m-Hypergeometric Solutions of Anti-Difference and q-Anti-Difference Equations

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Abstract

In this paper we consider the problem of finding m-hypergeometric solutions of anti-difference equations. We extend the greatest factorial factorization (GFF) of a polynomial, introduced by Paule (1995), to the m-greatest factorial factorization (mGFF). Equipped with the mGFF-concept, we present algebraically motivated approach to the problem. This approach requires only “gcd” operations but no factorization. Then, we solve the same problem for q-anti-difference equations.

Keywords : Gosper’s algorithm, m-hypergeometric solution, m-greatest factorial factorization, q-Gosper algorithm, qm-hypergeometric solution, qm-greatest factorial factorization.

الحلول الهايبرجيومترية لمعادلات ضد الفروقات
ومعادلات ضد الفروقات- q

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البصرة، العراق

الخلاصة

في هذا البحث نتأمل مسالة ايجاد الحلول الهايبرجيومترية -m لمعادلات ضد الفروقات. نوسع تحليل العوامل الاعظم (GFF) لمتعددة الحدود، المقدم من قبل باول (1995)، الى تحليل العوامل الاعظم -m (mGFF). باستخدام مفهوم ، نقدم اسلوب جبري للمسالة. هذا الاسلوب يتطلب فقط عمليات gcd ولايتطلب التحليل الى عوامل. ثم نحل نفس المسالة لمعادلات الفروقات-q .

1. Introduction

Let m denotes a positive integer, $\mathbb{N}$ be the set of natural numbers, K be the field of characteristic zero, $K(n)$ be the field of rational functions over K, $K[n]$ be the ring of polynomials over K, F denotes the transcendental extension of K by the indeterminate q, i.e., $F = K(q)$, E denotes the shift operator on $K[n]$, i.e., $(Ep)(n) = p(n+1)$ for any $p \in K[n]$, ε denotes the q-shift operator on $F[n]$ and $F(n)$, i.e., $(\varepsilon u)(n) = u(qn)$ for any $u \in F[n]$ or $u \in F(n)$, deg(p) denotes the polynomial degree (in n) of any $p \in K[n]$ or $p \in F[n]$, $p \neq 0$. We define deg $(0) = -1$. We assume the result of any gcd (greatest common divisor) computation in $K[n]$ or $F[n]$ as being normalized to a monic polynomial p, i.e., the leading coefficient of p being 1. Recall that a non-zero term t_n is called a hypergeometric term over K if there exist a rational function $r(n) \in K(n)$ such that

$$\frac{t_{n+1}}{t_n} = r(n).$$

Gosper's algorithm (Goper, 1978) (also see Graham *et al.*, 1989, Koepf, 1998, Petkovšek *et al.,* 1996) has been extensively studied and widely used to prove hypergeometric identities. Given a hypergeometric term t_n, Gosper's algorithm is a procedure to find a hypergeometric term z_n satisfying

$$z_{n+1} - z_n = t_n. \tag{1.1}$$

if it exists, or confirm the nonexistence of any solution of (1.1). In Paule (1995), Paule introduced the GFF-concept. Equipped with the GFF-concept, he presented a new and algebraically motivated approach to Gosper's algorithm.

A non-zero term a_n is called an m-hypergeometric over K if there exist a rational function $w(n) \in K[n]$ such that

$$\frac{a_{n+m}}{a_n} = w(n). \tag{1.2}$$

In Koepf (1995), Koepf extends Gosper's algorithm to find m-hypergeometric solutions h_n of

$$h_{n+m} - h_n = a_n, \tag{1.3}$$

where a_n is a given m-hypergeometric term. In Petkovšek and Bruno (1993), Petkovšek and Bruno described an algorithm to find m-hypergeometric solutions of homogeneous linear recurrences with polynomial coefficients. Their algorithm reduces to algorithm **Hyper** (Petkovšek, 1992) when $m = 1$.

