Sixth and Fourth Order Compact Finite Difference Schemes for Two and Three Dimension Poisson Equation with Two Methods to derive These Schemes

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Abstract

In this paper we improve the accuracy of the numerical approximation used to solve the two and three dimension Poisson equation. We get this improvement by using the compact finite difference schemes. These schemes give high order accuracy; therefore we derive five schemes to solve two and three dimension Poisson equation: central finite difference, fourth and sixth order compact finite difference by using Taylor expansion scheme, fourth and sixth order compact finite difference by using compact operators scheme. Numerical experiments are conducted to test accuracy of these schemes.

Keyword: compact finite difference, Poisson equation, fourth and sixth order.

Introduction

The Poisson equation is one of the fundamental equations in mathematical physics. It occurs in a broad range of applications including acoustics, electromagnetism and fluid mechanics. There are many different approaches to this problem in literature. (Marchuk et al , 1983) give a finite difference solution to 3D Poisson equation has only second

order of accuracy. (Khoromskij, 1980) solve the 2D Poisson equation with fourth order accuracy. Also, (Samarskij, 1984) derive other fourth order difference scheme.

The compact finite difference method (CFD) one of the methods used to increase the accuracy of the numerical solutions of the partial differential equation. Therefore, we can use many methods to derive the compact finite difference scheme for any problem. (Spotz, 1995) use the Taylor series to drive a fourth and sixth compact finite difference for 2D and 3D Poisson equation. (Zhang, 2001) derive a fourth order compact finite difference for 2D Poisson equation by using compact operators method.

In this paper we have done the following:
1- Use the compact operators method to get the sixth order compact finite difference for 2D Poisson equation.
2- Develop the compact operators method to be used it for 3D Poisson equation.
3- Obtain a fourth and sixth compact finite difference for 2D and 3D Poisson equation by using the compact operators method
4- Testing the accuracy of all schemes which are derived in this paper

1. Compact Scheme by Using Taylor Expansion

Spotz (1995) derive the compact scheme to solve Poisson equation using this method as follows:

1.1 Two Dimension

Poisson equation of the form

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = f(x, y), \qquad (x, y) \in \Omega \qquad \qquad \text{…(1)}$$

Where Ω is a rectangular domain, or a union of rectangular domains, with suitable boundary condition defined on $\partial\Omega$. The solution $\varphi(x, y)$ and the forcing function f(x,y) are assumed to be sufficiently smooth and have required continuous partial derivatives.

For convenience, let us consider a rectangular domain $\Omega = [0, Lx] \times [0, Ly]$. Here subscripts are obviously not derivatives. We discretize Ω with uniform mesh sizes Δx and Δy respectively in the x and y coordinate directions. Denote $Nx = Lx/\Delta x$ and $Ny = Ly/\Delta y$ the numbers of uniform intervals along the x and y coordinate directions, respectively. The mesh point are (x_i, y_j) with $x_i = i\Delta x$ and $y_j = j\Delta y$, $0 \le i \le Nx$, $0 \le j \le Ny$.

In the sequel, we may also use the index pair (i,j) to represent the mesh point (x_i, y_j). In this paper we take $\Delta x = \Delta y = h$ .also $Nx = Ny = N$.

The standard second order central difference operators define at grid point (i,j) can be written as

$$\left. \begin{array}{l} \delta_x^2 \varphi_{ij} = \dfrac{\varphi_{i+1j} - 2\varphi_{ij} + \varphi_{i-1j}}{h^2} \\[4mm] \delta_y^2 \varphi_{ij} = \dfrac{\varphi_{ij+1} - 2\varphi_{ij} + \varphi_{ij-1}}{h^2} \end{array} \right\} \qquad \text{…(2)}$$

The derivatives in Eq.(1) can be approximated to second order accuracy as

$$
\left.\begin{array}{l}
\left.\dfrac{\partial^2\varphi}{\partial x^2}\right|_{ij} = \delta_x^2\varphi_{ij} - \dfrac{h^2}{12}\dfrac{\partial^4\varphi}{\partial x^4} - \dfrac{h^4}{360}\dfrac{\partial^6\varphi}{\partial x^6} + O(h^6) \\[4mm]
\left.\dfrac{\partial^2\varphi}{\partial y^2}\right|_{ij} = \delta_y^2\varphi_{ij} - \dfrac{h^2}{12}\dfrac{\partial^4\varphi}{\partial y^4} - \dfrac{h^4}{360}\dfrac{\partial^6\varphi}{\partial y^6} + O(h^6)
\end{array}\right\} \quad \ldots(3)
$$

Using the finite difference operators in (2) and (3), Eq.(1) can be discretized at a given grid point (x_i, y_i) as

$$\delta_x^2\varphi_{ij} + \delta_y^2\varphi_{ij} - \tau_{ij} = f_{ij}. \qquad \ldots(4)$$

Where the truncation error is

$$\tau_{ij} = \dfrac{h^2}{12}[\dfrac{\partial^4\varphi}{\partial x^4} + \dfrac{\partial^4\varphi}{\partial y^4}]_{ij} + \dfrac{h^4}{360}[\dfrac{\partial^6\varphi}{\partial x^6} + \dfrac{\partial^6\varphi}{\partial y^6}]_{ij} + O(h^6) \qquad \ldots(5)$$

Equation (4) is called a high Central Difference Scheme (CDS). We have include both $O(h^2)$ and $O(h^4)$ term in Eq.(5) because we wish to approximate all of them in order to construct an $O(h^6)$ scheme.

To obtain compact approximation to the $O(h^2)$ terms in Eq.(5), we simply take the appropriate derivatives of Eq.(1),

$$\dfrac{\partial^4\varphi}{\partial x^4} = \dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial^4\varphi}{\partial x^2\partial y^2} \qquad \ldots(6)$$

$$\dfrac{\partial^4\varphi}{\partial y^4} = \dfrac{\partial^2 f}{\partial y^2} - \dfrac{\partial^4\varphi}{\partial x^2\partial y^2} \qquad \ldots(7)$$

Substituting Eq.(6) and Eq.(7) into Eq.(5) yields

$$\tau_{ij} = \dfrac{h^2}{12}[\dfrac{\partial^2 f}{\partial x^2} - 2\dfrac{\partial^4\varphi}{\partial x^2\partial y^2} + \dfrac{\partial^2 f}{\partial y^2}]_{ij} + \dfrac{h^4}{360}[\dfrac{\partial^6\varphi}{\partial x^6} + \dfrac{\partial^6\varphi}{\partial y^6}]_{ij} + O(h^6) \qquad \ldots(8)$$

Note that all term on the right hand side of Eq.(8) have compact $O(h^2)$ approximations at noted ij, and the approximation of these terms has the following form:

$$\left.\dfrac{\partial^4\varphi}{\partial x^2\partial y^2}\right|_{ij} = \delta_x^2\delta_y^2\varphi_{ij} - \dfrac{h^2}{12}\left[\left(\dfrac{\partial^6\varphi}{\partial x^4\partial y^2} + \dfrac{\partial^6\varphi}{\partial x^2\partial y^4}\right)\right]_{ij} + O(h^4) \qquad \ldots(9)$$

Also,

$$\partial_x^2 \partial_y^2 \varphi_{ij} = \frac{1}{h^4}\Big[4\varphi_{ij} - 2(\varphi_{i-1j} + \varphi_{i+1j} + \varphi_{ij-1} + \varphi_{ij+1}) + \varphi_{i-1j-1} + \varphi_{i+1j-1} + \varphi_{i+1j+1} + \varphi_{i-1j+1}\Big]$$

$$\dots(10)$$

We can easily get an $O(h^4)$ method in the same way by substituting difference expressions for the $O(h^2)$ term in Eq.(8) and including these in the finite difference approximation (4). The resulting higher-order scheme follows from

$$\delta_x^2\varphi_{ij} + \delta_y^2\varphi_{ij} + \frac{h^2}{6}\delta_x^2\delta_y^2\varphi_{ij} - \tau_{ij} = f_{ij} + \frac{h^2}{12}(\delta_x^2 f_{ij} + \delta_y^2 f_{ij}) \qquad \dots(11)$$

Equation (11) is called a high order compact difference scheme of order 4 (HOC-4).

