Journal of Basrah Researches ((Sciences)) Vol. 37, No. 1. 15 February ((2011)) 32-38

Analysis of variance and covariance Repeated Measurement Model

ISSN 1817 – 2695

Huda Zeki Naji , Renna Dawood and Rana Rony Faik

Dept.of Mathematics , College of Science ,University of Basrah
Basrah –Iraq
((Received 14/11/2010, Accepted 16/1/2011))

Abstract

In this article we consider one-way Multivariate repeated measurements analysis of covariance model (MRM ANCOVA), which contains one between-units factor (Group with q levels) ,one within-units factor (Time with p levels) and one covariate (Z_1) . For this model the covariate is time-independent, that is measured only once . The test statistics of various hypotheses on between-unites factors, within-units factors and the interaction between them are given.

Key Words: (One-Way MRM): One-Way Multivariate Repeated Measures Model, (Λ_r): Wilks distribution , (U^*) is $P \times P$ orthogonal matrix , the multivariate- Wishart distribution , (ANCOVA) analysis of variance and covariance contain the covariate (Z_1).

Introduction:

Repeated measurements analysis is widely used in many fields , for example, the health and life science, epidemiology , biomedical, agricultural, industrial, psychological, educational research and so on (see, Huynh and Feldt (1970)[10], and Vonesh and Chinchilli (1997)[8]). Repeated measurements analysis of variance, often referred to as randomized block and split-plot designs (see Bennett and Franklin (1954)[7], Sendeecor and Cochran (1967)[9])."Repeated measurements" is a term used to describe data in which the response variable for each experimental unit is observed on multiple occasions and possibly under different experimental conditions (see,Vonesh and Chinchilli (1997)[8]). This thesis is devoted to study of one-way Multivariate repeated measurements analysis of covariance model (MRM ANCOVA), which contains one between-units factor (Group with q levels) ,one within-units factor (Time with p levels) and one covariate (Z_1).For such model, the observations are transformed by an orthogonal matrix to Minimize the probability of presence a correlation between senses of a level of the experiment [3]. The ANCOVA which is based on the first set of transformed observations provides the ANCOVA for the between – units factor effects, while ANCOVA which is based on the k^{th} set of transformed observations, for each k=2,3,…,p provides the ANCOVA for within – units effect [4] .

(1.1) Covariate in One-Way Multivariate Repeated Measurements Design :

There is a variety of possibilities for the between units factors in a one-way design. In a randomized one-way MRM experiment, the experimental units are randomized to one between-units factor (Group with q levels) ,one within-units factor (Time with p levels) and one covariate (Z_1)[3]. For this model the covariate is time-independent, that is measured only once. For convenience ,we define the following linear model and parameterization for the one-way repeated measurements design with one between units factor incorporation two covariates :-

$$Y_{ijk} = \mu + \tau_j + (\delta_i)_j + \gamma_k + (\tau\gamma)_{jk} + (Z_{1ij} - \bar{Z}_{1..})\beta_1 + e_{ijk} \qquad \ldots(1.1)$$

Where

$(i = 1,\ldots,n_j)$ is an index for experimental unit within group (j)

$(j = 1,\ldots,q)$ is an index for levels of the between-units factor (Group).

$(k = 1,\ldots,p)$ is an index for levels of the within-units factor (Time).

$Y_{ijk} = (Y_{ijk1},\ldots,Y_{ijkr})'$ is the response measurement of within-units factors (Time) for unit i within treatment factors (Group).

$\mu = (\mu_1,\ldots,\mu_r)'$ is the over all mean.

$\tau_j = (\tau_{j1},\ldots,\tau_{jr})'$ is the added effect of the j^{th} level of the treatment factor (Group).

$\delta_{i(j)} = (\delta_{i(j)1},\ldots,\delta_{i(j)r})'$ is the random effect due to the experimental unit i within treatment group j.

$\gamma_k = (\gamma_{k1},\ldots,\gamma_{kr})'$ is the added effect of the k^{th} level of Time.

$(\tau\gamma)_{jk} = \left((\tau\gamma)_{jk1},\ldots,(\tau\gamma)_{jkr}\right)'$ is the added effect of the interaction between the units factor (Group) at the level of (Time).

$Z_{1ij} = (Z_{1ij1},\ldots,Z_{1ijr})'$ is the value of covariate Z_1 at time k for unit i within group j .