A non-zero term b_k is called a q-hypergeometric over F if there exists a rational function $\sigma \in F(q^k)$ such that

$$\frac{b_{k+1}}{b_k} = \sigma(q^k).$$

In Paule and Riese (1997), Paule and Riese introduced the q-greatest factorial factorization (qGFF) of polynomials, which is a q-analogue of the GFF-concept. Equipped with the qGFF, they presented a new approach to find q-hypergeometric solutions l_k of

$$l_{k+1} - l_k = b_k, \tag{1.4}$$

where b_k is a given q-hypergeometric term. Paule-Riese's approach can be viewed as an q-analogue of Gosper's algorithm.

A non-zero term f_k is called a qm-hypergeometric over F if there exist a rational function $\rho \in F(q^k)$ such that

$$\frac{f_{k+m}}{f_k} = \rho(q^k).$$

Let us define the dispersion $\text{dis}_m(a,b)$ of the polynomials $a(n), b(n) \in K[n]$ to be the greatest nonnegative integer k (if it exists) such that $a(n)$ and $b(n+mk)$ have a nontrivial common divisor, i.e.,

$$\text{dis}_m(a,b) = \max \{ k \in \mathbb{N} \mid \deg \gcd (a(n), b(n+mk)) \geq 1 \}.$$

If k does not exist then we set $\text{dis}_m(a,b) = \text{-}1$. Recall that the pair $\langle c,d \rangle$, $c,d \in K[n]$, is called the reduced form of $r \in K(n)$ if $r = c/d$, d is monic, and $\gcd(c,d) = 1$.

The contents of this paper are as follows: In Section 2, we give the Fundamental mGFF Lemma, which is an extension of the Fundamental Lemma given by Paule (1995). In Section 3, we extend Paule's approach to find m-hypergeometric solutions of anti-difference equations. In Section 4, we give the Fundamental qmGFF Lemma, which is an extension of the Fundamental qGFF Lemma given by Paule and Riese (1997). Finally, In Section 5, we extend Paule-Riese's approach to find qm-hypergeometric solutions of q-anti-difference equations.

2. m-Greatest Factorial Factorization

In this section we define the mGFF of a polynomial, which is an extension of the GFF-concept introduced by Paule.

2.1 Basic Definitions

Definition 2.1. *For any monic polynomial* $p \in K[n]$ *and* $i \in \mathbb{N}$, *the* i*-th* m*-falling factorial* $[p]_m^i$ *of* p *is defined as*

$$[p]_m^i = p \cdot E^{-m} p \cdot E^{-2m} p \cdot \ldots \cdot E^{(-i+1)m} p.$$

For $i = 0$, *we let* $[p]_m^0 = 1$.

Definition 2.2. *We say that* $\langle p_1, p_2, ..., p_s \rangle$, $p_i \in K[n]$, *is an* mGFF-*form of a monic polynomial* $p \in K[n]$ *if the following conditions hold:*

(mGFF1) $p = [p_1]_m^1 \cdot [p_2]_m^2 \cdots [p_s]_m^s$,
(mGFF2) *each* p_i *is monic and* $s > 0$ *implies* deg(p_s)>0,
(mGFF3) gcd($[p_i]_m^i$, $E^m p_j$)= 1 *for* $1 \le i \le j \le s$,
(mGFF4) gcd($[p_i]_m^i$, $E^{-jm} p_j$)= 1 *for* $1 \le i \le j \le s$.

If $\langle p_1, p_2, ..., p_s \rangle$ *is an* mGFF-*form of a monic* $p \in K[n]$ *we sometimes express this fact for short by* mGFF(p)=$\langle p_1, p_2, ..., p_s \rangle$.

2.2 The Fundamental mGFF Lemma

In this section we give the Fundamental mGFF Lemma, which is an extension of the Fundamental Lemma given by Paule. The $\gcd(p, E^m p)$ for $p \in K[n]$ plays a basic role in finding m-hypergeometric solutions of anti-difference equation (1.3).