We can, however, obtain an $O(h^6)$ scheme for this governing equation. Note that the approximation to the cross derivative of φ in Eq.(8) introduces additional $O(h^4)$ term that we must be careful to include in our derivation of an $O(h^6)$ scheme. More specifically, substituting the finite difference expression for the cross derivative of φ and its truncation error terms into Eq.(8) give us

$$\tau_{ij} = \frac{h^2}{12}\Big[\frac{\partial^2 f}{\partial x^2} - 2\partial_x^2\partial_y^2\varphi + \frac{\partial^2 f}{\partial y^2}\Big]_{ij} + \frac{h^4}{360}\Big[\frac{\partial^6\varphi}{\partial x^6} + 5\frac{\partial^6\varphi}{\partial x^4\partial y^2} + 5\frac{\partial^6\varphi}{\partial x^2\partial y^4} + \frac{\partial^6\varphi}{\partial y^6}\Big]_{ij} + O(h^6)$$

$$\dots(12)$$

Clearly, to get a compact $O(h^6)$ approximation, we require compact expressions for the four derivative of order 6 in Eq.(12) . This can actually be done by further differentiating (1). The required expressions are

$$\frac{\partial^6\varphi}{\partial x^6} = \frac{\partial^4 f}{\partial x^4} - \frac{\partial^6\varphi}{\partial x^4\partial y^2} \qquad \dots(13)$$

$$\frac{\partial^6\varphi}{\partial y^6} = \frac{\partial^4 f}{\partial y^4} - \frac{\partial^6\varphi}{\partial x^2\partial y^4} \qquad \dots(14)$$

And

$$\frac{\partial^6\varphi}{\partial x^4\partial y^2} + \frac{\partial^6\varphi}{\partial x^2\partial y^4} = \frac{\partial^4 f}{\partial x^2\partial y^2} \qquad \dots(15)$$

The key here is that we can use Eqs. (13) and (14) to algebraically eliminate all the derivatives of φ. The two dimensions, $O(h^6)$ compact approximation to (1) is therefore

$$\delta_x^2\varphi_{ij} + \delta_y^2\varphi_{ij} + \frac{h^2}{6}\delta_x^2\delta_y^2\varphi_{ij} - \tau_{ij} = f_{ij} + \frac{h^2}{12}(\delta_x^2 f_{ij} + \delta_y^2 f_{ij}) + \frac{h^4}{360}(\delta_x^4 f_{ij} + \delta_y^4 f_{ij}) + \frac{h^4}{90}\delta_x^2\delta_y^2 f_{ij}$$

$$\dots(16)$$

Where $\tau_{ij} = O(h^6)$

Equation (16) is called a high order compact difference scheme of order 6 (HOC-6).

1.2 Three Dimension

Now, we are interested in the high accuracy numerical solution of three dimensional (3D) Poisson equation of the form

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = f(x,y), \qquad (x,y,z) \in \Omega \qquad \qquad …(17)$$

Where Ω is a rectangular domain, or a union of rectangular domains, with suitable boundary condition defined on $\partial\Omega$. The solution $\varphi(x,y,z)$ and the forcing function f(x, y, z) are assumed to be sufficiently smooth and have required continuous partial derivatives.

For convenience, let us consider a rectangular domain $\Omega = [0,Lx] \times [0,Ly] \times [0,Lz]$. Here subscripts are obviously not derivatives. We discretize Ω with uniform mesh sizes Δx, Δy and Δz, respectively in the x , y and z coordinate directions. Denote Nx = Lx/Δx , Ny =Ly/Δy and Nz =Lz/Δz the numbers of uniform intervals along the x , y and z coordinate directions ,respectively . The mesh point are (x_i, y_j, z_k) with $x_i = i\Delta x$ and $y_j = j\Delta y$, $z_k = k\Delta z$, $0 \le i \le Nx$, $0 \le j \le Ny$, $0 \le k \le N_z$.

In the sequel, we may also use the index pair (i, j, k) to represent the mesh point (x_i, y_i, z_i).In this paper we take $\Delta x = \Delta y = \Delta z = h$ .also Nx = Ny=Nz = N .
The standard second order central difference operators define at grid point (i, j, z) can be written as

$$\left. \begin{aligned} \delta_x^2 \varphi_{ijk} &= \frac{\varphi_{i+1jk} - 2\varphi_{ijk} + \varphi_{i-1jk}}{h^2} \\[2mm] \delta_y^2 \varphi_{ijk} &= \frac{\varphi_{ij+1k} - 2\varphi_{ijk} + \varphi_{ij-1k}}{h^2} \\[2mm] \delta_z^2 \varphi_{ijk} &= \frac{\varphi_{ijk+1} - 2\varphi_{ijk} + \varphi_{ijk-1}}{h^2} \end{aligned} \right\} \qquad …(18)$$

The derivatives in Eq.(17) can be approximated to second order accuracy as

$$\left. \begin{aligned} \frac{\partial^2 \varphi}{\partial x^2}\bigg|_{ijk} &= \delta_x^2 \varphi_{ijk} - \frac{h^2}{12}\frac{\partial^4 \varphi_{ijk}}{\partial x^4} - \frac{h^4}{360}\frac{\partial^6 \varphi_{ijk}}{\partial x^6} + O(h^6) \\[2mm] \frac{\partial^2 \varphi}{\partial y^2}\bigg|_{ijk} &= \delta_y^2 \varphi_{ijk} - \frac{h^2}{12}\frac{\partial^4 \varphi_{ijk}}{\partial y^4} - \frac{h^4}{360}\frac{\partial^6 \varphi_{ijk}}{\partial y^6} + O(h^6) \\[2mm] \frac{\partial^2 \varphi}{\partial z^2}\bigg|_{ijk} &= \delta_z^2 \varphi_{ijk} - \frac{h^2}{12}\frac{\partial^4 \varphi_{ijk}}{\partial z^4} - \frac{h^4}{360}\frac{\partial^6 \varphi_{ijk}}{\partial z^6} + O(h^6) \end{aligned} \right\} \qquad …(19)$$

The relative simplicity of the Poisson equation makes it a good candidate for our first attempt at an HOC scheme in three dimensions. The central difference scheme for (1) in three dimensions is

$$\delta_x^2 \varphi_{ijk} + \delta_y^2 \varphi_{ijk} + \delta_z^2 \varphi_{ijk} - \tau_{ijk} = f_{ijk}. \qquad \qquad \ldots(20)$$

Where the truncation error has the form

$$\tau_{ijk} = \frac{h^2}{12}[\frac{\partial^4 \varphi}{\partial x^4} + \frac{\partial^4 \varphi}{\partial y^4} + \frac{\partial^4 \varphi}{\partial z^4}]_{ijk} + \frac{h^4}{360}[\frac{\partial^6 \varphi}{\partial x^6} + \frac{\partial^6 \varphi}{\partial y^6} + \frac{\partial^6 \varphi}{\partial z^6}]_{ijk} + O(h^6) \qquad \ldots(21)$$

Equation (20) is called a Central Difference Scheme (CDS). We have include both $O(h^2)$ and $O(h^4)$ term in Eq.(21) because we wish to approximate all of them in order to yield an $O(h^6)$ scheme.

To obtain compact approximation to the $O(h^2)$ terms in Eq.(21), we take the appropriate derivatives of Eq.(17) to write

$$\frac{\partial^4 \varphi}{\partial x^4} = \frac{\partial^2 f}{\partial x^2} - \frac{\partial^4 \varphi}{\partial x^2 \partial y^2} - \frac{\partial^4 \varphi}{\partial x^2 \partial z^2} \qquad \qquad \ldots(22)$$

$$\frac{\partial^4 \varphi}{\partial y^4} = \frac{\partial^2 f}{\partial y^2} - \frac{\partial^4 \varphi}{\partial x^2 \partial y^2} - \frac{\partial^4 \varphi}{\partial y^2 \partial z^2} \qquad \qquad \ldots(23)$$

$$\frac{\partial^4 \varphi}{\partial z^4} = \frac{\partial^2 f}{\partial z^2} - \frac{\partial^4 \varphi}{\partial x^2 \partial z^2} - \frac{\partial^4 \varphi}{\partial y^2 \partial z^2} \qquad \qquad \ldots(24)$$

