Exact Solutions of Nonlinear Filtration Equation obtained by a Generalized Conditional Symmetry Method

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Abstract

In this paper, we find a useful result under certain types of generalized conditional symmetry, which is reduction of nonlinear filtration equation to different classes by classifying the filtration coefficient. As a consequence, of our result some new exact solutions for two classes of important filtration equation are obtained, for a third class of filtration equation, compatible condition for exact solution is also derived.

ا لاصة

هذه البحث، قمنا با اد نتيجة مفيدة ت بعض أنواع التناظر الشرطي ا ُعممِ، ال فض معادلةِ الترشيحِ اللاخطّيةِ إ الأصنافِ ا ختلفةِ بتصنيف معاملِ الترشيح. وكنتيجة من نتائج البحث هو ا صول على بعض ا لول ا ضبوطةِ ا ديدةِ لإثنان ِمن أصنافِ معادلـةِ الترشـيح ا همةِ , وبالنسبة عادلة الترشيح من الدرجة الثالثة ، اشتقاق شرط متوافق للحل ا ضبوطِ .

Introduction

Nonlinear partial differential equations (PDEs) occur in several places in the study of applied mathematics, physics and engineering. Hence, the study of these equations has become very important. Unfortunately, most of these equations are not integrable. Hence, finding some exact solutions or compatibility condition to obtain exact solution of these equations has become very important in the study of nonlinear PDEs.

The most famous and established method for finding exact solution of nonlinear PDEs is called group analysis (Olver, 1986). Several generalization of this method has been proposed in the literature. One such generalization is the method of generalized symmetry. This is further generalized by (Fokas,. and Liu, 1994) to generalized conditional symmetry. This concept provides an algorithm for constructing physically important exact solutions of nonlinear PDE.

In our work, we will apply the generalized conditional symmetry method to the well-known nonlinear PDEs, which is called filtration equation

$$u_t = D(u_x)\, u_{xx} \qquad (1.1)$$

eq(1.1) is used in mechanics as a mathematical model in studying shear currents of nonlinear viscoplastic media , processes of filtration of non-Newtonian fluids , as well as for describing the propagation of oscillations of temperature and salinity to depths in oceans.Eq(1.1) is transformed to the well-known diffusion equation ($v_t = h(v)\, v_x$) when $v = u_x$ (Ibragimov, 2004). Eq(1.1) is one of the classes obtained by group classification (Basara-Horwath.etal, 2001).The function $D(u_x)$ is known as a filtration coefficient. In general the filtration coefficient is not fixed, and we consider the family of equations of the form (1.1) with arbitrary functions $D(u_x)$.

generalized The remainder of this paper is arranged as follows: in section 2 we define the conditional symmetry and we are going to construct our theorem (2.1) . In section 3, as a consequence some exact solutions of different classes of eq.(1.1) are obtained via the method of generalized conditional symmetry . And finally, in section 4 we present some conclusion.

Special Forms of Filtration coefficient

In this section, we will derive three classes of filtration coefficient $D(u_x)$ to eq.(1.1).

Let

$$u_t = K(t,u) \qquad (2.1)$$

suppose that $K(t,u)$ denotes a function that depends on a differentiable manner of $t, u, u_x, u_{xx}, \ldots$

Definition (2.1) (Qu. 1997)

The function $\sigma\,(t,x,u)$ is called a generalized conditional symmetry of eq(2.1) if there exists a function $A(t,x,u,\sigma)$ such that,

$$\frac{\partial\sigma}{\partial t} + [K,\sigma] = A(t,x,u,\sigma) \quad , \quad A(t,x,u,0)=0 \qquad (2.2)$$

where , $[K,\sigma] = \sigma' K - K' \sigma$ (the prime denotes Frechet derivative) , σ is a differentiable function of $t, x, u, u_x, u_{xx}, \ldots$ and $A(t,x,u,\sigma)$ is a differentiable function of $t, x, u, u_x, u_{xx}, \ldots, \sigma, \sigma_x, \sigma_{xx}, \ldots$.

In case σ does not depend explicitly on t then eq.(2.2) implies $\sigma' K|_{\sigma=0} = 0$

Since $K'\sigma$ is a linear function of σ .

Now, we can state our main result .

Theorem (2.1): Suppose that eq(1.1) admits a generalized conditional symmetry $\sigma = u_{xx} + F(u)u_x + G(u)$, then

(1) $F = G = 0$, for arbitrary functions $D(u_x)$

(2) $F = \dfrac{1}{c - \frac{\alpha}{2}u}$ & $G = 0$, for $D(u_x) = b_1\, e^{\alpha\, u_x}$

(3) $F = 0$ & $G = r$, for $D(u_x) = b_1 u_x + b_2$

where F, G are functions of u , and $b_1 \neq 0, b_2, \alpha, c, r$ are arbitrary constants.

