\left[\frac{1}{24}\left(\delta_y^2\delta_z^2\left(\varepsilon^{n+1}+\varepsilon^n\right),w^{n+1}+w^n\right)+\frac{T_q}{12\Delta t}\left(\delta_y^2\delta_z^2\left(\varepsilon^{n+1}-\varepsilon^n\right),w^{n+1}+w^n\right)\right].$$

(23)

We can find each term on the right-hand side of (23). Using lemma1 and (22), we can get

$$\frac{1}{2}\left(\delta_x^2\left(\varepsilon^{n+1}+\varepsilon^n\right),\ w^{n+1}+w^n\right)=-\frac{1}{2}\left(\left\|\delta_x\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\varepsilon^{n+1}\right\|^2-\left\|\delta_x\varepsilon^n\right\|^2\right)\right),$$

$$\frac{T_q}{\Delta t}\left(\delta_x^2\left(\varepsilon^{n+1}-\varepsilon^n\right),\ w^{n+1}+w^n\right)=-\frac{T_q}{\Delta t}\left(\left\|\delta_x\varepsilon^{n+1}\right\|^2-\left\|\delta_x\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right),$$

$$\frac{1}{2}\left(\delta_y^2\left(\varepsilon^{n+1}+\varepsilon^n\right),\ w^{n+1}+w^n\right)=-\frac{1}{2}\left(\left\|\delta_y\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_y\varepsilon^n\right\|^2\right)\right),$$

$$\frac{T_q}{\Delta t}\left(\delta_y^2\left(\varepsilon^{n+1}-\varepsilon^n\right),\ w^{n+1}+w^n\right)=-\frac{T_q}{\Delta t}\left(\left\|\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_y\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right),$$

$$\frac{1}{2}\left(\delta_z^2\left(\varepsilon^{n+1}+\varepsilon^n\right),\ w^{n+1}+w^n\right)=-\frac{1}{2}\left(\left\|\delta_z\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_z\varepsilon^n\right\|^2\right)\right),$$

$$\frac{T_q}{\Delta t}\left(\delta_z^2\left(\varepsilon^{n+1}-\varepsilon^n\right),\ w^{n+1}+w^n\right)=-\frac{T_q}{\Delta t}\left(\left\|\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_z\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_z\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right),$$

$$\frac{1}{24}\left(\Delta x^2+\Delta y^2\right)\left(\delta_x^2\delta_y^2\left(\varepsilon^{n+1}+\varepsilon^n\right),w^{n+1}+w^n\right)$$
$$=-\frac{1}{24}\left(\Delta x^2+\Delta y^2\right)\left[\left\|\delta_x\delta_y\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_y\varepsilon^n\right\|^2\right)\right],$$

$$\frac{T_q}{12\Delta t}\left(\Delta x^2+\Delta y^2\right)\left(\delta_x^2\delta_y^2\left(\varepsilon^{n+1}-\varepsilon^n\right),w^{n+1}+w^n\right)$$
$$=-\frac{T_q}{12\Delta t}\left(\Delta x^2+\Delta y^2\right)\left[\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_y\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_y\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right],$$

$$\frac{1}{24}\left(\Delta y^2+\Delta z^2\right)\left(\delta_y^2\delta_z^2\left(\varepsilon^{n+1}+\varepsilon^n\right),w^{n+1}+w^n\right)$$
$$=-\frac{1}{24}\left(\Delta y^2+\Delta z^2\right)\left[\left\|\delta_y\delta_z\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_y\delta_z\varepsilon^n\right\|^2\right)\right],$$

$$\frac{T_q}{12\Delta t}\left(\Delta y^2+\Delta z^2\right)\left(\delta_y^2\delta_z^2\left(\varepsilon_{ij}^{n+1}-\varepsilon_{ij}^n\right),w^{n+1}+w^n\right)$$
$$=-\frac{T_q}{12\Delta t}\left(\Delta y^2+\Delta z^2\right)\left[\left\|\delta_y\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_y\delta_z\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\delta_z\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right].$$