Lemma 2.1. ("Fundamental mGFF Lemma") *Given a monic polynomial* $p \in K[n]$ *with* mGFF-*form* $\langle p_1, p_2, ..., p_s \rangle$. *Then*

$$\gcd(p, E^m p) = [p_2]_m^1 \cdot [p_3]_m^2 \cdots [p_s]_m^{s-1}.$$

Proof. Proceeding by induction on s the case $s = 0$ is trivial. For $s > 0$,

$$\gcd(p, E^m p) = [p_s]_m^{s-1} \cdot \gcd([p_1]_m^1 ... [p_{s-1}]_m^{s-1} \cdot E^{(-s+1)m} p_s, E^m([p_1]_m^1 ... [p_{s-1}]_m^{s-1} \cdot p_s)).$$
$$= [p_s]_m^{s-1} \cdot \gcd([p_1]_m^1 ... [p_{s-1}]_m^{s-1}, E^m([p_1]_m^1 ... [p_{s-1}]_m^{s-1})).$$

The first equality is obvious, the second is a consequence of mGFF3 and mGFF4 because for $i < s$ we have

$$\gcd([p_i]_m^i, E^m p_s) = \gcd(E^{(-s+1)m} p_s, E^m [p_i]_m^i) = E^m \gcd(E^{-sm} p_s, [p_i]_m^i) = 1.$$

together with $\gcd(E^{(-s+1)m} p_s, E^m p_s) \mid \gcd([p_s]_m^s, E^m p_s) = 1$. The rest of the proof follows from applying the induction hypothesis.
□

In the above lemma we see that from the m GFF-form of a polynomial p we can find the m GFF-form of $\gcd(p, E^m p)$.

3. m-Hypergeometric Solutions of Anti-Difference Equations

In this section we extend Paule approach to find m-hypergeometric solutions h_n of equation (1.3). Given an m-hypergeometric term a_n and suppose that there exists an m-hypergeometric term h_n satisfying equation (1.3), then by using (1.3) we find

$$\frac{h_n}{a_n} = \frac{h_n}{h_{n+m} - h_n} = \frac{1}{\frac{h_{n+m}}{h_n} - 1}.$$

Let $y(n) = {h_n}/{a_n}$. It follows that $y(n)$ is a rational function of n. Let $\langle a, b \rangle$ be the reduced form of $w(n) = {a_{n+m}}/{a_n}$. Substituting $y(n)a_n$ for h_n in (1.3) to obtain

$$a(n)y(n+m) - b(n)y(n) = b(n). \qquad (3.1)$$

This means, the problem of finding m-hypergeometric solution of (1.3) is equivalent to finding a rational solution $y(n)$ of (3.1). If a solution $y(n) \in K(n)$ of (3.1) with the reduced form $\langle u, v \rangle$ exists, assume we know v or a multiple $V \in K[n]$ of v. Then equation (3.1) can be written as

$$a(n) \cdot V(n) \cdot U(n+m) - b(n) \cdot V(n+m) \cdot U(n) = b(n) \cdot V(n) \cdot V(n+m), \qquad (3.2)$$

where $U(n) = u(n) \cdot {V(n)}/{v(n)}$ is unknown polynomial. Hence the problem reduces to finding a polynomial solution $U \in K[n]$ of equation (3.2). To solve (3.2) we try to find a suitable denominator polynomial V and then U can be computed as a polynomial solution of (3.2). Let

$$v_i(n) = \frac{v(n+i)}{\gcd(v, E^m v)} \quad \text{for } i \in \{0, m\}.$$

Then (3.1) is equivalent to

$$a(n)\cdot v_0(n)\cdot u(n+m)-b(n)\cdot v_m(n)\cdot u(n)=b(n)\cdot v_0(n)\cdot v_m(n)\cdot \gcd(v,E^m v). \quad (3.3)$$