Proof: we find the Frechet derivative of σ in the form

$$\sigma' = \frac{d^2}{dx^2} + F\,\frac{d}{dx} + \dot{F}\,u_x + \dot{G} \qquad (2.3)$$

from the definition (2.1) and eq(2.3) , it implies that eq(1.1) admits the generalized conditional symmetry iff

$$(3D'F\dot{F} - D\ddot{F} - D''F^3)u_x^3 + (4DF\dot{F} - D\ddot{G} - 2D'F^3 + 3GD'\dot{F} + 3FD'\dot{G} - 3D''GF^2)u_x^2 + (2DG\dot{F} - 4D'GF^2 + 3D'G\dot{G} - 3D''FG^2)u_x + G^2(D'' + 2D'F) = 0$$

Where $\bullet = \frac{d}{du}$ & $' = \frac{d}{du_x}$, Equating the coefficients of different powers of u_x to zero, yields the following overdetermined equations for D , F and G :

$$3D'F\dot{F} - D\ddot{F} - D''F^3 = 0 \qquad (2.4a)$$

$$4DF\dot{F} - D\ddot{G} - 2D'F^3 + 3D'\dot{F}G + 3D'F\dot{G} - 3D''GF^2 = 0 \qquad (2.4b)$$

$$2DG\dot{F} - 4D'GF^2 + 3D'G\dot{G} - 3D''FG^2 = 0 \qquad (2.4c)$$

$$G^2(D'' + 2D'F) = 0 \qquad (2.4d)$$

Now, we will show that a solution of this system is possible only if $D(u_x)$ takes the form (1) , (2) and (3) given in the theorem above.

From eq(2.4d) we get the two cases

<u>Case(1):</u> $G = 0$

It is easy to show that eq(2.4c) is satisfied. Eq(2.4b) will be reduced to the equation

$$F(4D\dot{F} - 2D'F^2) = 0$$

<u>subcase 1(i):</u> $F = 0$

Eq.(2.4a) is satisfied , then in this case we find that $D(u_x)$ is arbitrary function of u_x and $F = G = 0$ (*i.e* $\sigma = u_{xx}$)

<u>subcase 1(ii):</u> $4D\dot{F} - 2D'F^2 = 0$

In this case, it implies that there exist $\alpha \in R$, such that

$$\frac{D'}{D}=\alpha \quad , \; i.e \quad D=b_1 e^{\alpha u_x}$$

and

$$\frac{2\dot{F}}{F^2}=\alpha \quad , \; i.e \quad F=\frac{1}{c-(\frac{\alpha}{2})u}$$

where $b_1 \neq 0, c, \alpha$ are arbitrary constants.

Obviously , eq.(2.4a) is satisfied.

$$i.e \qquad \sigma = u_{xx} + \frac{u_x}{c-\frac{\alpha}{2}u}$$

Case (2): $D''+2D'F=0$

In this case, it implies that there exist $\beta \in R$ such that

$$\frac{D''}{D'}=\beta \qquad \& \qquad F=-\frac{\beta}{2}$$

From eq.(2.4a) , we get $D'' F^3 = 0$, $i.e$ $D''\beta^3=0$

Since $\beta=0$, it implies that $D''=0$, therefore $D=b_1 u_x+b_2$ & $F=0$

where $b_1 \neq 0, b_2$ are arbitrary constants.

from eq.(2.4b) , it implies that $\ddot{G}=0$, $i.e$ $G=r_1 u+r$

and finally, eq.(2.4c) become in the form

$$D' G \dot{G} = 0 \tag{2.5}$$

if we substitution D and G in eq.(2.5) then $r_1 = 0$, $i.e$ $G=r$

where , r is arbitrary constant.

$$i.e \qquad \sigma = u_{xx} + r$$

3.Exact Solutions of the Filtration Equations (1.1)

In this section we will show that the importance of theorem (2.1) to find an exact solution of three classes of nonlinear filtration equation (1.1) and some nonlinear diffusion equation which is equivalent to special class of eq.(1.1).

3.1 Class one, equation (1.1) with arbitrary filtration coefficient

Let the well-known filtration equation (1.1) , from theorem (2.1) ,it implies that the generalized conditional symmetry of eq.(1.1) is $\sigma = u_{xx}$

since $\sigma = 0$, $(i.e \quad u_{xx} = 0 \ )$
then $u = A(t)\ x + B(t)$
now , we will substitute the value of u above in eq.(1.1) we will get ,
$A(t)'\ x + B(t)' = 0$, it implies that $A(t)' = 0$, $B(t)' = 0$
$i.e \quad A(t) = a$, $B(t) = b$ where a, b are arbitrary constants and the general solution of eq.(1.1) is $u = a\,x + b$

