ON the number of double secants of (k,n)-arc in PG(2,q)

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Abstract

In this paper we discuss the number of double secants of (k,n)-arc in PG(2,q) for $k < q+2n-4$ and $k \geq q+2n-4$.

دراسة حول عدد القواطع المزدوجة للقوس- (k , n) في المستوي الاسقاطي PG(2, q)

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الخلاصـة

في هذا البحث تم دراسة القواطع المزدوجة للقوس- (k , n) في المستوي الاسقاطي PG(2,q) و مناقشة اعدادها عندما $k< q+2n-4$ و $k\geq q+2n-4$.

1. Introduction

A (k,n)-arc in a projective plane PG(2,q) is a set of k points at most n on every line. A (k,n)-arc is called complete if it cannot be extended to a ($k+1,n$)-arc. If the order of the plane is q, then $k \leq q\,n - q + n$ with equality if and only if every line intersects the arc in 0 or n points. Arcs realizing the upper bound are called maximal arcs. Equality in the bound implies that $n \mid q$ or $n=q+1$. If $1 < n < q$, then the maximal arc is called nontrivial. The only known examples of nontrivial maximal arcs in Desarguesian projective planes are hyperovals (n=2), and for n > 2 the Denniston (1969) arcs and an infinite family constructed by (Thas, 1974, 1980) . The Denniston arcs exist for all pair (n,q)=($2^a,2^b$), $0 < a < b$.

The maximal arcs do not exist in Desarguesian planes of odd order (Ball , Blokhuis and Mazzocca ,1997). The case (n,q)=(3,9) was proved by (Cossu, 1961) and (Thas, 1975) proved the cases (n,q)=($3,3^h$). A general proof the first appeared in (Ball , Blokhuis and Mazzocca ,1997), although a more accessible version of the proof appears in (Ball and Bolkhuis ,1998) and (Hirschfeld ,1997).

A (k,2)-arc is called for simply k-arc. An irreducible conic in PG(2,q) is (q+1)-arc. A line l in PG(2,q) is called i-secant of a (k,n)-arc K if | $l \cap$ K | = i. The point Q$\in$ PG(2,q) | K
is of index zero, if there is no n-secant passes through it.

In this paper we show that, if K containing a conic and $k < q+2n-4$ the number of double secants is exactly one and for $k \geq q+2n-4$ the number of double secants is at least one or not. Also, we show that for q odd there is a complete (k,m)- arc, $k = (n-1)(q+1)/2+n+(m-n)(q+n)$, $m \geq n$, $n= (q+3)/2$.

2. Basic result of double secants

In this section, we will give some result on the double secants and their numbers.

Suppose that K is a (k,n)-arc in PG(2,q) containing irreducible conic. An n-secant of K which is a secant (2-secant) of a conic will be called a *double secant* of K.

It is clear that, if n=2 then the number of double secants is exactly $k(k-1)/2$.

Lemma 2.1 If a (k,n>3)-arc K containing a conic and $k=q+n-1$, then every n-secant of K is a double secant and the number of double secants is exactly one.

Proof: Let l be an n-secant of K. If l is not a double secant, then l is either a tangent or an exterior line of a conic. If l is a tangent, then it contains a point of a conic and n-1 points not on the conic, hence $k=q+1+n-1=q+n$. If l is an exterior line, then l contains n points not on the conic and hence $k=q+n+1$. In both cases, we have a contradiction. So l must be a double secant. Suppose that there are two double secants of K, then every double secant must be contained n-2 points not on the conic and hence the number of points on K is $q+2n-3$ or $q+2n-4$ which is a contradiction.□

Lemma 2.2 If a (k,n>4)-arc K containing a conic and $k=q+n$, then every n-secant of K is a double secant or tangent of a conic and the number of double secants is exactly one.

Proof: Since K containing a conic, then the points of K can be partitioned into q+1 points of a conic and n-1 points not on the conic. Suppose that l is an n-secant of K, then l either contained n-2 points not on the conic and two points of the conic, in this case l is a double secant of K or l contains n-1 points not on the conic and one point of the conic, therefore l is a tangent. Hence the number of double secants is exactly one.□

Clearly, when $k<q+n+1$, then every n-secant is a double secant or tangent of a conic and when $k\geq q+n+1$, then every n-secant is a double secant , tangent or an exterior line of a conic. The following theorem is a generalized of lemmas (2.1) and (2.2).

Theorem 2.1 If K is a (k,n)-arc in PG$(2,q)$ containing a conic, then

(i) If $k<q+2n-4$, then the number of double secants is exactly one,

(ii) If $k\geq q+2n-4$, then the number of double secants is at least one or not.

Proof: (i) Suppose that, there are two double secants of K, then every double secant must be contained n-2 points not on the conic. So that the number of points on K is at least $q+1+2n-5 = q+2n-4$ which is a contradiction.