From this equation we immediately get that $v_0(n) \mid b(n)$ and that $v_m(n) \mid a(n)$. Let m GFF $(v) = \langle p_1, p_2,..., p_s \rangle$, by using the m GFF-concept and the Fundamental m GFF Lemma we get that

$$v_0 = \frac{v}{\gcd(v,E^m v)} = p_1.E^{-m}p_2...E^{(-s+1)m}p_s \mid b(n), \quad (3.4)$$

$$v_m = \frac{E^m v}{\gcd(v,E^m v)} = E^m p_1.E^m p_2...E^m p_s \mid a(n), \quad (3.5)$$

This observation gives rise to a simple algorithm for computing a multiple $V=[P_1]_m^1\cdot[P_2]_m^2\cdots[P_s]_m^s$ of v.

● Straightforward conclusion

$$p_i \mid \gcd\left(E^{-m}a, E^{(i-1)m}b\right) \quad \forall \ i\in\{1,...,s\}.$$

If $P_i=\gcd(E^{-m}a,E^{(i-1)m}b)$ then obviously $p_i \mid P_i$. Thus, we could take

$$m\,\text{GFF}(V)=\langle P_1,P_2,...,P_N\rangle,$$

where $N=\text{dis}_m(a,b)=\max\ \{i\in\mathbb{N} \mid \deg\ \gcd(a,E^{im}b)\ge 1\}$. If N is not defined then we set $V=1$.

● Refined conclusions

$$p_1 \mid \gcd\left(E^{-m}a,b\right).$$

If $P_1=\gcd(E^{-m}a,b)$ then $p_1 \mid P_1$ and

$$p_2 \mid \gcd\left(E^{-m}\left(\frac{a}{E^m(P_1)}\right),E^m\left(\frac{b}{P_1}\right)\right).$$

If $P_2=\gcd\left(E^{-m}\left(\frac{a}{E^m(P_1)}\right),E^m\left(\frac{b}{P_1}\right)\right)$, then $p_{12} \mid P_2$ and so on until we arrive at a P_N and we may again take $m\,\text{GFF}(V)=\langle P_1,P_2,...,P_N\rangle$.

□

The algorithm that we have just derived for (1.3) can be written, by using the "redefined conclusions", as follows:

Algorithm 3.1.

INPUT : $w(n) \in K(n)$ *such that* $a_{n+m} / a_n = w(n)$ *for all* $n \in \mathbb{N}$.

OUTPUT: *an* m*-hypergeometric solution* h_n *of* (1.3) *if it exists, otherwise* "*no* m*-hypergeomet-ric solution of* (1.3) *exists*".

(1) *Decompose* $w(n)$ *into the reduced form* $\langle a,b \rangle$.

(2) *Compute* $N = \operatorname{dis}_m(a,b) = \max\{ i \in \mathbb{N} \mid \deg \gcd(a, E^{im} b) \geq 1\}$.

If $N > 0$ *then compute for* j *from* 1 *to* N

$$P_j(n) = \gcd(E^{-m}a, E^{(j-1)m}b)$$
$$a = \frac{a}{E^m P_j(n)}$$
$$b = \frac{b}{E^{-(j-1)m} P_j(n)}$$
$$m\,GFF(V) = \langle P_1, P_2, \ldots, P_N \rangle$$

otherwise $V = 1$.

(3) *If equation* (3.2) *can be solved for* $U \in K[n]$ *then return* $h_n = \frac{U(n)}{V(n)} a_n$, *otherwise return* "*no* m*-hypergeometric solution of* (1.3) *exists*".

□

4. q-m-Greatest Factorial Factorization

In this section we define the "q-m-Greatest Factorial Factorization" (qmGFF) of a polynomial which is an extension of the qGFF-concept introduced by Paule and Riese. Also, it is a q-analogue of the mGFF-concept, defined with respect to the q-shift operator ε instead of the shift operator E as for mGFF. In Sections 4 and 5, we will use n as an abbreviation for q^k.