$$\frac{1}{24}\left(\Delta x^2+\Delta z^2\right)\left(\delta_x^2\delta_z^2\left(\varepsilon^{n+1}+\varepsilon^n\right),w^{n+1}+w^n\right)$$
$$=-\frac{1}{24}\left(\Delta x^2+\Delta z^2\right)\left[\left\|\delta_x\delta_z\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_z\varepsilon^n\right\|^2\right)\right],$$

$$\frac{T_q}{12\Delta t}\left(\Delta x^2+\Delta z^2\right)\left(\delta_x^2\delta_z^2\left(\varepsilon^{n+1}-\varepsilon^n\right),w^{n+1}+w^n\right) \tag{24}$$
$$=-\frac{T_q}{12\Delta t}\left(\Delta x^2+\Delta z^2\right)\left[\left\|\delta_x\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_z\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_z\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right],$$

Substituting (24) into (23), yield

$$
\begin{aligned}
&\frac{1}{\alpha\,\Delta t}\Bigg[\left(\|w^{n+1}\|^2-\|w^n\|^2\right)+\frac{\Delta x^2}{12}\left(\|\delta_x w^{n+1}\|^2-\|\delta_x w^n\|^2\right)+\frac{\Delta y^2}{12}\left(\|\delta_y w^{n+1}\|^2-\|\delta_y w^n\|^2\right)\\
&+\frac{\Delta z^2}{12}\left(\|\delta_z w^{n+1}\|^2-\|\delta_z w^n\|^2\right)\Bigg]=-\frac{1}{2}\left(\|\delta_x(\varepsilon^{n+1}+\varepsilon^n)\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_x\varepsilon^{n+1}\|^2-\|\delta_x\varepsilon^n\|^2\right)\right)\\
&\qquad-\frac{1}{2}\left(\|\delta_y(\varepsilon^{n+1}+\varepsilon^n)\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_y\varepsilon^{n+1}\|^2-\|\delta_y\varepsilon^n\|^2\right)\right)\\
&\qquad-\frac{1}{2}\left(\|\delta_z(\varepsilon^{n+1}+\varepsilon^n)\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_z\varepsilon^{n+1}\|^2-\|\delta_z\varepsilon^n\|^2\right)\right)\\
&\qquad-\frac{T_q}{\Delta t}\left(\|\delta_x\varepsilon^{n+1}\|^2-\|\delta_x\varepsilon^n\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_x(\varepsilon^{n+1}-\varepsilon^n)\|^2\right)\right)\\
&\qquad-\frac{T_q}{\Delta t}\left(\|\delta_y\varepsilon^{n+1}\|^2-\|\delta_y\varepsilon^n\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_y(\varepsilon^{n+1}-\varepsilon^n)\|^2\right)\right)\\
&\qquad-\frac{T_q}{\Delta t}\left(\|\delta_z\varepsilon^{n+1}\|^2-\|\delta_z\varepsilon^n\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_z(\varepsilon^{n+1}-\varepsilon^n)\|^2\right)\right)\\
&\qquad-\frac{1}{24}\left(\Delta x^2+\Delta y^2\right)\left[\|\delta_x\delta_y(\varepsilon^{n+1}+\varepsilon^n)\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_x\delta_y\varepsilon^{n+1}\|^2-\|\delta_x\delta_y\varepsilon^n\|^2\right)\right]\\
&\qquad-\frac{T_q}{12\Delta t}\left(\Delta x^2+\Delta y^2\right)\left[\|\delta_x\delta_y\varepsilon^{n+1}\|^2-\|\delta_x\delta_y\varepsilon^n\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_x\delta_y(\varepsilon^{n+1}-\varepsilon^n)\|^2\right)\right]\\
&\qquad-\frac{1}{24}\left(\Delta x^2+\Delta z^2\right)\left[\|\delta_x\delta_z(\varepsilon^{n+1}+\varepsilon^n)\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_x\delta_z\varepsilon^{n+1}\|^2-\|\delta_x\delta_z\varepsilon^n\|^2\right)\right]\\
&\qquad-\frac{T_q}{12\Delta t}\left(\Delta x^2+\Delta z^2\right)\left[\|\delta_x\delta_z\varepsilon^{n+1}\|^2-\|\delta_x\delta_z\varepsilon^n\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_x\delta_z(\varepsilon^{n+1}-\varepsilon^n)\|^2\right)\right]\\
&\qquad-\frac{1}{24}\left(\Delta y^2+\Delta z^2\right)\left[\|\delta_y\delta_z(\varepsilon^{n+1}+\varepsilon^n)\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_y\delta_z\varepsilon^{n+1}\|^2-\|\delta_y\delta_z\varepsilon^n\|^2\right)\right]\\
&\qquad-\frac{T_q}{12\Delta t}\left(\Delta y^2+\Delta z^2\right)\left[\|\delta_y\delta_z\varepsilon^{n+1}\|^2-\|\delta_y\delta_z\varepsilon^n\|^2+\frac{2T_q}{\Delta t}\left(\|\delta_y\delta_z(\varepsilon^{n+1}-\varepsilon^n)\|^2\right)\right].
\end{aligned}
\tag{25}
$$

