

***m* -Hypergeometric Solutions of Linear Recurrence
and *q* -Recurrence Equations**

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Abstract

In Koepf (1995), Koepf presents an algorithm to find an *m* -hypergeometric solution s_n of

$$s_{n+m} - s_n = a_n,$$

where a_n is a given *m* -hypergeometric term. We give a *q* -analogue of that algorithm. Also we generalize Koepf's algorithm to find *m* -hypergeometric solutions of linear recurrence equations without any restriction on the coefficients. Then we solve the same problem for linear *q* -recurrence equations.

Keywords : Gosper algorithm, *m* -hypergeometric solution, *q* -Gosper algorithm, *qm* -hypergeometric solution.

q-

m-

s_n m-

(1995)

$$s_{n+m} - s_n = a_n,$$

q-

a_n

m-

q-

1. Introduction

Let m denotes a positive integer, \mathbb{N} be the set of natural numbers, K be the field of characteristic zero, $K(n)$ be the field of rational functions over K , $K[n]$ be the ring of polynomials over K , and F denotes the transcendental extension of K by the indeterminate q , i.e., $F = K(q)$. In this paper, we will use x as an abbreviation for q^k . Recall that a non-zero term t_n is called a hypergeometric over K if there exist a rational function $r \in K(n)$ such that

$$\frac{t_{n+1}}{t_n} = r(n).$$

Gosper's algorithm (Gosper, 1978) (see also Graham *et al.*, 1989, Koepf, 1998, Petkovšek *et al.*, 1996) has been extensively studied and widely used to prove hypergeometric identities. Given a hypergeometric term t_n , Gosper's algorithm is a procedure to find a hypergeometric term z_n satisfying

$$z_{n+1} - z_n = t_n, \quad (1.1)$$

if it exists, or confirm the nonexistence of any solution of (1.1). A non-zero term a_n is called an m -hypergeometric over K if there exist a rational function $w(n) \in K(n)$ such that.

$$\frac{a_{n+m}}{a_n} = w(n),$$

In Koepf (1995), Koepf extends Gosper's algorithm to find m -hypergeometric solution s_n of

$$s_{n+m} - s_n = a_n, \quad (1.2)$$

where a_n is a given m -hypergeometric term. In Petkovšek and Bruno (1993), Petkovšek and Bruno described an algorithm to find m -hypergeometric solutions of the linear recurrence equation

$$\sum_{i=0}^d p_i s_{n+mi} = 0, \quad (1.3)$$

where d is a positive integer and $\{p_i(n)\}_{i=0}^d$ are given polynomials over K . Their algorithm reduces to algorithm **Hyper** (Petkovšek, 1992) when $m = 1$. Recall that a non-zero term h_k is called a q -hypergeometric over F if there exist a rational function $\sigma \in F(x)$ such that

$$\frac{h_{k+1}}{h_k} = \sigma(x).$$

q -Gosper's algorithm (see Böing and Koepf, 1999, Koornwinder, 1993, Paule and Riese, 1997, Paule and Strehl, 1995) determines if there exists a q -hypergeometric term anti-difference of a given q -hypergeometric term and computes this anti-difference provided that it exists. A non-zero term f_k is called a qm -hypergeometric over F if there exist a rational function $\rho \in F(x)$ such that

$$\frac{f_{k+m}}{f_k} = \rho(x).$$

The contents of this paper are as follows: In Section 2, we give a q -analogue of Koepf's algorithm. In Section 3, we show that Koepf's algorithm can be generalized to find m -hypergeometric solutions of non-homogenous linear recurrence equations without any restrictions on the coefficients. Finally, in Section 4, we solve the same problem in Section 3 for the q -case.

2. qm -Hypergeometric Solutions of Anti-Recurrence Equations

In this section, we give a q -analogue of Koepf's algorithm, i.e., find a qm -hypergeometric solution g_k of

$$g_{k+m} - g_k = f_k, \tag{2.1}$$

where f_k is a given qm -hypergeometric term.