Substituting Eq.(22) , Eq.(23) and Eq.(24) into Eq.(21) we obtain

$$\tau_{ijk} = \frac{h^2}{12}[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} - 2\frac{\partial^4 \varphi}{\partial y^2 \partial z^2} - 2\frac{\partial^4 \varphi}{\partial x^2 \partial z^2}]_{ijk} + \frac{h^4}{360}[\frac{\partial^6 \varphi}{\partial x^6} + \frac{\partial^6 \varphi}{\partial y^6} + \frac{\partial^6 \varphi}{\partial z^6}]_{ijk}$$
$$\ldots(25)$$

Note that all term on the right hand side of Eq.(25) have compact $O(h^2)$ approximations at noted ij, and the approximation of these term has the following form:

$$\left.\frac{\partial^4 \varphi}{\partial x^2 \partial y^2}\right|_{ijk} = \delta_x^2 \delta_y^2 \varphi_{ijk} - \frac{h^2}{12}\left[\left(\frac{\partial^6 \varphi}{\partial x^4 \partial y^2} + \frac{\partial^6 \varphi}{\partial x^2 \partial y^4}\right)\right]_{ijk} + O(h^4) \qquad \ldots(26)$$

$$\left.\frac{\partial^4 \varphi}{\partial y^2 \partial z^2}\right|_{ijk} = \delta_y^2 \delta_z^2 \varphi_{ijk} - \frac{h^2}{12}\left[\left(\frac{\partial^6 \varphi}{\partial y^4 \partial z^2} + \frac{\partial^6 \varphi}{\partial y^2 \partial z^4}\right)\right]_{ijk} + O(h^4) \qquad \ldots(27)$$

$$\left.\frac{\partial^4 \varphi}{\partial x^2 \partial z^2}\right|_{ijk} = \delta_x^2 \delta_z^2 \varphi_{ijk} - \frac{h^2}{12}\left[\left(\frac{\partial^6 \varphi}{\partial x^4 \partial z^2} + \frac{\partial^6 \varphi}{\partial x^2 \partial z^4}\right)\right]_{ijk} + O(h^4) \qquad \ldots(28)$$

Also,

$$\partial_x^2\partial_y^2\varphi_{ijk} = \frac{1}{h^4}[4\varphi_{ijk} - 2(\varphi_{i-1jk} + \varphi_{i+1jk} + \varphi_{ij-1k} + \varphi_{ij+1k}) + \varphi_{i-1j-1k} + \varphi_{i+1j-1k}$$

$$+ \varphi_{i+1j+1k} + \varphi_{i-1j+1k}] \qquad\qquad ...(29)$$

$$\partial_y^2\partial_z^2\varphi_{ijk} = \frac{1}{h^4}[4\varphi_{ijk} - 2(\varphi_{ij-1k} + \varphi_{ij+1k} + \varphi_{ijk-1} + \varphi_{ijk+1}) + \varphi_{ij-1k-1} + \varphi_{ij+1k-1}$$

$$+ \varphi_{ij+1k+1} + \varphi_{ij-1k+1}] \qquad\qquad ...(30)$$

$$\partial_x^2\partial_z^2\varphi_{ijk} = \frac{1}{h^4}[4\varphi_{ijk} - 2(\varphi_{i-1jk} + \varphi_{i+1jk} + \varphi_{ijk-1} + \varphi_{ijk+1}) + \varphi_{i-1jk-1} + \varphi_{i+1jk-1}$$

$$+ \varphi_{i+1jk+1} + \varphi_{i-1jk+1}] \qquad\qquad ...(31)$$

We can easily get an $O(h^4)$ method in the same way as before by substituting difference expressions for the $O(h^2)$ term in Eq.(25) and including these in the finite difference approximation (4). The resulting higher-order scheme follows from

$$[\delta_x^2 + \delta_y^2 + \delta_z^2]\varphi_{ijk} + \frac{h^2}{6}[\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_x^2\delta_z^2]\varphi_{ijk} - \tau_{ijk} = f_{ijk} + \frac{h^2}{12}(\delta_x^2 + \delta_y^2 + \delta_z^2)f_{ijk}$$

$$...(32)$$

Equation (32) is called a high order compact difference scheme of order 4 (HOC-4). Note that Eq.(32) corresponds to a 19-point stencil, encompassing all the adjacent nodes of the mesh located on the three grid planes that intersect the node ijk, but not the corner points of the surrounding cube. We can, however, obtain a $O(h^6)$ scheme for this governing equation. Note that the approximation to the cross derivative of φ in Eq.(25) introduces additional $O(h^4)$ term that we must be careful to include in our derivation of an $O(h^6)$ scheme. More specifically, substituting the finite difference expression for the cross derivative of φ and its truncation error terms into Eq.(25) give us

$$\tau_{ij} = \frac{h^4}{72}[\frac{\partial^6\varphi}{\partial x^4\partial y^2} + \frac{\partial^6\varphi}{\partial x^2\partial y^4} + \frac{\partial^6\varphi}{\partial y^4\partial z^2} + \frac{\partial^6\varphi}{\partial y^2\partial z^4}\frac{\partial^6\varphi}{\partial x^4\partial z^2} + \frac{\partial^6\varphi}{\partial x^2\partial z^4}] + \frac{h^4}{360}[\frac{\partial^6\varphi}{\partial x^6} + \frac{\partial^6\varphi}{\partial y^6} +$$

$$\frac{\partial^6\varphi}{\partial z^6}]_{ijk} + O(h^6) \qquad\qquad ...(33)$$

Clearly, to get a compact $O(h^6)$ approximation, we require compact expressions for the four derivative of order six in Eq.(33) . This can actually be done by further differentiating (17). The required expressions are

$$\frac{\partial^6\varphi}{\partial x^6} = \frac{\partial^4 f}{\partial x^4} - \frac{\partial^6\varphi}{\partial x^4\partial y^2} - \frac{\partial^6\varphi}{\partial x^4\partial z^2} \qquad\qquad ...(34)$$

$$\frac{\partial^6\varphi}{\partial y^6} = \frac{\partial^4 f}{\partial y^4} - \frac{\partial^6\varphi}{\partial x^2\partial y^4} - \frac{\partial^6\varphi}{\partial y^4\partial z^2} \qquad\qquad ...(35)$$

$$\frac{\partial^6\varphi}{\partial z^6} = \frac{\partial^4 f}{\partial z^4} - \frac{\partial^6\varphi}{\partial x^2\partial z^4} - \frac{\partial^6\varphi}{\partial y^2\partial z^4} \qquad \ldots(36)$$

And

$$\frac{\partial^6\varphi}{\partial x^2\partial y^2\partial z^2} = \frac{\partial^4 f}{\partial y^2\partial z^2} - \frac{\partial^6\varphi}{\partial y^4\partial z^2} - \frac{\partial^6\varphi}{\partial y^2\partial z^4} \qquad \ldots(37)$$

$$\frac{\partial^6\varphi}{\partial x^2\partial y^2\partial z^2} = \frac{\partial^4 f}{\partial x^2\partial z^2} - \frac{\partial^6\varphi}{\partial x^4\partial z^2} - \frac{\partial^6\varphi}{\partial x^2\partial z^4} \qquad \ldots(38)$$

$$\frac{\partial^6\varphi}{\partial x^2\partial y^2\partial z^2} = \frac{\partial^4 f}{\partial x^2\partial y^2} - \frac{\partial^6\varphi}{\partial x^4\partial y^2} - \frac{\partial^6\varphi}{\partial x^2\partial y^4} \qquad \ldots(39)$$

The key here is that we can use Eqs. (34) - (39) to algebraically eliminate all the derivatives of φ expect $\dfrac{\partial^6\varphi}{\partial x^2\partial y^2\partial z^2}$, which has a compact approximation as follows

$$\delta_x^2\delta_y^2\delta_z^2\varphi_{ijk} = \frac{1}{h^6}[-8\varphi_{ijk} + 4(\varphi_{i-1jk} + \varphi_{i+1jk} + \varphi_{ij-1k} + \varphi_{ij+1k} + \varphi_{ijk-1} + \varphi_{ijk+1}) -$$

$$2(\varphi_{ij-1k-1} + \varphi_{ij+1k-1} + \varphi_{i-1jk-1} + \varphi_{i+1jk-1} + \varphi_{i-1j-1k} + \varphi_{i+1j-1k} + \varphi_{i+1j+1k}) -$$

$$2(\varphi_{ijk+1} + \varphi_{ij-1k+1} + \varphi_{ij+1k+1} + \varphi_{i-1jk+1} + \varphi_{i+1jk+1}) + \varphi_{i-1j-1k-1} + \varphi_{i+1j-1k-1} +$$

$$\varphi_{i-1j+1k-1} + \varphi_{i+1j+1k-1} + \varphi_{i-1j-1k+1} + \varphi_{i+1j-1k+1} + \varphi_{i-1j+1k+1} + \varphi_{i+1j+1k+1}]$$
$$\ldots(40)$$