after dropping the twelve negative terms from the right-hand side of (25) we have

$$\frac{1}{\alpha\Delta t}\left[\left(\left\|w^{n+1}\right\|^2-\left\|w^n\right\|^2\right)+\frac{\Delta x^2}{12}\left(\left\|\delta_x w^{n+1}\right\|^2-\left\|\delta_x w^n\right\|^2\right)+\frac{\Delta y^2}{12}\left(\left\|\delta_y w^{n+1}\right\|^2-\left\|\delta_y w^n\right\|^2\right)\right.$$
$$\left.+\frac{\Delta z^2}{12}\left(\left\|\delta_z w^{n+1}\right\|^2-\left\|\delta_z w^n\right\|^2\right)\right]\le-\frac{2T_q}{\Delta t}\left[\left(\left\|\delta_x\varepsilon^{n+1}\right\|^2-\left\|\delta_x\varepsilon^n\right\|^2\right)-\left(\left\|\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_y\varepsilon^n\right\|^2\right)\right.$$
$$\left.-\left(\left\|\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_z\varepsilon^n\right\|^2\right)\right]-\frac{T_q}{6\Delta t}\left[\left(\Delta x^2+\Delta y^2\right)\left(\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_y\varepsilon^n\right\|^2\right)\right.$$
$$\left.-\left(\Delta x^2+\Delta z^2\right)\left(\left\|\delta_x\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_z\varepsilon^n\right\|^2\right)-\left(\Delta y^2+\Delta z^2\right)\left(\left\|\delta_y\delta_z\varepsilon^{n+1}\right\|^2-\left\|\delta_y\delta_z\varepsilon^n\right\|^2\right)\right].$$

This implies the following

$$\frac{1}{\alpha}\left[\left\|w^{n+1}\right\|^2+\frac{\Delta x^2}{12}\left\|\delta_x w^{n+1}\right\|^2+\frac{\Delta y^2}{12}\left\|\delta_y w^{n+1}\right\|^2+\frac{\Delta z^2}{12}\left\|\delta_z w^{n+1}\right\|^2\right]+2T_q\left\|\delta_x\varepsilon^{n+1}\right\|^2$$
$$+2T_q\left\|\delta_y\varepsilon^{n+1}\right\|^2+2T_q\left\|\delta_z\varepsilon^{n+1}\right\|^2+\frac{T_q}{6}\left(\Delta x^2+\Delta y^2\right)\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2+\frac{T_q}{6}\left(\Delta x^2+\Delta z^2\right)\left\|\delta_x\delta_z\varepsilon^{n+1}\right\|^2$$
$$+\frac{T_q}{6}\left(\Delta y^2+\Delta z^2\right)\left\|\delta_y\delta_z\varepsilon^{n+1}\right\|^2\le\frac{1}{\alpha}\left[\left\|w^n\right\|^2+\frac{\Delta x^2}{12}\left\|\delta_x w^n\right\|^2+\frac{\Delta y^2}{12}\left\|\delta_y w^n\right\|^2+\frac{\Delta z^2}{12}\left\|\delta_z w^n\right\|^2\right]$$
$$+2T_q\left\|\delta_x\varepsilon^n\right\|^2+2T_q\left\|\delta_y\varepsilon^n\right\|^2+2T_q\left\|\delta_z\varepsilon^n\right\|^2+\frac{T_q}{6}\left(\Delta x^2+\Delta y^2\right)\left\|\delta_x\delta_y\varepsilon^n\right\|^2+$$
$$\frac{T_q}{6}\left(\Delta x^2+\Delta z^2\right)\left\|\delta_x\delta_z\varepsilon^n\right\|^2+\frac{T_q}{6}\left(\Delta y^2+\Delta z^2\right)\left\|\delta_y\delta_z\varepsilon^n\right\|^2. \qquad (26)$$