Let us define the dispersion $\operatorname{dis}_m(a(n), b(n))$ of the q-monic polynomials $a(n), b(n) \in F[n]$ is the greatest nonnegative integer i (if it exists) such that $a(n)$ and $b(q^{mi}n)$ have a nontrivial common divisor, i.e.,

$$\operatorname{dis}_m(a,b) = \max\{ i \in \mathbb{N} \mid \deg \gcd\,(a(n), b(q^{mi}n)) \geq 1\}.$$

A polynomial $p \in F[n]$ is said to be q-monic if $p(0) = 1$. Any polynomial $p \in F[n]$ has a unique factorization, the q-monic decomposition, in the form

$$p = z \cdot n^{\alpha} \cdot \hat{p}\,,$$

where $z \in F$, $\alpha \in \mathbb{N}$, and $\hat{p} \in F[n]$ q-monic. We will write $\gcd_q$ instead of "gcd", indicating that the $\gcd_q$ of two q-monic polynomials is understood to be q-monic.

More generally, if $p_1 = z_1 \cdot n^{\alpha_1} \cdot \hat{p}_1$ and $p_2 = z_2 \cdot n^{\alpha_2} \cdot \hat{p}_2$ are q-monic decompositions of $p_1, p_2 \in F[n]$, we define

$$\gcd_q(p_1, p_2) = \gcd(n^{\alpha_1}, n^{\alpha_2}) \cdot \gcd_q(\hat{p}_1, \hat{p}_2).$$

4.1 Basic Definitions

Definition 4.1. *For any q-monic polynomial $p \in F[n]$ and $i \in \mathbb{N}$, the i-th m-falling q-factorial $[p]_{m_q}^{\underline{i}}$ of p is defined as*

$$[p]_{m_q}^{\underline{i}} = p \cdot \varepsilon^{-m} p \cdot \varepsilon^{-2m} p \cdot \ldots \cdot \varepsilon^{(-i+1)m} p.$$

For $i = 0$, we let $[p]_{m_q}^{\underline{0}} = 1$.

Definition 4.2. *We say that $\langle p_1, p_2, \ldots, p_s \rangle$, $p_i \in F[n]$, is an qm GFF-form of a q-monic polynomial $p \in F[n]$ if the following conditions hold:*

(qm GFF1) $p = [p_1]_{m_q}^{\underline{1}} \cdot [p_2]_{m_q}^{\underline{2}} \cdots [p_s]_{m_q}^{\underline{s}}$,
(qm GFF2) *each* p_i *is* q-*monic and* $s > 0$ *implies* $\deg(p_s) > 0$,
(qm GFF3) $\gcd_q([p_i]_{m_q}^{\underline{i}}, \varepsilon^{m} p_j) = 1$ *for* $1 \le i \le j \le s$,
(qm GFF4) $\gcd_q([p_i]_{m_q}^{\underline{i}}, \varepsilon^{-jm} p_j) = 1$ *for* $1 \le i \le j \le s$.

If $\langle p_1, p_2, \ldots, p_s \rangle$ is the qm GFF-form of a q-monic $p \in F[n]$ we also denote this fact for short by $qm\,\mathrm{GFF}(p) = \langle p_1, p_2, \ldots, p_s \rangle$.

4.2 The Fundamental qm GFF Lemma

In this section we give the Fundamental qm GFF Lemma which is an extension of the Fundamental q GFF Lemma given by Paule and Riese. In finding m-hypergeometric solutions of anti-difference equations (i.e., $q=1$) the gcd $(p, E^m p)$ for $p \in K[n]$ plays a basic role. The same is true for qm-hypergeometric solutions of q-anti-difference equations with respect to the q-shift operator ε instead of E.