This operator brings the eight corner points into our $O(h^6)$ approximation, which follows from

$$[\delta_x^2 + \delta_y^2 + +\delta_z^2]\varphi_{ijk} + \frac{h^2}{6}[\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_x^2\delta_z^2]\varphi_{ijk} + \frac{h^4}{30}\delta_x^2\delta_y^2\delta_z^2\varphi_{ijk} - \tau_{ijk} = f_{ij} + \frac{h^2}{12}[\delta_x^2$$

$$+ \delta_y^2 + \delta_z^2]f_{ijk} + \frac{h^4}{360}[\delta_x^4 + \delta_y^4 + \delta_z^4]f_{ijk} + \frac{h^4}{90}[\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_x^2\delta_z^2]f_{ijk}$$
$$\ldots(41)$$

Where $\tau_{ijk} = O(h^6)$

Equation (41) is called a high order compact difference scheme of order 6 (HOC-6).

2- Compact Operator Scheme

For an illustration purpose, we consider the analogous one dimensional problem first

$$\frac{d^2\varphi}{dx^2} = f \qquad \ldots(42)$$

The second derivative $\dfrac{d^2\varphi}{dx^2}$ at a grid i can be approximated using central difference operator as

$$\left.\dfrac{d^2\Phi}{dx^2}\right|_i = \delta_x^2\varphi_{ij} - \dfrac{h^2}{12}\dfrac{d^4\Phi_i}{dx^4} - \dfrac{h^4}{360}\dfrac{d^6\Phi_i}{dx^6} + O(h^6) \qquad \ldots(43)$$

Dropping the last two terms in the right hand side of Eq.(43) yields the standard second order difference scheme. The idea behind the high order compact approximation scheme is to approximate the term $\dfrac{d^4\Phi_i}{dx^4}$ to second order accuracy and to achieve an overall truncation accuracy of fourth order in (43). To this end, we double differentiate Eq.(42) to have

$$\dfrac{d^4\Phi}{dx^4} = \dfrac{d^2f}{dx^2} \qquad \ldots(44)$$

Substituting (44) into (43) yields a fourth compact approximation for the second derivative

$$\left.\dfrac{d^2\Phi}{dx^2}\right|_i = \delta_x^2\varphi_i - \dfrac{h^2}{12}\dfrac{d^2f_i}{dx^2} - \dfrac{h^4}{360}\dfrac{d^6\Phi_i}{dx^6} + O(h^6) \qquad \ldots(45)$$

Hence the fourth order compact approximation of the one dimension Poisson (42) is

$$\delta_x^2\varphi_i - \dfrac{h^2}{12}\dfrac{d^2f_i}{dx^2} = f_i \qquad \ldots(46)$$

We note that this fourth order compact approximation is different from the second order approximation only in the approximation of the right hand side function f. We can rewrite (46) as

$$\delta_x^2\varphi_i = (1+\dfrac{h^2}{12}\dfrac{d^2}{dx^2})f_i \qquad \ldots(47)$$

Also, by using the same process we get

$$\dfrac{d^6\Phi}{dx^6} = \dfrac{d^4f}{dx^4} \qquad \ldots(48)$$

Substituting (48) into (43) yields a sixth compact approximation for the second derivative

$$\left.\dfrac{d^2\Phi}{dx^2}\right|_i = \delta_x^2\varphi_i - \dfrac{h^2}{12}\dfrac{d^2f_i}{dx^2} - \dfrac{h^4}{360}\dfrac{d^4f_i}{dx^4} + O(h^6) \qquad \ldots(49)$$

Hence the sixth order compact approximation of the one dimension Poisson (42) is

$$\delta_x^2\varphi_i = (1+\dfrac{h^2}{12}\dfrac{d^2}{dx^2}+\dfrac{h^4}{360}\dfrac{d^4}{dx^4})f_i \qquad \ldots(50)$$

We denote to the fourth and sixth approximation as

$$A_x\varphi_i = L_xf_i \qquad \ldots(51)$$

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Where $A_x = \delta_x^2$ in fourth and sixth order and $L_x = (1 + \dfrac{h^2}{12}\dfrac{d^2}{dx^2})$ in fourth order

and $L_x = (1 + \dfrac{h^2}{12}\dfrac{d^2}{dx^2} + \dfrac{h^4}{360}\dfrac{d^4}{dx^4})$ in sixth order.

System (51) can be rewrite as

$$L_x^{-1} A_x \varphi_i = f_i \qquad \qquad \text{…(52)}$$

Here the operator L_x^{-1} has symbolic meaning only. In application, the fourth and sixth order compact difference scheme is given by Eq.(51), not by Eq.(52). Analogous symbolic fourth and sixth order compact approximation operator can be obtained for the y variable. For two dimension, we can apply the symbolic fourth order compact approximation operators to the second derivative $\dfrac{d^2\varphi}{dx^2}$ and $\dfrac{d^2\varphi}{dy^2}$ in Eq.(42), respectively.

This yields symbolically

$$L_x^{-1} A_x \varphi_{ij} + L_y^{-1} A_y \varphi_{ij} = f_{ij} \qquad \qquad \text{…(53)}$$

Applying to both sides of Eq.(53) with the operator $L_x L_y$, we obtain

$$L_y A_x \varphi_{ij} + L_x A_y \varphi_{ij} = L_x L_y f_{ij} \qquad \qquad \text{…(54)}$$

Applying the symbolic operators for the fourth order compact approximation, we have

$$(1 + \frac{h^2}{12}\frac{\partial^2}{\partial y^2})\delta_x^2 \varphi_{ij} + (1 + \frac{h^2}{12}\frac{\partial^2}{\partial x^2})\delta_y^2 \varphi_{ij} = (1 + \frac{h^2}{12}\frac{\partial^2}{\partial x^2})(1 + \frac{h^2}{12}\frac{\partial^2}{\partial y^2})f_{ij} \qquad \text{…(55)}$$

Simplifying (55) we get

$$\delta_x^2 \varphi_{ij} + \delta_y^2 \varphi_{ij} + \frac{h^2}{6}\delta_x^2\delta_y^2 \varphi_{ij} = f_{ij} + \frac{h^2}{12}(\delta_x^2 f_{ij} + \delta_y^2 f_{ij}) + \frac{h^4}{144}\delta_x^2\delta_y^2 f_{ij} \qquad \text{…(56)}$$

We called Eq.(56) a modified high order compact difference scheme of order 4 (MHOC-4). Applying the symbolic operators for the sixth order compact approximation, we have

$$(1 + \frac{h^2}{12}\frac{\partial^2}{\partial y^2} + \frac{h^4}{360}\frac{\partial^4}{\partial y^4})\delta_x^2\varphi_{ij} + (1 + \frac{h^2}{12}\frac{\partial^2}{\partial x^2} + \frac{h^4}{360}\frac{\partial^4}{\partial x^4})\delta_y^2\varphi_{ij} = (1 + \frac{h^2}{12}\frac{\partial^2}{\partial x^2} +$$

$$\frac{h^4}{360}\frac{\partial^4}{\partial x^4})(1 + \frac{h^2}{12}\frac{\partial^2}{\partial y^2} + \frac{h^4}{360}\frac{\partial^4}{\partial y^4})f_{ij} \qquad \qquad \text{...(57)}$$

Simplifying (57) and use Eq.(15) we get

$$\delta_x^2 \varphi_{ij} + \delta_y^2 \varphi_{ij} + \frac{h^2}{6}\delta_x^2\delta_y^2 \varphi_{ij} = f_{ij} + \frac{h^2}{12}(\delta_x^2 f_{ij} + \delta_y^2 f_{ij}) + \frac{h^4}{360}(\delta_x^4 f_{ij} + \delta_y^4 f_{ij}) +$$

$$\frac{h^4}{90}\delta_x^2\delta_y^2 f_{ij} + \frac{h^6}{4320}(\delta_x^2 \delta_y^4 f_{ij} + \delta_x^4 \delta_y^2 f_{ij}) + \frac{h^8}{129600}\delta_x^4\delta_y^4 f_{ij} \qquad \text{...(58)}$$

We called Eq.(58) a modified high order compact difference scheme of order 6 (MHOC-6).