$\bar{Z}_{1..} = (\bar{Z}_{1..1},\ldots,\bar{Z}_{1..r})'$ is the mean of covariate Z_1 over all experimental units.

$\beta_1 = (\beta_{11},\ldots,\beta_{1r})'$ is the slope corresponding to covariate Z_1.

$e_{ijk} = (e_{ijk1},\ldots,e_{ijkr})'$ is the random error at time k for unit i within group j.

For the parameterization to be of full rank, we impose the following set of conditions :

$$\sum_{k=1}^{p} \gamma_k = 0 \;,\; \sum_{j=1}^{q} \tau_j = 0$$

$$\sum_{j=1}^{q} (\tau\gamma)_{jk} = 0 \quad \text{for each} \quad k = 1,\ldots,p \;,\; \sum_{k=1}^{p} (\tau\gamma)_{jk} = 0 \quad \text{for each} \quad j = 1,\ldots,q$$

$$\sum_{i,j=1}^{n_j,q} (Z_{1ij}) = 0$$

We assume that e_{ijk}'s and $\delta_{i(j)}$'s are independent with

$$e_{ijk} = (e_{ijk1},\ldots,e_{ijkr})' \sim \; i.i.d \;\; N_r(0,\Sigma_e)$$

$$\delta_{i(j)} = [\delta_{i(j)1},\ldots,\delta_{i(j)r}]' \sim i.i.d \;\; N_r(0,\Sigma_\delta) \qquad \ldots(1.2)$$

Where N_r is denoted to the multivariate normal distribution , and Σ_e, Σ_δ are $r \times r$ positive definite matrices [2] .

Where the variance and covariance matrix Σ of the model (1.1) satisfy the assumption of compound symmetry, i.e.

$$i.e \quad \Sigma = I_p \otimes \Sigma_e + J_p \otimes \Sigma_\delta$$

$$= \begin{bmatrix} \Sigma_e + \Sigma_\delta & \Sigma_\delta & \cdots & \Sigma_\delta \\ \Sigma_\delta & \Sigma_e + \Sigma_\delta & \cdots & \Sigma_\delta \\ \vdots & \vdots & \vdots & \vdots \\ \Sigma_\delta & \Sigma_\delta & \cdots & \Sigma_e + \Sigma_\delta \end{bmatrix} \quad \ldots(1.3)$$

Where

I_p denotes the $P \times P$ identity matrix.

J_p denotes the $P \times P$ matrix of one's. and

$\otimes$ be the Kroneker product operation of two matrices.

j_p denotes the $P \times 1$ vector of one's .

(1.2) Transforming the one-way Repeated measurements Analysis of Covariance (ANCOVA) model :

In this section we use an orthogonal matrix to transform the observations

Y_{ijk} for $i = 1, \ldots, n_j$, $j = 1, \ldots, q$, $k = 1, \ldots, p$

Let U^* be any $P \times P$ orthogonal matrix and U denotes $p \times (p-1)$ matrix, which is partitioned as follows [4] :

$$U^* = (p^{-\frac{1}{2}} j_p \quad U) \quad \ldots(1.4)$$

$$Y_{ij}^* = Y_{ij} U^*$$

$$\left[Y_{ij1}^* \cdots Y_{ijp}^*\right]' = \left[Y_{ij1} \cdots Y_{ijp}\right]' U^*$$

$$\operatorname{cov}(\overrightarrow{Y_{ij}^*}) = (U^{*'} \otimes I_r)(I_p \otimes \Sigma_e + J_p \otimes \Sigma_\delta)(U^* \otimes I_r)$$

$$= I_p \otimes \Sigma_e + U^{*'} J_p U^* \otimes \Sigma_\delta \quad \ldots(1.5)$$

$$= \begin{bmatrix} \Sigma_e + p\Sigma_\delta & 0 & \cdots & 0 \\ 0 & \Sigma_e & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_e \end{bmatrix}$$

(1.3) Analysis of Covariance (ANCOVA) for the One-Way Repeated Measurements Model :

In this section, we study the ANCOVA for the effects of between-units factors and within-units factors for the One-way RM model (1.1). Also we give the null hypotheses which are concerned with these effects and the interaction between them, and the test statistics for them [8] .