Lemma 4.1. ("Fundamental qm GFF Lemma") *Given a q-monic polynomial $p \in F[n]$ with qm GFF-form $\langle p_1, p_2, ..., p_s \rangle$. Then*

$$\gcd_q (p, \varepsilon^m p) = [p_2]^{\underline{1}}_{m_q} \cdot [p_3]^{\underline{2}}_{m_q} \cdots [p_s]^{\underline{s-1}}_{m_q}.$$

Proof. Proceeding by induction on s the case $s=0$ is trivial. For $s>0$,

$$\gcd_q (p, \varepsilon^m p) = [p_s]^{\underline{s-1}}_{m_q} \cdot \gcd_q ([p_1]^{\underline{1}}_{m_q} ... [p_{s-1}]^{\underline{s-1}}_{m_q} \cdot \varepsilon^{(-s+1)m} p_s, \varepsilon^m ([p_1]^{\underline{1}}_{m_q} ... [p_{s-1}]^{\underline{s-1}}_{m_q} \cdot p_s)).$$
$$= [p_s]^{\underline{s-1}}_{m_q} \cdot \gcd_q ([p_1]^{\underline{1}}_{m_q} ... [p_{s-1}]^{\underline{s-1}}_{m_q}, \varepsilon^m ([p_1]^{\underline{1}}_{m_q} ... [p_{s-1}]^{\underline{s-1}}_{m_q})).$$

The first equality is obvious, the second is a consequence of qm GFF3 and qm GFF4 because for $i < s$ we have

$$\gcd_q ([p_i]^{\underline{i}}_{m_q}, \varepsilon^m p_s) = \gcd_q (\varepsilon^{(-s+1)m} p_s, \varepsilon^m [p_i]^{\underline{i}}_{m_q}) = \varepsilon^m \gcd_q (\varepsilon^{-sm} p_s, [p_i]^{\underline{i}}_{m_q}) = 1.$$

together with $\gcd_q (\varepsilon^{(-s+1)m} p_s, \varepsilon^m p_s) \mid \gcd_q ([p_s]^{\underline{s}}_{m_q}, \varepsilon^m p_s) = 1.$ The rest of the proof follows from applying the induction hypothesis.
□

In the above lemma we see that from the qm GFF-form of a q-monic polynomial p one directly can extract the qm GFF-form of $\gcd_q (p, \varepsilon^m p)$.

5. qm-Hypergeometric Solutions of q-Anti-Difference Equations

In this section we extend Paule-Riese's approach to find qm-hypergeometric solutions g_k of the q-anti-difference equation

$$g_{k+m} - g_k = f_k, \tag{5.1}$$

where f_k is a given a qm-hypergeometric term. Given a qm-hypergeometric term f_k and suppose that there exists a qm-hypergeometric term f_k satisfying equation (5.1), then by using (5.1) we find

$$\frac{g_k}{f_k} = \frac{g_k}{g_{k+m} - g_k} = \frac{1}{\frac{g_{k+m}}{g_k} - 1}.$$

Let $\tau = g_k / f_k$. It follows that τ is a rational function over F. Substituting $\tau \cdot f_k$ for g_k in (5.1) to obtain

$$\rho \cdot \varepsilon^m \tau - \tau = 1, \tag{5.2}$$

where $\rho = f_{k+m} / f_k \in F(n)$ is a rational function. Let $\rho = z \cdot n^{\alpha} \cdot a / b$ with $z \in F$, α integer, and $a, b \in F[n]$ relatively prime and q-monic. For any integer α we define $\alpha_+ = \max(\alpha, 0)$ and $\alpha_- = \max(-\alpha, 0)$, thus equation (5.2) is equivalent to

$$z \cdot n^{\alpha_+} \cdot a \cdot \varepsilon^m \tau - n^{\alpha_-} \cdot b \cdot \tau = n^{\alpha_-} \cdot b. \tag{5.3}$$