For three dimension, we can apply the symbolic fourth order compact approximation operators to the second derivative $\dfrac{d^2\varphi}{dx^2}$, $\dfrac{d^2\varphi}{dy^2}$ and $\dfrac{d^2\varphi}{dz^2}$ in Eq.(51), respectively. This yields symbolically

$$L_x^{-1}A_x\varphi_{ijk} + L_y^{-1}A_y\varphi_{ijk} + L_z^{-1}A_z\varphi_{ijk} = f_{ijk} \qquad \ldots(59)$$

Applying to both sides of Eq.(59) with the operator $L_x L_y$, we obtain

$$L_y L_z A_x\varphi_{ijk} + L_x L_z A_y\varphi_{ijk} + L_x L_y A_z\varphi_{ijk} = L_x L_y L_z f_{ijk} \qquad \ldots(60)$$

Applying the symbolic operators for the fourth order compact approximation, we have

$$(1+\frac{h^2}{12}\frac{\partial^2}{\partial y^2})(1+\frac{h^2}{12}\frac{\partial^2}{\partial z^2})\delta_x^2\varphi_{ijk} + (1+\frac{h^2}{12}\frac{\partial^2}{\partial x^2})(1+\frac{h^2}{12}\frac{\partial^2}{\partial z^2})\delta_y^2\varphi_{ijk} + (1+\frac{h^2}{12}\frac{\partial^2}{\partial x^2})$$

$$(1+\frac{h^2}{12}\frac{\partial^2}{\partial y^2})\delta_z^2\varphi_{ijk} = (1+\frac{h^2}{12}\frac{\partial^2}{\partial x^2})(1+\frac{h^2}{12}\frac{\partial^2}{\partial y^2})(1+\frac{h^2}{12}\frac{\partial^2}{\partial z^2})f_{ijk}$$

$$\ldots(61)$$

Simplifying (61) we get

$$(\delta_x^2 + \delta_y^2 + \delta_z^2)\varphi_{ijk} + \frac{h^2}{6}(\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_x^2\delta_z^2)\varphi_{ijk} + \frac{h^4}{36}\delta_x^2\delta_y^2\delta_z^2\varphi_{ijk} = f_{ijk} +$$

$$\frac{h^2}{12}(\delta_x^2 + \delta_y^2 + \delta_z^2)f_{ijk} + \frac{h^4}{144}(\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_x^2\delta_z^2)f_{ijk} + \frac{h^6}{1728}\delta_x^2\delta_y^2\delta_z^2 f_{ijk}$$

$$\ldots(62)$$

We called Eq.(62) a modified high order compact difference scheme of order 4 (MHOC-4). Applying the symbolic operators for the sixth order compact approximation, we have

$$(1+\frac{h^2}{12}\frac{\partial^2}{\partial y^2}+\frac{h^4}{360}\frac{\partial^4}{\partial y^4})(1+\frac{h^2}{12}\frac{\partial^2}{\partial z^2}+\frac{h^4}{360}\frac{\partial^4}{\partial z^4})\delta_x^2\varphi_{ijk} + (1+\frac{h^2}{12}\frac{\partial^2}{\partial x^2}+\frac{h^4}{360}\frac{\partial^4}{\partial x^4})(1+$$

$$\frac{h^2}{12}\frac{\partial^2}{\partial z^2}+\frac{h^4}{360}\frac{\partial^4}{\partial z^4})\delta_y^2\varphi_{ijk} + (1+\frac{h^2}{12}\frac{\partial^2}{\partial x^2}+\frac{h^4}{360}\frac{\partial^4}{\partial x^4})(1+\frac{h^2}{12}\frac{\partial^2}{\partial y^2}+\frac{h^4}{360}\frac{\partial^4}{\partial y^4})\delta_z^2\varphi_{ijk}$$

$$= (1+\frac{h^2}{12}\frac{\partial^2}{\partial x^2}+\frac{h^4}{360}\frac{\partial^4}{\partial x^4})(1+\frac{h^2}{12}\frac{\partial^2}{\partial y^2}+\frac{h^4}{360}\frac{\partial^4}{\partial y^4})(1+\frac{h^2}{12}\frac{\partial^2}{\partial z^2}+\frac{h^4}{360}\frac{\partial^4}{\partial z^4})f_{ijk}$$

$$\ldots(63)$$

Simplifying (63) and removing the term which include the derivatives of φ except the derivatives $\delta_x^2, \delta_y^2, \delta_z^2, \delta_x^2\delta_y^2, \delta_y^2\delta_z^2, \delta_x^2\delta_z^2$ and $\delta_x^2\delta_y^2\delta_z^2$ we get

$$(\delta_x^2 + \delta_y^2 + \delta_z^2)\varphi_{ijk} + \frac{h^2}{6}(\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_x^2\delta_z^2)\varphi_{ijk} + \frac{h^4}{48}\delta_x^2\delta_y^2\delta_z^2\varphi_{ijk} = f_{ijk} + \frac{h^2}{12}(\delta_x^2$$

$$+ \delta_y^2 + \delta_z^2)f_{ijk} + \frac{h^4}{360}(\delta_x^4 + \delta_y^4 + \delta_z^4)f_{ijk} + \frac{h^4}{144}(\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_x^2\delta_z^2)f_{ijk} + \frac{h^6}{4320}(\delta_x^4\delta_y^2$$

$$+ \delta_y^4\delta_z^2 + \delta_x^4\delta_z^2 + \delta_x^2\delta_y^4 + \delta_y^2\delta_z^4 + \delta_x^2\delta_z^4)f_{ijk} + \frac{h^6}{1728}\delta_x^2\delta_y^2\delta_z^2 f_{ijk} + \frac{h^8}{51840}(\delta_x^2\delta_y^4\delta_z^2 + \delta_x^2\delta_y^2\delta_z^4$$

$$+ \delta_x^4\delta_y^2\delta_z^2)f_{ijk} + \frac{h^8}{129600}(\delta_x^4\delta_y^4 + \delta_y^4\delta_z^4 + \delta_x^4\delta_z^4)f_{ijk} + \frac{h^{10}}{1555200}(\delta_x^2\delta_y^4\delta_z^4 + \delta_x^4\delta_y^2\delta_z^4$$

$$+ \delta_x^4\delta_y^4\delta_z^2)f_{ijk} + \frac{h^{12}}{46656000}\delta_x^4\delta_y^4\delta_z^4 f_{ijk}$$

$$\dots(64)$$

We called Eq.(64) a modified high order compact difference scheme of order 6 (MHOC-6).

3. Numerical Solver

we will now show the procedure and algorithm that is used to solve the general linear system of the sixth order compact finite difference scheme as example, where this system can be written in the following general form :

$$[\delta_x^2 + \delta_y^2 + \delta_z^2]\varphi_{ijk} + \frac{h^2}{6}[\delta_x^2\delta_y^2 + \delta_y^2\delta_z^2 + \delta_x^2\delta_z^2]\varphi_{ijk} + \frac{h^4}{30}\delta_x^2\delta_y^2\delta_z^2\varphi_{ijk} = F_{ijk} \qquad \dots(65)$$

And to solve the linear system (65) , we use the Gauss-Seidel iterative method .Also to apply this iterative method to the above linear system , we must arrange this system . Firstly this system can be written as follows:-