Now

$$Y_{ij1}^* = Y_{ij} p^{-\frac{1}{2}} j_p = \left[\frac{1}{\sqrt{p}} \sum_{k=1}^{p} Y_{ijk1} \quad \frac{1}{\sqrt{p}} \sum_{k=1}^{p} Y_{ijk2} \cdots \frac{1}{\sqrt{p}} \sum_{k=1}^{p} Y_{ijkr} \right]'$$

From (1.1), we obtain [3]:

$$Y_{ij1}^{*} = \mu^{*} + \tau_{j}^{*} + \delta_{i(j)}^{*} - \overline{Z}_{1..}^{*}\beta_{1}^{*} + e_{ij1}^{*}$$

Then the set of vectors

$(Y_{111}^{*},\cdots,Y_{n_1 11}^{*})',(Y_{121}^{*},\cdots,Y_{n_2 21}^{*})',\cdots,(Y_{1q1}^{*},\cdots,Y_{n_q q1}^{*})'$

Have mean vectors [11] :

$X_1 = \sqrt{P}\mu + \sqrt{P}\tau_1 - \sqrt{P}\overline{Z}_{1..}\beta_1$

$X_2 = \sqrt{P}\mu + \sqrt{P}\tau_2 - \sqrt{P}\overline{Z}_{1..}\beta$

$\vdots$

$X_q = \sqrt{P}\mu + \sqrt{P}\tau_q - \sqrt{P}\overline{Z}_{1..}\beta_1$

Respectively, and each of them has covariance matrix $p\sum_{\delta} + \sum_{e}$.

So, the null hypothesis of the same treatment effects is [1]:

$$H_{01} = \tau_1 - \overline{Z}_{1..} = \cdots = \tau_q - \overline{Z}_{1..} = 0$$

The ANCOVA based on the set of transformed observations above the $Y_{ij1's}^{*}$ provides the ANCOVA for between-units effects. This leads to the following form for the sum square terms SS_G , SS_{Z_1} ، $SS_{u(GroupZ_1)}$ [3] :

$$SS_G = \sum_{j=1}^{q} n_j(\overline{Y}_{j1}^{*} - \overline{Y}_{1}^{*})(\overline{Y}_{j1}^{*} - \overline{Y}_{1}^{*})', \quad \overline{Y}_{j1}^{*} = \frac{\sum_{i=1}^{n_j} Y_{ij1}^{*}}{n_j} \quad , \quad \overline{Y}_{1}^{*} = \frac{\sum_{j=1}^{q}\sum_{i=1}^{n_j} Y_{ij1}^{*}}{n}$$

$$SS_{Z_1} = \sum_{j=1}^{q}\sum_{i=1}^{n_j}\frac{1}{\beta_1}\left[(\overline{Y}_{ij1}^{*} - \overline{Y}_{j1}^{*})(\overline{Y}_{ij1}^{*} - \overline{Y}_{j1}^{*})'\right]$$

$$SS_{u(GroupZ_1)} = \sum_{j=1}^{q}\sum_{i=1}^{n_j}\left((Y_{ij1}^{*} - \overline{Y}_{ij.} + \overline{Y}_{1})(Y_{ij1}^{*} - \overline{Y}_{ij.} + \overline{Y}_{1})'\right)$$

Thus

$SS_G \sim W_r(q-1, p\sum_{\delta} + \sum_{e})$

$SS_{Z_1} \sim W_r(1, p\sum_{\delta} + \sum_{e})$

$SS_{u(GroupZ_1Z_2)} \sim W_r(n-q-2, p\sum_{\delta} + \sum_{e})$

Where W_r denotes the multivariate-Wishart distribution.