Theorem 2.1. *Given a qm -hypergeometric term f_k . If the equation*

$$h_{k+1} - h_k = f_{km}, \tag{2.2}$$

has a q -hypergeometric solution with respect to h_k , then equation (2.1) has a qm -hypergeometric solution with respect to g_k given by $g_k = h_{k/m}$, otherwise equation (2.1) has no qm -hypergeometric solution.

Proof. If g_k is a qm -hypergeometric solution of equation (2.1), then by using (2.1), we find

$$\frac{g_k}{f_k} = \frac{g_k}{g_{k+m} - g_k} = \frac{1}{\frac{g_{k+m}}{g_k} - 1}.$$

Let $\tau(x) = \frac{g_k}{f_k}$. It follows that $\tau(x)$ is a rational function of x . Substituting $\tau(x)f_k$ for g_k in (2.1) to obtain

$$\rho(x)\tau(q^k m) - \tau(x) = 1, \tag{2.3}$$

where $\rho(x) = \frac{f_{k+m}}{f_k}$ is a rational function of x . Analogously, from (2.2) we get

$$\rho(x^m)\mu(qx) - \mu(x) = 1, \tag{2.4}$$

where $\mu(x) = \frac{h_k}{f_{km}}$ is a rational function of x . Clearly (2.3) and (2.4) are either both solvable and have solutions such that $\mu(x) = \tau(x^m)$, or are both unsolvable. So either (2.2) has no q -hypergeometric solution and (2.1) has no qm -hypergeometric solution, or (2.2) has a q -hypergeometric solution $h_k = \mu(x)f_{km}$ and (2.1) has the desired solution. □

Algorithm 2.1.

INPUT : a qm -hypergeometric term f_k .

OUTPUT : a qm -hypergeometric solution g_k of (2.1) if it exists, otherwise “no qm -hypergeometric solution of (2.1) exists”.

- (1) Compute the q -hypergeometric solution h_k of equation (2.2) if it exists, otherwise return “no qm -hypergeometric solution of (2.1) exists”.
- (2) Compute the qm -hypergeometric solution g_k of equation (2.1) by the following relation:

$$g_k = h_{k/m}.$$

3. m -Hypergeometric Solutions of Linear Recurrence Equations

In this section, we generalize Koepf's algorithm to find m -hypergeometric solutions of the linear recurrence equation

$$\sum_{i=0}^d p_i(n)s_{n+mi} = a_n, \tag{3.1}$$

where $\{p_i(n)\}_{i=0}^d$ are given polynomials and a_n is a given m -hypergeometric term.

Theorem 3.1. Given an m -hypergeometric term a_n . If the equation

$$\sum_{i=0}^d p_i(mn)t_{n+i} = a_{mn}, \tag{3.2}$$

has a hypergeometric solution with respect to t_n , then equation (3.1) has an m -hypergeometric solution with respect to s_n given by $s_n = t_{n/m}$, otherwise equation (3.1) has no m -hypergeometric solution.

Proof. If s_n is an m -hypergeometric solution of equation (3.1), then the left hand-side of (3.1) can

be written as a rational function multiple of s_n . Let $S(n) = s_n/a_n$. It follows that $S(n)$ is a rational

function of n . Substituting $S(n)a_n$ for s_n in (3.1) to obtain

$$\sum_{i=0}^d p_i(n)S(n+mi) \prod_{j=0}^{i-1} w(n+mj) = 1, \tag{3.3}$$

where $w(n) = a_{n+m}/a_n$ is a rational function of n . Analogously, from (3.2) we get

$$\sum_{i=0}^d p_i(mn)T(n+i) \prod_{j=0}^{i-1} w(m(n+i)) = 1, \tag{3.4}$$

where $T(n) = t_n/a_{mn}$ is a rational function of n . Clearly (3.3) and (3.4) are either both solvable and have solutions such that $T(n) = S(mn)$, or are both unsolvable. So either (3.2) has no hypergeometric solution and (3.1) has no m -hypergeometric solution, or (3.2) has a hypergeometric solution $t_n = T(n)a_{mn}$ and (3.1) has the desired solution.