Substituting the value of $\partial_x^2, \partial_y^2, \partial_z^2, \partial_x^2\partial_y^2, \partial_x^2\partial_z^2, \partial_y^2\partial_z^2$ and, $\partial_x^2\partial_y^2\partial_z^2$ in Eq(65) yield

$$[\frac{\varphi_{i+1jk} - 2\varphi_{ijk} + \varphi_{i-1jk}}{h^2}] + [\frac{\varphi_{ij+1k} - 2\varphi_{ijk} + \varphi_{ij-1k}}{h^2}] + [\frac{\varphi_{ijk+1} - 2\varphi_{ijk} + \varphi_{ijk-1}}{h^2}] +$$

$$\frac{1}{6h^2}[\{4\varphi_{ij} - 2(\varphi_{i-1j} + \varphi_{i+1j} + \varphi_{ij-1} + \varphi_{ij+1}) + \varphi_{i-1j-1} + \varphi_{i+1j-1} + \varphi_{i+1j+1} + \varphi_{i-1j+1}\} +$$

$$\{4\varphi_{ijk} - 2(\varphi_{ij-1k} + \varphi_{ij+1k} + \varphi_{ijk-1} + \varphi_{ijk+1}) + \varphi_{ij-1k-1} + \varphi_{ij+1k-1} + \varphi_{ij+1k+1} + \varphi_{ij-1k+1}\} +$$

$$\{4\varphi_{ijk} - 2(\varphi_{i-1jk} + \varphi_{i+1jk} + \varphi_{ijk-1} + \varphi_{ijk+1}) + \varphi_{i-1jk-1} + \varphi_{i+1jk-1} + \varphi_{i+1jk+1} + \varphi_{i-1jk+1}\}] +$$

$$\frac{1}{30h^2}[-8\varphi_{ijk} + 4(\varphi_{i-1jk} + \varphi_{i+1jk} + \varphi_{ij-1k} + \varphi_{ij+1k} + \varphi_{ijk-1} + \varphi_{ijk+1}) - 2(\varphi_{ij-1k-1} + \varphi_{ij+1k-1} +$$

$$\varphi_{ij-1k-1} + \varphi_{i+1jk-1} + \varphi_{i-1j-1k} + \varphi_{i+1j-1k} + \varphi_{i+1j+1k}) - 2(\varphi_{i-1j+1k} + \varphi_{ij-1k+1} + \varphi_{ij+1k+1} +$$

$$\varphi_{i-1jk+1} + \varphi_{i+1jk+1}) + \varphi_{i-1j-1k-1} + \varphi_{i+1j-1k-1} + \varphi_{i-1j+1k-1} + \varphi_{i+1j+1k-1} + \varphi_{i-1j-1k+1} +$$

$$\varphi_{i+1j-1k+1} + \varphi_{i-1j+1k+1} + \varphi_{i+1j+1k+1}] = F_{ijk}$$

$$\dots(66)$$

Simplifying Eq. (66) and applying the Gauss – Seidel on it we get the following iterative formula

$$
\begin{aligned}
\varphi_{ijk}^{(n+1)} = \frac{15}{64}&[[\varphi_{i+1jk}^{(n)} + \varphi_{i-1jk}^{(n+1)}] + [\varphi_{ij+1k}^{(n)} + \varphi_{ij-1k}^{(n+1)}] + [\varphi_{ijk+1}^{(n)} + \varphi_{ijk-1}^{(n+1)}] + \\
\frac{1}{6}&[\{-2(\varphi_{i-1jk}^{(n+1)} + \varphi_{i+1jk}^{(n)} + \varphi_{ij-1k}^{(n+1)} + \varphi_{ij+1k}^{(n)}) + \varphi_{i-1j-1k}^{(n+1)} + \varphi_{i+1j-1k}^{(n)} + \varphi_{i+1j+1k}^{(n)} + \varphi_{i-1j+1k}^{(n)}\} + \\
&\{-2(\varphi_{ij-1k}^{(n+1)} + \varphi_{ij+1k}^{(n)} + \varphi_{ijk-1}^{(n+1)} + \varphi_{ijk+1}^{(n)}) + \varphi_{ij-1k-1}^{(n+1)} + \varphi_{ij+1k-1}^{(n)} + \varphi_{ij+1k+1}^{(n)} + \varphi_{ij-1k+1}^{(n)}\} + \\
&\{-2(\varphi_{i-1jk}^{(n+1)} + \varphi_{i+1jk}^{(n)} + \varphi_{ijk-1}^{(n+1)} + \varphi_{ijk+1}^{(n)}) + \varphi_{i-1jk-1}^{(n+1)} + \varphi_{i+1jk-1}^{(n)} + \varphi_{i+1jk+1}^{(n)} + \varphi_{i-1jk+1}^{(n)}\}] + \\
\frac{1}{30}&[4(\varphi_{i-1jk}^{(n+1)} + \varphi_{i+1jk}^{(n)} + \varphi_{ij-1k}^{(n+1)} + \varphi_{ij+1k}^{(n)} + \varphi_{ijk-1}^{(n+1)} + \varphi_{ijk+1}^{(n)}) - 2(\varphi_{ij-1k-1}^{(n+1)} + \varphi_{ij+1k-1}^{(n)} + \\
&\varphi_{ij-1k-1}^{(n+1)} + \varphi_{i+1jk-1}^{(n)} + \varphi_{i-1j-1k}^{(n+1)} + \varphi_{i+1j-1k}^{(n)} + \varphi_{i+1j+1k}^{(n)}) - 2(\varphi_{i-1j+1k}^{(n)} + \varphi_{ij-1k+1}^{(n)} + \varphi_{ij+1k+1}^{(n)} + \\
&\varphi_{i-1jk+1}^{(n)} + \varphi_{i+1jk+1}^{(n)}) + \varphi_{i-1j-1k-1}^{(n+1)} + \varphi_{i+1j-1k-1}^{(n)} + \varphi_{i-1j+1k-1}^{(n)} + \varphi_{i+1j+1k-1}^{(n)} + \varphi_{i-1j-1k+1}^{(n)} + \\
&\varphi_{i+1j-1k+1}^{(n)} + \varphi_{i-1j+1k+1}^{(n)} + \varphi_{i+1j+1k+1}^{(n)}] - h^2 F_{ijk}]
\end{aligned}
$$

$$\dots(67)$$

Where $\varphi_{ijk}^{(n)}$ represents the value of the function φ at the point (x_i, y_j, z_k) and at iterative step (n). Firstly we give an initial condition to the function at every node as follows

$$\varphi_{ijk}^{(0)} = 1, \qquad i, j, k = 0(1)N \qquad\qquad \dots(68)$$

And then calculate the value of the function at the step (1) from Eq.(67) and check the convergence criteria

$$\sum_{k=1}^{N-1} \sum_{j=1}^{N-1} \sum_{i=1}^{N-1} \left| \varphi_{ijk}^{(n+1)} - \varphi_{ijk}^{(n)} \right| < \gamma \qquad\qquad \dots(96)$$

where $\gamma = 10^{-6}$

if they are satisfied stop the computation and out put the results. Otherwise, use the newly obtained results as initial guess and repeat the computation. The absolute error evaluated by using the following equation

$$\text{Error} = \frac{\displaystyle\sum_{k=1}^{N-1} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \left| \varphi_{eijk} - \varphi_{ijk} \right|}{(N-1)^3} \qquad\qquad \dots(70)$$

Where φ_{ijk} represents the approximate value and φ_{eijk} represents the exact value.

4. Numerical Results
4.1 Two Dimension

We now consider two dimension model problems to test the high order compact formulation for Poisson equation.

Test 1: The first test problem

$$\left.\begin{aligned}
\varphi(x,0) = e^{x}, && \varphi(x,1) = e^{x+1}, && 0 \leq x \leq 1, \\
\varphi(0,y) = e^{y}, && \varphi(1,y) = e^{1+y}, && 0 \leq y \leq 1,
\end{aligned}\right\} \quad \ldots(71)$$

Combined with the following forcing function,

$$f = 2\, e^{x+y}$$

The resulting exact solution is

$$\varphi_{ex} = e^{x+y} \qquad\qquad \ldots(72)$$

Surface of the exact solution are shown in Figure 1. Figure 2 show surface for CDS , HOC-4, MHOC-4, HOC-6 and MHOC-6 solution respectively, to test problem 1 with h=1/12. Table 1 show the value of the absolute error computed for different value of N (N=2, 4, 6, 8, 10, 12)
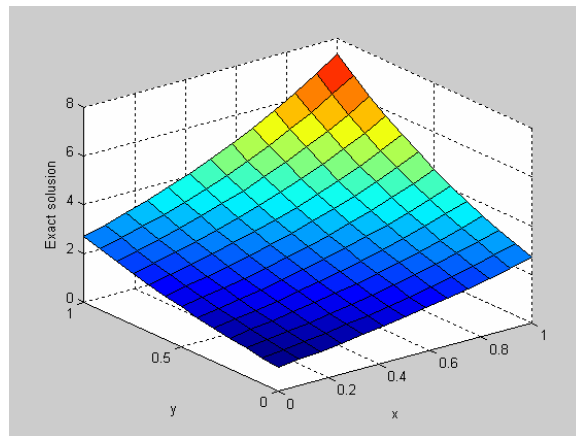

