Homotopy Perturbation Method for Solving Time-Delay Burgers Equation

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Abstract

In this paper, we use homotopy perturbation method to find numerical solutions for time- delay Burgers equation. In three examples for time-delay Burgers equation, we will compute the results with the use of homotopy perturbation method and compare with results of the exact solution to show the accuracy of this method.

1-Introduction

Time-delay Burgers equation is widely used in physical processes and do not have a precise analytical solution. So approximate methods are used to solve this equation. Fahmy[3] solved the generalized time delay Burgers-Huxley equation, the generalized time delay convective Fishers equation and the generalized time delay Burgers--Fisher equation by using factorization method to find explicit particular travelling wave solutions and yields to the general solution. Fahmy et al [2]used the improved tanh function method to construct exact multiple soliton and triangular periodic solutions and calculated the numerical solutions of this equation by using the adomian decomposition method and variational iteration method. While Kim and Sakthivel[6] used the $\frac{G'}{G}-$ expansion method, which is implemented to establish travelling wave solutions for time-delay

Burgers-Fisher equation. This solutions are expressed by hyperbolic functions and trigonometric functions. And by using the first –integral which is based on the ring theory of commutative algebra.

The homotopy perturbation method proposed by Ji-Huan He[8,9,10]. Many authers try to improved this method to solve various nonlinear problems [4,7,11,12,13,15]. HPM yields a very rapid convergence of the solution series and some time one iteration leads to high accuracy of the solution.

The generalized time delayed Burgers-Fisher equation takes in the following form[2]:

$$\tau u_{tt} + [1 - \tau f_u] u_t = u_{xx} - \delta u^s u_x + f(u), \qquad f(u) = qu(1-u) \qquad (1)$$

where τ, p are any real numbers and $s \in \mathbb{N}$.

When $q = \tau = 0$ and $\delta = s = 1$, Eq.(1) assumes the form of the classical Burgers equation:

$$u_t - u_{xx} + \delta u\, u_x = 0 \qquad (2)$$

In this work, we use the HPM to find the numerical solution for time-delay Burgers equation. The improved tanh function method is used by Fahmy and et al. [2] to find exact travelling wave solution for Eq.(1) for $q = 0$ and different values of $\tau, p, s; i.e.$, we solve the equation:

$$\tau u_{tt} + u_t = u_{xx} - \delta u^s u_x \qquad (3)$$

The paper is organized as follows: in the next section, the Homotopy perturbation method is introduced. In section 3, the improved tanh function method is presented. The application of homotopy perturbation method for solving time-delay Burgers equation is introduced in section 4. The numerical solutions of the problem are obtained in section 5.Section 6 ends this paper in conclusion.

2-Homotopy perturbation method[7,9]

To illustrate the HPM, Ji-Huan He considered the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega \qquad (4)$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma \qquad (5)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is a boundary of the domain Ω.

The operator A can be generally divided in to two parts L and N,where L is linear, and N is nonlinear, therefore equation(4) can be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \qquad (6)$$

The homotopy technique which is constructed by He[8], $v(r,p)$: $\Omega \times [0,1] \rightarrow \mathbb{R}$ satisfied:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \qquad (7a)$$

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \quad (7b)$$

where $r \in \Omega$ and $p \in [0,1]$ that is called homotopy parameter, and u_0 is an initial approximation of (4), which is satisfies the boundary conditions. Obviously, from equation(7),we have:

$$H(v,0) = L(v) - L(u_0) = 0, \quad (8)$$

$$H(v,1) = A(v) - f(r) = 0, \quad (9)$$

and the changing process of p from 0 to 1, is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic.

The embedding parameter $p \in [0,1]$ as a "small parameter" is used and assume that the solution of equation(6) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \quad (10)$$

Setting $p = 1$ results in the approximate solution of equation(4):

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots \quad (11)$$

The series (11) is convergent for most cases, however, the convergent rate depends upon the nonlinear operator $A(v)$ (the following opinions are suggested by He[8])

(1)The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e., $p \to 1$.

(2)The norm of $L^{-1}(\partial N/\partial v)$ must be smaller than one so that the series converges.

Theorem

Suppose that X and Y be Banach space and $N: X \to Y$ is a contraction nonlinear mapping, that is

$$\forall\, v, \tilde{v} \in X: \ \|N(v) - N(\tilde{v})\| \le \gamma \|v - \tilde{v}\|, \quad 0 < \gamma < 1$$

Which according to Banach's fixed point theorem, having the fixed point u, that is $N(u) = u$.