The multivariate Wilks test [12] :

$T_{w_1} = \dfrac{\left|SS_{u(GroupZ_1)}\right|}{\left|SS_{u(GroupZ_1)} + SS_G\right|}$, When H_{01} is true

$T_{w_1} \sim \Lambda_r(n-q-2, q-1)$

$T_{w_2} = \dfrac{\left|SS_{u(GroupZ_1)}\right|}{\left|SS_{u(GroupZ_1)} + SS_{Z_1}\right|}$, When H_{01} is true

$T_{w_2} \sim \Lambda_r(n-q-2, 1)$

The ANCOVA based on the set of transformed observations the $Y^*_{ijk's}$ for each $k = 2,\ldots,p$ has the model which is partitioned as follows[3]:

$$Y^*_{ijk} = \sum_{k'=1}^{p} u_{kk'} Y_{ijk'} \qquad , \quad k = 2,\cdots,p$$

$$= \sum_{k'=1}^{p} u_{kk'} \left(\mu + \tau_j + \gamma_{k'} + (\tau\gamma)_{jk'} + (Z_{1ij} - \bar{Z}_{1..})\beta_1 + (Z_{2ij} - \bar{Z}_{2..})\beta_2 + e_{ijk'} \right)$$

$$Y^*_{ijk} = \gamma^*_k + (\tau\gamma)^*_{jk} - \bar{Z}^*_{1..}\beta^*_1 + e^*_{ijk}$$

Then from above the analysis we test the following hypotheses[4] :

$$H_{02} : \gamma^*_2 - \bar{Z}^*_{2..} = \cdots = \gamma^*_p - \bar{Z}^*_{1..} = 0$$

$$H_{03} : (\tau\gamma)^*_{j2} - \bar{Z}^*_{1..} = \cdots = (\tau\gamma)^*_{jp} - \bar{Z}^*_{1..} = 0$$

The ANCOVA based on the set of transformed observations above , the $Y^*_{ij2},\ldots,Y^*_{ijp}$ provides the ANCOVA for within-units effects.

This leads to the following forms for the sum square terms[3] :

$$SS_{Time} = n\sum_{k=2}^{p} (\bar{Y}^*_k (\bar{Y}^*_k)')$$

$$SS_{Time\times Group} = \sum_{k=2}^{p}\sum_{j=1}^{q} n_j \left((\bar{Y}^*_{jk} - \bar{Y}^*_k)(\bar{Y}^*_{jk} - \bar{Y}^*_k)' \right), \quad \bar{Y}^*_{jk} = \frac{\sum_{i=1}^{n_j} Y^*_{ijk}}{n_j} \qquad , k = 2,\ldots,p$$

$$SS_{Time\times Umit(GroupZ_1)} = \sum_{k=2}^{p}\sum_{j=1}^{q}\sum_{i=1}^{n_j} \left((Y^*_{ijk} - \bar{Y}_{ij.} + \bar{Y}_k)(Y^*_{ijk} - \bar{Y}_{ij.} + \bar{Y}_k)' \right)$$

Then from the above sum square terms , we have [13] :

$$SS_{Time} \sim W_r\left((p-1), p\Sigma_\delta + \Sigma_e \right)$$

$$SS_{Time\times Group} \sim W_r\left((p-1)(q-1), p\Sigma_\delta + \Sigma_e \right)$$

$$SS_{Time\times Unit(GroupZ_1)} \sim W_r\left((p-1)(n-q)\, p\Sigma_\delta + \Sigma_e \right)$$

The multivariate Wilks test [12] :

$$T_{w_3} = \frac{\left| SS_{Time\times Unit(GroupZ_1)} \right|}{\left| SS_{Time\times Unit(GroupZ_1)} + SS_{Time} \right|} \quad , \quad \text{when } H_{02} \text{ is true}$$

$$T_{w_3} \sim \Lambda_r\left((p-1)(n-q), (p-1) \right)$$

$$T_{w_4} = \frac{\left| SS_{Time\times Unit(GroupZ_1)} \right|}{\left| SS_{Time\times Unit(GroupZ_1)} + SS_{Time\times Group} \right|} \quad , \quad \text{when } H_{02} \text{ is true}$$

$$T_{w_4} \sim \Lambda_r\left((p-1)(n-q), (p-1)(q-1) \right)$$

The one-way MRM ANCOVA with one between-unit factor (Group) and two covariates (Z_1) that are time-independent.