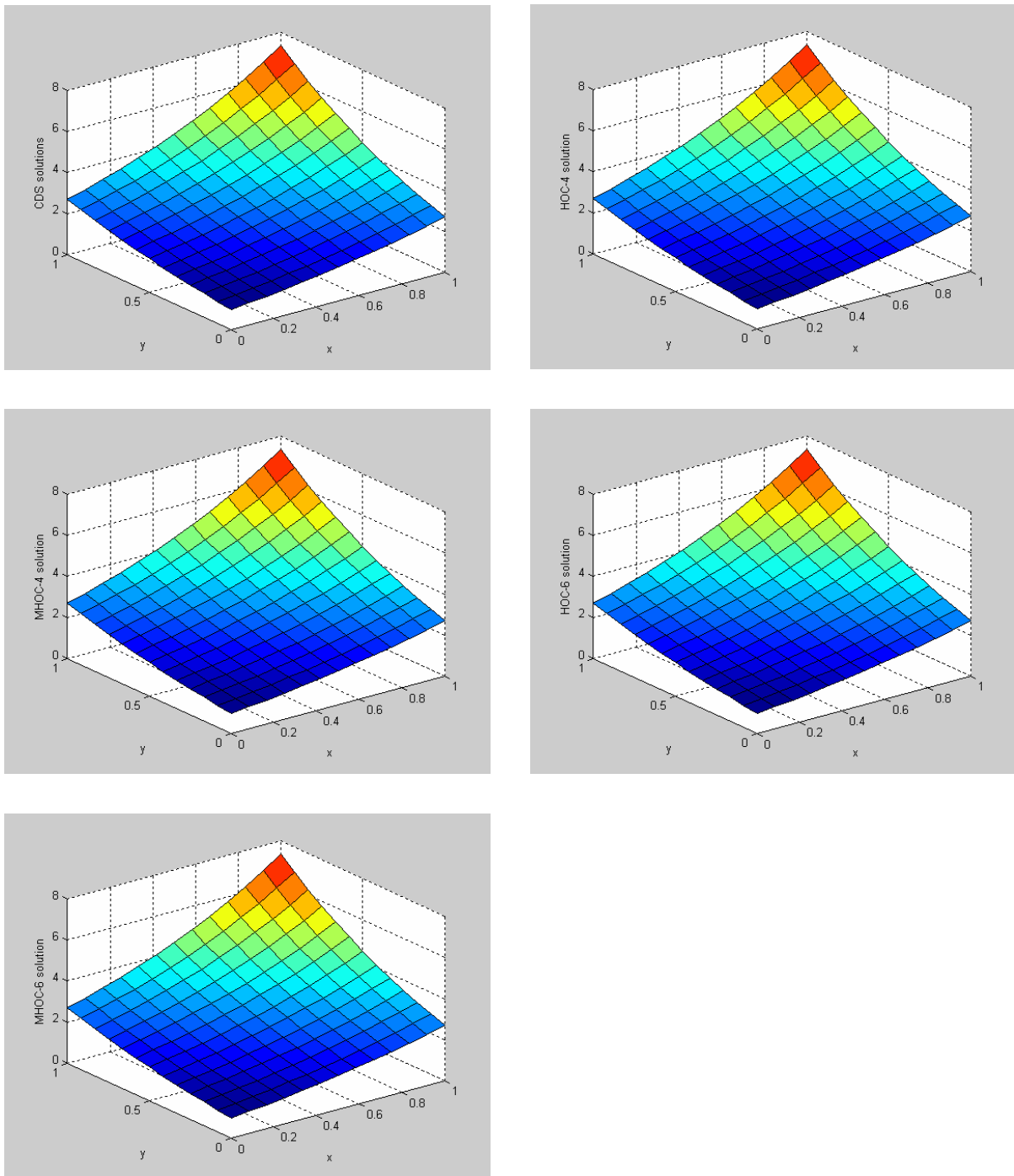Figure 1: Surface of the exact solution to 2D test problem 1 for h=1/12.

Figure 2: Surface of the CDS , HOC-4, MHOC-4, HOC-6 and MHOC-6 solutions to 2D test problem 1 for h=1/12.

Table 1 : The absolute error computed by CDS, HOC-4, MHOC-4, HOC-6 and MHOC-6 for test problem 1.

	N=2	N=4	N=6	N=8	N=10	N=12
CDS	7.138E-03	1.510E-03	6.105E-04	3.245E-04	2.001E-04	1.354E-04
HOC-4	4.318E-04	1.989E-05	3.475E-06	1.028E-06	4.038E-07	1.892E-07
MHOC-4	2.548E-04	1.164E-05	2.030E-06	6.003E-07	2.357E-07	1.104E-07
HOC-6	7.029E-06	8.128E-08	6.306E-09	1.041E-09	2.538E-10	7.558E-11
MHOC-6	4.067E-06	4.685E-08	3.629E-09	5.951E-10	1.417E-10	3.910E-11

Test 2: The second test problem

$$\varphi(x,0) = e^x + 1, \qquad \varphi(x,1) = e^x + e, \qquad 0 \le x \le 1,$$
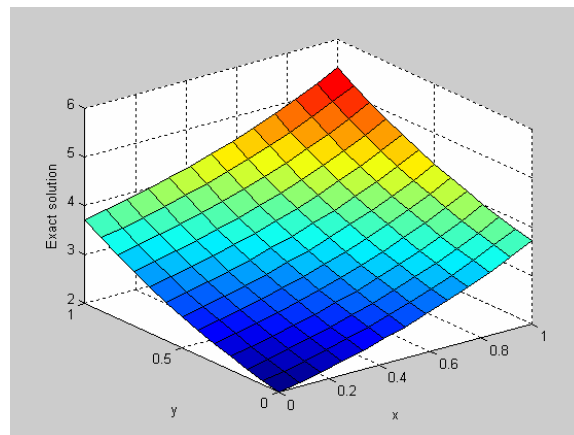$$\varphi(0,y) = e^y + 1, \qquad \varphi(1,y) = e^y + e, \qquad 0 \le y \le 1 \qquad \dots(73)$$

combined with the following forcing function,

$$f = e^x + e^y \qquad \dots(74)$$

The resulting exact solution is

$$\varphi_{ex} = e^x + e^y \qquad \dots(75)$$

Surface of the exact solution are shown in Figure 3. Figure 4 show surface for CDS , HOC-4, MHOC-4, HOC-6 and MHOC-6 solution respectively, to test problem 1 with h=1/12. Table 2 shows the value of the absolute error computed for different value of N (N=2, 4, 6, 8, 10, 12)
.