(18) follows from (26) by recursion with respect to n.

Numerical Results

Numerical experiments are conducted to validate the proposed discretization scheme. A model problem is constructed by setting $T_q=1,\ \alpha=1/3,\ S=0.0$. The boundary and initial conditions are set to satisfy the exact solution as

$T(x,y,t)=e^{x+y+z+t},\ 0\le t\le 1,\quad 0\le x,y,z\le 1$. The absolute error evaluated by using the following equation

$$Error=\frac{\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1}\sum_{k=1}^{N_z-1}\left|T_{eijk}-T_{ijk}\right|}{(N_x-1)(N_y-1)(N_z-1)}$$

where T_{ijk} represents the approximate value and T_{eijk} represents the exact value. The errors of the fourth order and the second schemes are compared in Fig. 1 and Fig. 2 for $0.001 \le \Delta t \le 0.02$ and two choices $\Delta x = 0.1$ and $\Delta x = 0.05$. The errors of the fourth order scheme are shown to be smaller than those of the second order scheme in both cases.

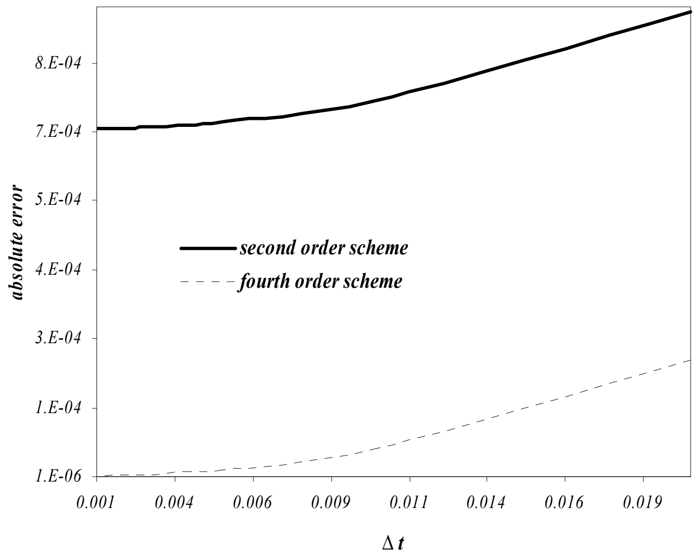

Fig. 1: Absolute error comparison of second and fourth order schemes at $T_q = 1$, $\alpha = 1/3$, $S = 0.0$, $\Delta x = \Delta y = \Delta z = 0.1$ and t =1.0.