(ii) Since K containing a conic, then the points of K can be partitioned into $q+1$ points of the conic and at least $2n-5$ points not on the conic. Suppose that l is an n-secant of K, if l is a double secant of K, then l must be contained n-2 points not on the conic and two points of the conic and hence every other secant of conic will be contained at least n-3 points not on the conic. If l is a tangent of the conic, then every secant of the conic will be contained at least n-4 points not on the conic, if every secant of the conic contains at most n-3 points not on the conic, then there is no double secant of K. Finally, if l is an exterior line of the conic then every secant of the conic will be contained at least n-5 points not on the conic, if every secant of the conic contains at most n-4 points not on the conic, then there is no double secant.□

The following theorem is the main result of this paper.

Theorem 2.2 If K is a (k,n)-arc in PG$(2,q)$ containing a conic, q is odd and $P_1,P_2\in$ K be two points not on the conic. Let l_i and m_i be the double secants of K such that $\bigcap_i l_i=P_1$ and $\bigcap_i m_i=P_2$, ($i=1,\ldots,(q+1)/2$) and let t_i be an n-secants of K such that one of them is a double secant of K, ($i=1,\ldots,n-1$) and the intersection of any two of l_i, m_j and t_r is a point of K, ($i,j=1,\ldots,(q+1)/2$), ($r=1,\ldots,n-1$), then

(i) The number of m-secants is at least $q+n$,

(ii) $k=(n-1)(q+1)/2+n+(m-n)(q+n)$ is a complete (k,m)-arc, where $m\geq n$, $n=(q+3)/2$.

Proof: It is easy to show (i).

(ii) Suppose that K is not complete, then there is a point $Q\in$ PG$(2,q)$| K of index zero such that K contained in K_1= K$\cup$ {Q}. Counting the points of K_1, we have $k_1=k+1=$

$(n-1)(q+1)/2+n+(m-n)(q+n)+1$. Since the number of m-secants is $q+n$ and the intersection of any two m-secants is a point of K, then counting the points of PG(2,q)| K_1, we have

$t=(q+1-m)(q+n)$, so

$t+k_1=(q+n)(q+1-m)+(n-1)(q+1)/2+n+(m-n)(q+n)+1$

$= (2q^2+q+nq+5n-2n^2+1)/2= (4q^2+4q+8)/4=q^2+q+2.$

Which is impossible, so K is complete.□

Now, we refer to (Hameed and Abdul Hussain ,1999) and determine the number of interior and exterior points on or not on the (k,m)-arc K when $m=n=(q+3)/2$.

The following system was suggested in (Hameed and Abdul Hussain,1999),

$$\begin{aligned} &x+y=q^2+q+1-k-r,\\ &x_1+y_1=k-t,\\ &x+x_1=q(q+1)/2 \qquad \ldots \quad (2.1)\\ &y+y_1=q(q-1)/2\\ &x, x_1, y, y_1 \geq 0. \end{aligned}$$

In the case of theorem (2.2) the number of points on the $(k,(q+3)/2)$-arc is $k=(q^2+4q+7)/4$ and $r=0$, $t=q+1$. If we compare the value of k with the bounds of k in (Hameed and Abdul Hussain,1999) we find that the cases (1,4,5,7,9) does not occurs, so the remaining cases are satisfying and give the solution of system (2.1).

Corollary 2.1 If a $(k,(q+3)/2)$-arc, $k=(q^2+4q+7)/4$ containing a conic, then the solution of system (2.1) is of one of the cases,

(1) $x>x_1$, $y>y_1$,

(2) $x>x_1$, $y<y_1$,

(3) $x=x_1$, $y>y_1$,

(4) $x>x_1$, $y=y_1$.

3. Small planes

In this section, we give two examples of the planes PG(2,3) and PG(2,5).

Example 3.1 In PG(2,3) a (7,3)-arc is complete and represent the Fano plane where the 6 double secant are the lines of Fano plane minus one.

Example 3.2 The set of points,

k={
(1,0,0),(0,1,0),(0,0,1),(1,0,1),(1,1,1),(1,2,4),(1,4,4),(1,4,2),(1,3,3),(1,2,0),(0,1,2),(1,0,4
),

(1,4,1)}, form a complete (13,4)-arc in PG(2,5) with secant distribution t_o=0, t_1=6, t_2=12, t_3=4, t_4=9.

A 6-arc in PG(2,5) is a conic, so the set of points C={(1,0,0),(0,1,0),(0,0,1),(1,1,1),(1,2,4),
(1,4,2)}, form a conic in PG(2,5) which is a subset of K. The required double secants are given in table (3.1).

Table (3.1)

l_1: $x+4z=0$,	l_3: $y=0$,	l_5: $4y+z=0$,	l_7: $3x+4y+3z=0$.
l_2: $3x+2y+2z=0$,	l_4: $x+z=0$,	l_6: $x+y=0$,	

The intersection of the double secants l_1, l_2 and l_3 is the point (1,0,1) and the intersection of the double secants l_4, l_5 and l_6 is the point (1,4,4). The number of 4-secant is 9. The set K|C is the set of interior and exterior points of conic on K and the solution of system (2.1) is x=12, y=6, x_1=3, y_1=4.

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