3.2 Class two, equation (1.1) with linear filtration coefficient

Let

$$u_t = (b_1\, u_x + b_2\,)\ u_{xx} \tag{3.1}$$

from theorem (2.1) , eq.(3.1) admits a generalized conditional symmetry
$\sigma = u_{xx} + r$, since $\sigma = 0 \quad (\ i.e \quad u_{xx} + r = 0 \ )$, it implies that

$$u = A(t)\,x - \frac{r}{2}\ x^2 + B(t) \tag{3.2}$$

then we substitute the value of u in eq.(3.1), we will get

$$A(t)'\,x + B(t)' = r^2\, b_1\, x - r\,(b_1\, A(t) + b_2\,)$$

i.e

$$A(t)' = r^2\, b_1 \qquad \& \qquad B(t)' = -r\ (b_1\, A(t) + b_2\,) \tag{3.3}$$

a general solution of eq.(3.3) is

$$A(t) = r^2\ b_1\, t + k_1 \qquad \& \qquad B(t) = k_2 - \frac{r^3\, b_1^{\,2}}{2}\, t^2 - r\,(b_1\, k_1 + b_2\,)\ t$$

where k_1 & k_2 are arbitrary constants.
the exact solution of eq.(3.1) is

$$u = (r^2\, b_1\, t + k_1\,)\ x - \frac{r}{2}\, x^2 - \frac{r^3\, b_1^{\,2}}{2}\, t^2 - r\,(b_1\, k_1 + b_2\,)\ t + k_2$$

Note: As a consequence, the exact solution of diffusion equation
$v_t = \big((b_1\, v + b_2\,)\ v_x\big)_x$ which is equivalent to eq.(3.1) is $v = A(t) - r\,x$

3.3 Class three, equation (1.1) with exponintional filtration coefficient:

In this section, we present proposition as a compatibility condition to obtain an exact solution of nonlinear filtration equation

$$u_t = b_1\ e^{\alpha\ u_x}\ u_{xx} \tag{3.4}$$

because we find complicate to write the solution u of eq.(3.4) explicitly .

Proposition (3.1): The equation (3.4) have the exact solution

$$E\,i\,(1, -\ln(\,A(t)\ (c - \frac{\alpha}{2}\ u\ )))\, = A(t)\ (\ x + B(t)\ ) \tag{3.5}$$

if and only if $A(t)$ & $B(t)$ satisfying the following equation

$$A(t)' + \frac{d}{dt}(A(t)\, B(t)) = (\frac{\alpha}{2}\, u - c)\left(b_1\, A(t)^3 + \frac{A(t)'}{\ln(A(t)\,(c - \frac{\alpha}{2}\, u)} \right)$$

(3.6)

where $A(t) \neq 0$ & $B(t)$ are functions of t (integral constant) , and $\alpha \neq 0$

Ei is called an Exponential Integral .

Proof:

The prove has two way,

At the first, if we let eq.(3.5) is a solution of eq.(3.4),we need to show that the eq.(3.6) is satisfying .

But from theorem (2.1) ,eq.(3.4) admits a generalized conditional symmetry

$$\sigma = u_{xx} + \frac{u_x}{(c - \frac{\alpha}{2} u)}$$

since $\sigma = 0$

i.e

$$u_{xx} + \frac{u_x}{(c - \frac{\alpha}{2} u)} = 0 \qquad (3.7)$$

the first integral of eq.(3.7) is

$$u_x = \frac{2}{\alpha} \ln(A(t)\,(c - \frac{\alpha}{2} u)) \qquad (3.8)$$

from eq.(3.8) ,it implies that

$$\frac{\alpha}{2} \int \frac{\partial u}{\ln(A(t)\,(c - \frac{\alpha}{2} u))} = x + B(t) \qquad (3.9)$$

then , the general solution of eq.(3.7) , which implies that from eq.(3.9) and given as in eq.(3.5) (because eq.(3.4) and eq.(3.7) are compatible)

now, in order to determine eq.(3.6) , we differentiate both sides of eq.(3.5) with respect to t , we find

$$\frac{\frac{\alpha}{2}\, A(t)\, u_t \;-\; A(t)'\,(c - \frac{\alpha}{2}\, u\,)}{\ln\left(A(t)\,(c - \frac{\alpha}{2}\, u\,)\right)} = A(t)'\,x + \frac{d}{dt}(\,A(t)\,B(t)\,)$$

hence

$$u_t = \frac{2}{\alpha\, A(t)}\left(A(t)'\;(c - \frac{\alpha}{2}\, u\,) + \ln(\,A(t)\,(c - \frac{\alpha}{2}\,))\,(\,A(t)'\;x \;+ \frac{d}{dt}(A(t)\,B(t))\right) \tag{3.10}$$

if , we substitute u_{xx} , u_x & u_t from eq.(3.7) , eq(3.8) and eq.(3.10), respectively in eq.(3.4) , then eq.(3.6) holds.

And conversely, if we let the eq.(3.6) is satisfying , we need to show that eq.(3.5) is the exact solution of eq.(3.4).

Since, the eq.(3.6) is satisfying and from theorem (2.1) we find the exact solution of eq.(3.7) given as in eq.(3.5) , therefore we will get directly that the exact solution of eq.(3.4) is the eq.(3.5) □

Conclusion

In this paper, we find the exact solutions to some non-integrable forms of filtration and diffusion equation. We conclude, the existence of such exact solution for non-integrable PDE can be traced back to the relation of these equations with integrable ones, i.e , the usefulness of the generalized conditional symmetry definition follows from the fact that it implies that the eq.(2.1) and $\sigma = 0$ are compatible .

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