This means the problem of finding a qm-hypergeometric solutions of (5.1) is equivalent to finding rational solutions τ of (5.3). Let $\tau = u / v$ where $u, v \in F[n]$ be two unknown relatively prime polynomials with $v = n^{\beta} \cdot \hat{v}$ the q-monic decomposition of v. If a solution τ of (5.3) exists, assume we know v or a multiple $V \in F[n]$ of v. Then equation (5.3) can be written as

$$z \cdot n^{\alpha_+} \cdot a \cdot V \cdot \varepsilon^m U - n^{\alpha_-} \cdot b \cdot \varepsilon^m V \cdot U = n^{\alpha_-} \cdot b \cdot V \cdot \varepsilon^m V. \tag{5.4}$$

where $U(n) = u(n) \cdot V(n) / v(n)$ is unknown polynomial. Hence the problem reduces to finding a polynomial solution $U \in F[n]$ of equation (5.4). To solve (5.4) we try to find a suitable denominator polynomial V and then U can be computed as a polynomial solution of (5.4). Let

$$v_i = \frac{\varepsilon^i v}{\gcd_q(v, \varepsilon^m v)} \quad \text{for } i \in \{0, m\}.$$

Then (5.3) is equivalent to

$$z \cdot n^{\alpha_+} a \cdot v_0 \cdot \varepsilon^m u - n^{\alpha_-} \cdot b \cdot v_m \cdot u = n^{\alpha_-} \cdot b \cdot v_0 \cdot v_m \cdot \gcd_q(v, \varepsilon^m v). \tag{5.5}$$

From this equation we immediately get that $v_0 \mid b$ and that $v_m \mid a$. Let $qm\,\text{GFF}(\hat{v}) = \langle p_1, p_2, \ldots, p_s \rangle$, by using the qm GFF-concept and the Fundamental qm GFF Lemma we get that

$$v_0 = \frac{v}{\gcd_q(v, \varepsilon^m v)} = \frac{\hat{v}}{\gcd_q(\hat{v}, \varepsilon^m \hat{v})} = p_1 . \varepsilon^{-m} p_2 \ldots \varepsilon^{(-s+1)m} p_s \mid b, \tag{5.6}$$

$$v_m = \frac{\varepsilon^m v}{\gcd_q(v, \varepsilon^m v)} = \frac{q^{m\beta} \cdot \varepsilon^m \hat{v}}{\gcd_q(\hat{v}, \varepsilon^m \hat{v})} = q^{m\beta} \cdot \varepsilon^m p_1 . \varepsilon^m p_2 ... \varepsilon^m p_s \;\Big|\; a, \tag{5.7}$$

This observation give rise to a simple algorithm for computing a multiple $\hat{V} = [P_1]^1_{m_q} \cdot [P_2]^2_{m_q} \cdots [P_s]^s_{m_q}$ of $\hat{v}$.

- Straightforward conclusion

$$p_i \;\Big|\; \gcd_q\left(\varepsilon^{-m} a, \varepsilon^{(i-1)m} b\right) \quad \forall \;\; i \in \{1, ..., s\}.$$

If $P_i = \gcd_q(\varepsilon^{-m} a, \varepsilon^{(i-1)m} b)$ then obviously $p_i \;|\; P_i$. Thus, we could take

$$\hat{V} = [P_1]^1_{m_q} \cdot [P_2]^2_{m_q} \cdots [P_N]^N_{m_q},$$

where $N = \text{dis}_m(a,b) = \max \;\{ i \in \mathbb{N} \;|\; \deg \; \gcd_q \; (a, \varepsilon^{im} b) \geq 1 \}$. If N is not defined then we set $\hat{V} = 1$.

- Refined conclusions

$$p_1 \;\Big|\; \gcd_q\left(\varepsilon^{-m} a, b\right).$$

if we take $P_1 = \gcd_q(\varepsilon^{-m} a, b)$ then $p_1 \;|\; P_1$ and

$$p_2 \;\Big|\; \gcd_q\left(\varepsilon^{-m}\left(\frac{a}{\varepsilon^m(P_1)}\right), \varepsilon^m\left(\frac{b}{P_1}\right)\right).$$