	Source	D.F	SS	Wilks Criterion
Between	$Group$	$q-1$	SS_G	$T_{W_1} = \dfrac{\lvert SS_{Unit(GroupZ_1)} \rvert}{\lvert SS_{Unit(GroupZ_1)} + SS_G \rvert}$
	Z_1	1	SS_{Z_1}	$T_{W_2} = \dfrac{\lvert SS_{Unit(GroupZ_1)} \rvert}{\lvert SS_{Unit(GroupZ_1)} + SS_{Z_1} \rvert}$
	$Unit(GroupZ_1)$	$n-q-2$	$SS_{Unit(GroupZ_1)}$	
Within	$Time$	$p-1$	SS_{Time}	$T_{W_3} = \dfrac{\lvert SS_{Time\times Unit(GroupZ_1)} \rvert}{\lvert SS_{Time\times Unit(GroupZ_1)} + SS_{Time} \rvert}$
	$Time \times Group$	$(p-1)\times(q-1)$	$SS_{Time\times Group}$	$T_{W_4} = \dfrac{\lvert SS_{Time\times Unit(GroupZ_1)} \rvert}{\lvert SS_{Time\times Unit(GroupZ_1)} + SS_{Time\times Group} \rvert}$
	$Time{*}Unit(GroupZ_1)$	$(p-1)\times(n-q)$	$SS_{Time\times Unit(GroupZ_1)}$	

References

[1] A.S., AL-Mouel, and, J.L., Wang "one-way multivariate Repeated measurements Analysis of variance model", Applied mathematics a journal of Chinese universities, 19, 435-448,(2004).

[2] A.S., AL-Mouel, and H.Z. Naji "The Spherisity test for Multivariate Repeated measurement Model has two Covariates" , Vol.2 , No.2, 143-159,(2010).

[3] A.S.AL-Mouel, and H.Z.Naji "Two Covariates in Repeated measurement " Basrah Researches (Sciences), Vol.36 part 1.19-30 ,(2010).

[4] A.S.AL-Mouel, , and J.M. Jassim, "two-way multivariate Repeated measurements Analysis of variance model" J. Basrah Researches (Sciences) Vol.32. part 1. -17-31,(2006).

[5]A.S.AL-Mouel, , and R.R.Faik "multivariate Repeated measurements with a Kronecker product structured covariance matrix" , Vol.1 , No.1, 241-247 ,(2009).

[6] A.S.AL-Mouel, " Nested Repeated measures model sufficiency and Estimation". Chinese Journal of Applied probability and statistics,20, 133-146 ,(2004).

[7] C.A .Bennett, and N.L .Franklin, "Statistical Analysis in Chemistry and the Chemical Industry", John Wiley & Sons, Inc., New York, (1954).

[8] E.F.Vonesh, and V.M .Chinchill "linear and non linear models for the Analysis of Repeated measurements", Marcel Dakker Inc., New York (1997).

[9] G.W. Snedecor, and W.G .Cochran, "Statistical Methods ", 6^{th} edition, Ames, Iowa State University press, Iowa, (1967).

[10] H .Huynh, and L.S. Feldt, "Conditions Under Which Mean Square Ratios in Repeated measurements Designs Have Exact F-Distributions", Journal of the American Statistical Association, 65,1970, 1582-1589.

[11] M.J. Jawad "Multivariate Repeated measures model and sphericity test ", Ph.D. Thesis, University of Basrah, (2006).

[12] S.S .Wilks "Sample Criteria for Testing Equality of Means of Variances and Equality of Covariances in a Normal Multivariate distribution" Ann. Math. Statist.17,257-281,(1946).

[13] T.W .Anderson, "An introduction to multivariate statistical Analysis", Wily, New York , (1984).

[14] V.M, Chinchilli, and W.H. Carter , "A likelihood Ratio Test for a patterned Covariance Matrix in Multivariate Growth Curve Model" , Biometrics, 40,151-156, (1984).

المستخلص

تناول هذا البحث دراسة نموذج من نماذج القياسات المتكررة متعددة المتغيرات للبيانات الكاملة ذات الاتجاه الواحد (One-way MRM) لتحليل التباين المشترك، والنموذج الذي تم دراسته هنا ذا اتجاه واحد ، وهذا يدل على وجود عامل واحد بين الوحدات يدعى (المجموعة) وعامل واحد داخل الوحدات يدعى الزمن هذا فضلاً عن عامل مرافق (مستقل).

كون هذا النموذج يحتوي على عامل مرافق (مستقل) واحد ، لذا يطلق على النموذج العام اسم (ANCOVA) ،ويشترط في هذا النموذج أن يكون العامل المرافق Z_1 (ثابت و مستقل زمنيا (أي يقاس مرة واحدة في كل مستوى من مستويات التجربة).