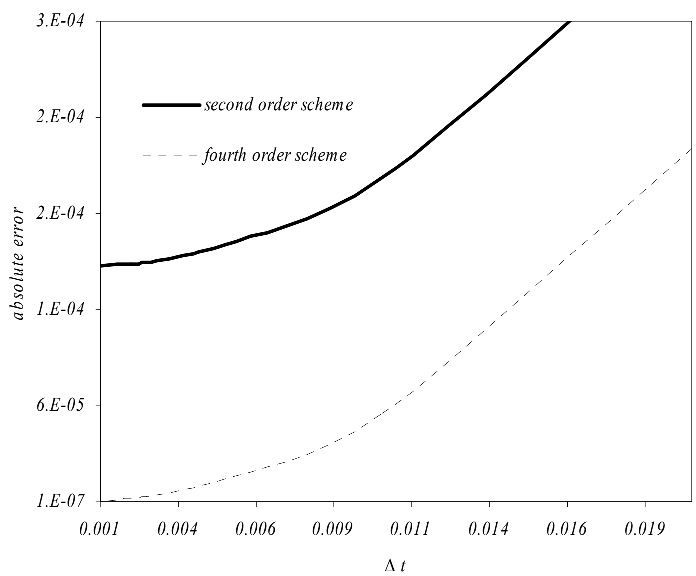

Fig. 1: Absolute error comparison of second and fourth order schemes at and t =1.0. $\Delta x = \Delta y = \Delta z = 0.05$, $S = 0.0$ $T_q = 1$, $\alpha = 1/3$

Note that the truncation error is of order $O(\Delta t^2, \Delta x^4, \Delta y^4, \Delta z^4)$ for the fourth order scheme and of order $O(\Delta t^2, \Delta x^2, \Delta y^2, \Delta z^2)$ for the second order scheme. Thus, if Δt is large and the temporal error component dominates, the difference in error magnitude between the fourth order scheme and the second order scheme will decrease.

The results of errors for T_{ij}^n at $t = 1.0$, computed for $\Delta x = \Delta y = \Delta z = 0.05$ and two choices $\Delta t = 0.002$ and $\Delta t = 0.001$ using the both the fourth order and second order finite difference schemes are listed in Table 1. Also, we note that the errors of the fourth order scheme are to be smaller than those of the second order scheme in both cases.

Table1: The absolute error computed by second order and fourth order scheme with $\Delta x = \Delta y = \Delta z = 0.05$.

		Δt=0.002		Δt=0.001	
x	y	second order	fourth order	second order	fourth order
0.05	0.05	4.02E-06	4.62E-07	3.67E-06	1.16E-07
0.1	0.1	1.69E-05	5.60E-07	1.65E-05	1.44E-07
0.15	0.15	3.97E-05	6.89E-07	3.92E-05	1.81E-07
0.2	0.2	7.17E-05	8.49E-07	7.11E-05	2.28E-07
0.25	0.25	1.12E-04	1.04E-06	1.11E-04	2.84E-07
0.3	0.3	1.57E-04	1.26E-06	1.56E-04	3.49E-07
0.35	0.35	2.05E-04	1.51E-06	2.04E-04	4.21E-07
0.4	0.4	2.53E-04	1.78E-06	2.52E-04	5.00E-07
0.45	0.45	2.97E-04	2.07E-06	2.96E-04	5.83E-07
0.5	0.5	3.34E-04	2.39E-06	3.32E-04	6.69E-07
0.55	0.55	3.61E-04	2.72E-06	3.59E-04	7.59E-07
0.6	0.6	3.75E-04	3.08E-06	3.72E-04	8.51E-07
0.65	0.65	3.73E-04	3.46E-06	3.70E-04	9.45E-07
0.7	0.7	3.54E-04	3.86E-06	3.51E-04	1.04E-06
0.75	0.75	3.16E-04	4.31E-06	3.13E-04	1.15E-06
0.8	0.8	2.61E-04	4.81E-06	2.58E-04	1.26E-06
0.85	0.85	1.91E-04	5.37E-06	1.87E-04	1.38E-06
0.9	0.9	1.13E-04	6.04E-06	1.09E-04	1.53E-06
0.95	0.95	4.14E-05	6.84E-06	3.63E-05	1.72E-06

Conclusions

We devised some numerical techniques for solving a three dimensional governing microscale heat transport equation. We proposed a fourth order accurate finite difference scheme to discretize the governing equation.. The system resulting from this scheme is solved by using Gauss-Seidel iterative method. The finite difference scheme has been proved to be unconditionally stable with respect to the initial values. Our Numerical results showed that the fourth order compact scheme is computationally more efficient and more accurate than the second order scheme.

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