If we take $P_2 = \gcd_q\left(\varepsilon^{-m}\left(\frac{a}{\varepsilon^m(P_1)}\right), \varepsilon^m\left(\frac{b}{P_1}\right)\right)$, then $p_2 \;|\; P_2$ and so on until we arrive at a P_N and we may again take $\hat{V} = [P_1]^1_{m_q} \cdot [P_2]^2_{m_q} \cdots [P_N]^N_{m_q}$.
□

With $\hat{V}$ in hand, all what is left for solving (5.4), and thus finding the a qm-hypergeometric solution of equation (5.1), is to determine an appropriate value of γ such that

$$v(n) = n^{\beta} \cdot \hat{v}(n) \;\Big|\; V(n) = n^{\gamma} \cdot \hat{V}(n).$$

For that we will follow the approach given by Paule and Riese (1997). Consider equation (5.5): (i) Assume that $\alpha \neq 0$ then either $\alpha_- \neq 0$ or $\alpha_+ \neq 0$. In the first case we have $\alpha_+ = 0$ and $n^{\alpha_-} \;|\; u$, hence β must be zero because of $\gcd_q(u,v) = 1$. This means we can choose $\gamma = 0$. In the second case we have $\alpha_- = 0$ and $n^{\min(\alpha_+,\beta)} \mid u$, because of $\gcd_q(v, \varepsilon^m v) = n^\beta \cdot \gcd_q(\hat{v}, \varepsilon^m \hat{v})$. Again β must be zero, and again we can choose $\gamma = 0$. (ii) Assume that $\alpha = 0$. In this case equation (5.5) evaluated at $n = 0$ turns into

$$(z - q^{\beta m})u(0) = q^{\beta m} \cdot \delta_{0,\beta},$$

where $\delta_{0,\beta}$ denotes the Kronecker symbol. This means if $\beta > 0$ we obtain, observing that $u(0) \neq 0$ in this case, as a condition for β that $z = q^{\beta m}$. Hence in case $\alpha = 0$, we choose $\gamma = \frac{1}{m} \cdot \log_q(z)$ if z is a positive integer power of q, or $\gamma = 0$ otherwise.

The algorithm that we have just derived for (5.1) can be written, by using the "redefined conclusions", as follows:

Algorithm 5.1.
INPUT : $\rho \in F(n)$ *such that* $f_{k+m} / f_k = \rho(q^k)$ *for all* $n \in \mathbb{N}$.
OUTPUT: *a* qm*-hypergeometric solution* g_k *of* (5.1) *if it exists, otherwise* "*no* qm*-hypergeometric solution* g_k *of* (5.1) *exists*".

(1) *Decompose* ρ *into the form* $\rho = z \cdot n^\alpha \cdot a/b$ *such that* $z \in F$, α *integer, and* $a, b \in F[n]$ *relatively prime and* q*-monic.*

(2) *Compute* $N = \mathrm{dis}_m(a,b) = \max\{ i \in \mathbb{N} \mid \deg \gcd q\ (a, \varepsilon^{im} b) \geq 1\}$.

If $N > 0$ *then compute for* j *from 1 to* N

$$P_j(n) = \gcd_q(\varepsilon^{-m} a, \varepsilon^{(j-1)m} b)$$
$$a = \frac{a}{\varepsilon^m P_j(n)}$$
$$b = \frac{b}{\varepsilon^{-(j-1)m} P_j(n)}$$

$$\hat{V} = [P_1]^1_{m_q} \cdot [P_2]^2_{m_q} \cdots [P_N]^N_{m_q}$$

otherwise $\hat{V} = 1$.

(3) *Determine the value of* γ *as follows*:

$$\gamma = \begin{cases} \frac{1}{m} \cdot \log_q(z) & \textit{if } \alpha = 0 \textit{ and } z \textit{ a positive integer power of } q, \\ 0 & \textit{otherwise.} \end{cases}$$

(4) Take $V = n^{\gamma} \cdot \hat{V}$. If equation (5.4) can be solved for $U \in F[n]$ then return $g_k = \frac{U(n)}{V(n)} f_k$, otherwise return "no qm-hypergeometric solution of (5.1) exists". □

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