Figure 3: Surface of the exact solution to 2D test problem 2 for h=1/12.
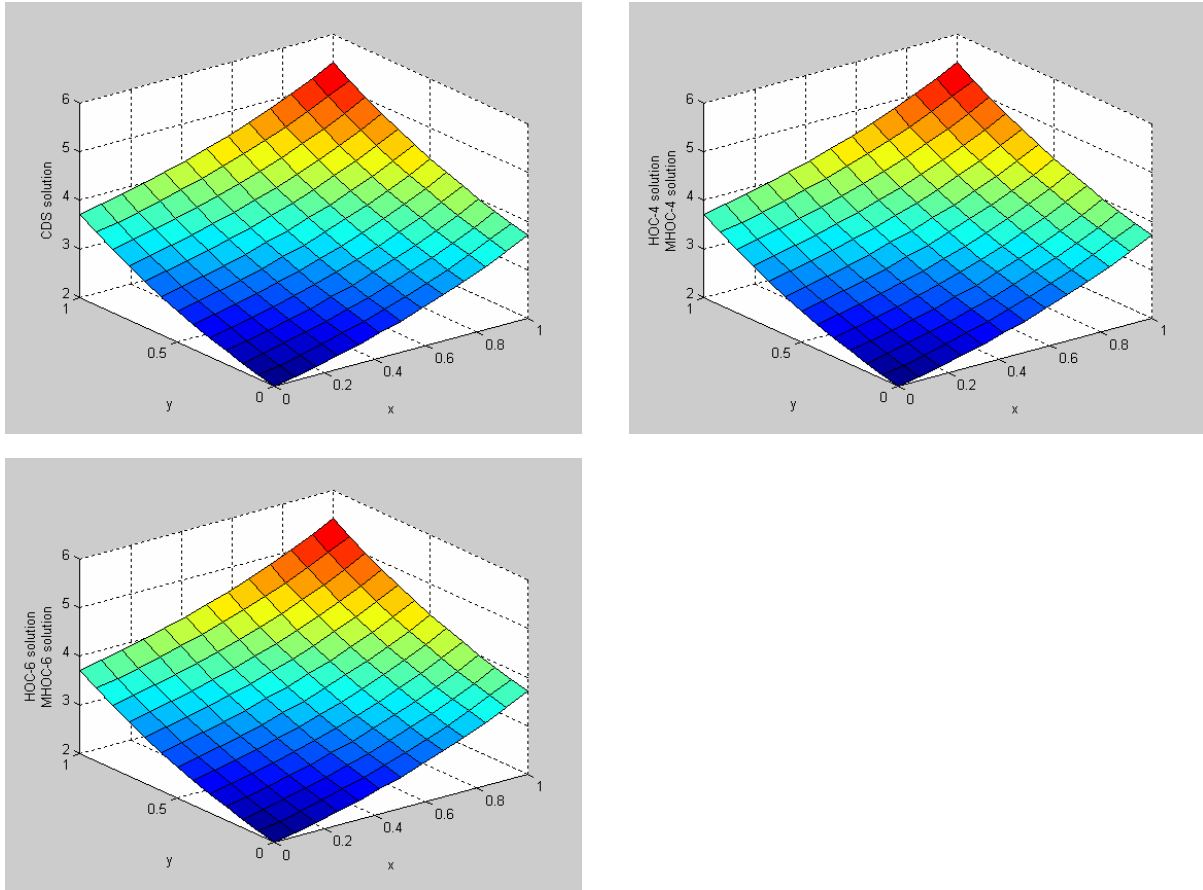
Figure 4: Surface of the CDS , HOC-4, MHOC-4, HOC-6 and MHOC-6 solutions to 2D test problem 2 for h=1/12.

Table 2 : The absolute error computed by CDS, HOC-4, MHOC-4, HOC-6 and MHOC-6 for test problem 2.

	N=2	N=4	N=6	N=8	N=10	N=12
CDS	4.330E-03	8.983E-04	3.619E-04	1.921E-04	1.184E-04	8.006E-05
HOC-4	4.313E-05	1.967E-06	3.429E-07	1.014E-07	3.979E-08	1.863E-08
MHOC-4	4.313E-05	1.967E-06	3.429E-07	1.014E-07	3.979E-08	1.863E-08
HOC-6	1.922E-07	2.186E-09	1.613E-10	1.787E-11	3.216E-12	7.634E-12
MHOC-6	1.922E-07	2.186E-09	1.613E-10	1.787E-11	3.216E-12	7.634E-12

In this example we note the HOC-4 and HOC-6 have the same results of MHOC-4 and MHOC-6 respectively this occur because the addition terms in the MHOC finished because of the derivatives of its equal to zero.

4.2 Three Dimension

We now consider three Test problems to test the high order compact formulation for Poisson equation.

Test 1: The first test problem

$$\left.\begin{array}{lll} \varphi(x, y, 0) = e^{x+y}, & \varphi(x, y, 1) = e^{x+y+1}, & 0 \leq x, y \leq 1, \\ \varphi(0, y, z) = e^{y+z}, & \varphi(1, y, z) = e^{1+y+z}, & 0 \leq y, z \leq 1, \\ \varphi(x, 0, z) = e^{x+z}, & \varphi(x, 1, z) = e^{x+1+z}, & 0 \leq x, z \leq 1. \end{array}\right\} \qquad \ldots(76)$$

combined with the following forcing function,

$$f = 3\, e^{x+y+z} \qquad \ldots(77)$$

The resulting exact solution is

$$\varphi_{ex} = e^{x+y+y} \qquad \ldots(78)$$

Table 3 show the value of the absolute error computed by use CDS, HOC-4, MHOC-4, HOC-6 and MHOC-6 schemes for different value of N (N=2, 4, 6, 8, 10, 12)

Table 3: The absolute error computed by CDS, HOC-4, MHOC-4, HOC-6 and MHOC-6 for test problem 1.

	N=2	N=4	N=6	N=8
CDS	1.177E-02	2.507E-03	9.933E-04	5.190E-04
HOC-4	1.633E-03	6.397E-05	1.062E-05	3.047E-06
MHOC-4	9.937E-04	3.992E-05	6.607E-06	1.892E-06
HOC-6	6.176E-05	6.181E-07	3.116E-08	6.207E-09
MHOC-6	3.729E-05	3.621E-07	1.280E-08	8.534E-09

Test 2: The second test problem

$$\varphi(x, y, 0) = \varphi(x, y, 1) = \varphi(0, y, z) = \varphi(1, y, z) = \varphi(x, 0, z) = \varphi(x, 1, z) = 0, \qquad 0 \leq x, y, z \leq 1,$$
$$\ldots(79)$$

Combined with the following forcing function,

$$f = -3\pi^2 \sin(\pi x)\, \sin(\pi y)\, \sin(\pi z) \qquad \ldots(80)$$

The resulting exact solution is

$$\varphi_{ex} = \sin(\pi x)\, \sin(\pi y)\, \sin(\pi z) \qquad \ldots(81)$$

Table 4 show the value of the absolute error computed by use CDS , HOC-4, MHOC-4, HOC-6 and MHOC-6 schemes for different value of N (N=2, 4, 6, 8, 10, 12)

Table 4 : The absolute error computed by CDS, HOC-4, MHOC-4, HOC-6 and MHOC-6 for test problem 2.

	N=2	N=4	N=6	N=8
CDS	2.337E-01	2.764E-02	9.632E-03	4.798E-03
HOC-4	2.910E-01	6.775E-03	1.004E-03	2.768E-04
MHOC-4	1.212E-01	3.858E-03	6.004E-04	1.684E-04
HOC-6	1.048E-01	6.628E-04	4.402E-05	6.856E-06
MHOC-6	5.743E-02	3.897E-04	2.617E-05	4.096E-06

Test 3: The third test problem

$$\left.\begin{array}{lll} \varphi(x,y,0) = \cos(x+y), & \varphi(x,y,1) = \cos(x+y+1), & 0 \le x, y \le 1, \\ \varphi(0,y,z) = \cos(y+z), & \varphi(1,y,z) = \cos(1+y+z), & 0 \le y, z \le 1, \\ \varphi(x,0,z) = \cos(x+z), & \varphi(x,1,z) = \cos(x+1+z), & 0 \le x, z \le 1. \end{array}\right\} \quad \text{…(82)}$$

combined with the following forcing function,
$$f = -3\cos(x+y+z) \qquad\qquad \text{…(83)}$$
The resulting exact solution is
$$\varphi_{ex} = \cos(x+y+z) \qquad\qquad \text{…(84)}$$

Table 5 show the value of the absolute error computed by use CDS, HOC-4, MHOC-4, HOC-6 and MHOC-6 schemes for different value of N (N=2, 4, 6, 8, 10, 12)

Table 5 : The absolute error computed by CDS, HOC-4, MHOC-4, HOC-6 and MHOC-6 for test problem 3.

	N=2	N=4	N=6	N=8
CDS	1.827E-04	9.119E-05	3.856E-05	2.061E-05
HOC-4	2.489E-05	2.404E-06	4.153E-07	1.193E-07
MHOC-4	1.417E-05	1.470E-06	2.526E-07	7.191E-08
HOC-6	9.341E-07	3.978E-08	1.914E-08	1.510E-08
MHOC-6	5.557E-07	3.213E-08	1.887E-08	1.506E-08

Conclusions

In this paper, the high order compact finite difference schemes are derived to solve the Poisson equation subject to Dirichlet boundary condition on a bounded two and three dimension region. We use two methods two drive the fourth and sixth order finite difference schemes, and these methods give four schemes. We use two problems to test the accuracy of these schemes in two dimension case and three problem to test the accuracy of these schemes in three dimension case. We note that the accuracy of the

sixth order schemes is very high compare with the fourth order schemes; also the accuracy of the fourth order schemes is very high compare with the central finite difference scheme. Moreover, the schemes which derived by use the compact operators method have more accuracy compare with scheme drive by use Taylor series expansion method. The linear system resulted from these schemes is solve by use the Gauss-Seidel iterative method.

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