□

Algorithm 3.1.

INPUT : $\{p_i(n)\}_{i=0}^d \in K[n]$ and an m -hypergeometric term a_n .

OUTPUT : an m -hypergeometric solution s_n of (3.1), if it exists, otherwise “no m -hypergeometric solution of (3.1) exists”.

- (1) Compute the hypergeometric solution t_n of equation (3.2) if it exists, otherwise return “no m -hypergeometric solution of (3.1) exists”.

(2) Compute the *m*-hypergeometric solution s_n of equation (3.1) by the following relation:

$$s_n = t_{n/m}.$$

Example 3.1. We want to find all 2-hypergeometric solutions of

$$s_{n+4} + (n^3 + 15n^2 + 3)s_{n+2} - n(n+5)(n+10)s_n = 4 - 50n. \tag{3.5}$$

By equation (3.2), t_n is a hypergeometric term, which satisfies

$$t_{n+2} + (8n^3 + 60n^2 + 3)t_{n+1} - 2n(2n+5)(2n+10)t_n = 4 - 100n.$$

The only hypergeometric solution of this equation is $t_n = 1$. Thus $s_n = t_{n/2} = 1$ is the only 2-hypergeometric solution of (3.5).
□

4. *qm*-Hypergeometric Solutions of Linear *q*-Recurrence Equations

We present an algorithm to find a *qm*-hypergeometric term g_k satisfying

$$\sum_{i=0}^d \lambda_i(x) g_{k+mi} = f_k, \tag{4.1}$$

where $\{\lambda_i(x)\}_{i=0}^d$ are given polynomials and f_k is a given *qm*-hypergeometric term. This algorithm is a generalization of the one given in Section 2. Also, it is a *q*-analogue of the one given in Section 3.

Theorem 4.1. Given a *qm*-hypergeometric term f_k . If the equation

$$\sum_{i=0}^d \lambda_i(x^m) h_{k+i} = f_{km}, \tag{4.2}$$

has a *q*-hypergeometric solution with respect to h_k , then equation (4.1) has a *qm*-hypergeometric solution with respect to g_k given by $g_k = h_{k/m}$, otherwise equation (4.1) has no *qm*-hypergeometric solution.

Proof. Let $\tau(x), \rho(x), \mu(x)$ be defined as in Section 2. It follows that $\rho(x)$ is a rational function of x . If g_k is a qm -hypergeometric solution of equation (4.1), then the left hand-side of (4.1) can be written as a rational function multiple of g_k . It follows that $\tau(x)$ is a rational function of x . Substituting $\tau(x)f_k$ for g_k in (4.1) to obtain

$$\sum_{i=0}^d \lambda_i(x) \tau(q^{mi}x) \prod_{j=0}^{i-1} \rho(q^{mj}x) = 1. \tag{4.3}$$

Analogously, from (4.2) we get

$$\sum_{i=0}^d \lambda_i(x^m) \mu(q^i x) \prod_{j=0}^{i-1} \rho(q^{mj} x^m) = 1, \tag{4.4}$$

where $\mu(x)$ is a rational function of x . Clearly (4.3) and (4.4) are either both solvable and have

solutions such that $\mu(x) = \tau(x^m)$, or are both unsolvable. So either (4.2) has no q -hypergeometric solution and (4.1) has no qm -hypergeometric solution, or (4.2) has a q -hypergeometric solution $h_k = \mu(x)f_{km}$ and (4.1) has the desired solution.

□

Algorithm 4.1.

INPUT : $\{\lambda_i(x)\}_{i=0}^d \in F[x]$ and a qm -hypergeometric term f_k .

OUTPUT: a qm -hypergeometric solution g_k of (4.1) if it exists, otherwise “no qm -hypergeometric solution of (4.1) exists”.

- (1) Compute the q -hypergeometric solution h_k of equation (4.2) if it exists, otherwise return “no qm -hypergeometric solution of (4.1) exists”.
- (2) Compute the qm -hypergeometric solution g_k of equation (4.1) by the following relation:

$$g_k = h_{k/m}.$$

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