The sequence generated by the homotopy perturbation method will be regarded as

$$V_n = N(V_{n-1}), \quad V_{n-1} = \sum_{i=0}^{n-1} u_i, \quad n = 1,2,3 \ldots.$$

and suppose that $V_0 = v_0 = u_0 \in B_r(u)$ where $B_r(u) = \{u^* \in X / \|u^* - u\| < r\}$, then we have the following statements:

(i) $\|V_n - u\| \le \gamma^n \| v_0 - u\|$.

(ii) $V_n \in B_r(u)$.

(iii) $\lim_{n\to\infty} V_n = u$.

Proof: (i) By the induction method on n, for $n = 1$ we have

$$\|V_1 - u\| = \|N(V_0) - N(u)\| \le \gamma \|v_0 - u\|.$$

Assume that $\|V_{n-1} - u\| \le \gamma^{n-1} \|v_0 - u\|$ as an induction hypothesis, then

$$\|V_n - u\| = \|N(V_{n-1}) - N(u)\| \le \gamma \|V_{n-1} - u\| \le \gamma\gamma^{n-1}\|v_0 - u\| = \gamma^n \|v_0 - u\|.$$

(ii) Using (i), we have

$$\|V_n - u\| \le \gamma^n \|v_0 - u\| \le \gamma^n r < r \Rightarrow V_n \in B_r(u).$$

(iii) Because of $\|V_n - u\| \le \gamma^n \|v_0 - u\|$, and $\lim_{n\to\infty} \gamma^n = 0$, we drive

$\lim_{n\to\infty} \|V_n - u\| = 0$, that is $\lim_{n\to\infty} V_n = u$.

3-The improved tanh function method[2]

The improved tanh function method is used to construct exact multiple soliton and triangular periodic solutions to the time-delay Burgers equation in the form:

$$u_1(x,t) = [Q \mp Q(tanh\zeta \pm I\ sech\zeta)]^{1/s}, \zeta = k_1(x-\omega t). \quad (12)$$

$$u_2(x,t) = [Q \mp QI(coth\zeta \pm cosh\zeta)]^{1/s}, \zeta = k_2(x-\omega t). \quad (13)$$

$$u_3(x,t) = [Q \mp QI(tanh\zeta)]^{1/s}, \zeta = k_3(x-\omega t). \quad (14)$$

$$u_4(x,t) = [Q \mp Q(tanh\zeta)]^{1/s}, \zeta = k_4(x-\omega t). \quad (15)$$

$$u_5(x,t) = [Q \mp Q(tan\zeta)]^{1/s}, \zeta = k_5(x-\omega t). \quad (16)$$

$$u_6(x,t) = [Q \mp QI(cot\zeta)]^{1/s}, \zeta = k_6(x-\omega t). \quad (17)$$

Where

$I = \sqrt{-1}$

$Q = (1+s)\omega/2p$

$k_1 = \mp s\omega/(\tau\omega^2 - 1)$

$k_2 = \pm s\omega I/(\tau\omega^2 - 1)$

$k_3 = \mp s\omega I/(\tau\omega^2 - 1)$

$k_4 = \pm s\omega/(\tau\omega^2 - 1)$

$k_5 = \pm s\omega I/(\tau\omega^2 - 1)$

$k_6 = \pm s\omega I/(\tau\omega^2 - 1)$

When $\tau \to 0$ the solution of the equation(2) in the following form:

$$u_1(x,t) = [Q \mp Q(tanh\zeta \pm I\ sech\zeta)]^{1/s}, \zeta = \pm s\omega(x-\omega t). \quad (18)$$

$$u_2(x,t) = [Q \mp QI(coth\zeta \pm cosh\zeta)]^{1/s}, \zeta = \mp s\omega I(x-\omega t). \quad (19)$$

$$u_3(x,t) = [Q \mp QI(tanh\zeta)]^{1/s}, \zeta = \pm s\omega I(x-\omega t). \quad (20)$$

$$u_4(x,t) = [Q \mp Q(tanh\zeta)]^{1/s}, \zeta = \mp s\omega(x-\omega t). \quad (21)$$

$$u_5(x,t) = [Q \mp Q(tan\zeta)]^{1/s}, \zeta = \mp s\omega I(x-\omega t). \quad (22)$$

$$u_6(x,t) = [Q \mp QI(cot\zeta)]^{1/s}, \zeta = \pm s\omega I(x-\omega t). \quad (23)$$

Such that $u_4(x,t)$ is the real solution but the others are the complex solutions for time-delay Burgers equation.

4- Application of homotopy perturbation method

At first, we construct a homotopy perturbation method for equation(3) as follows:

$$(1-p)\left(\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 u_0}{\partial t^2}\right) + p\left(\frac{\partial^2 v}{\partial t^2} + \frac{1}{\tau}\left(\frac{\partial v}{\partial t} + \delta v^s\frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2}\right)\right) = 0 \quad (24)$$

By substituting (10) into (24) and equating the coefficients of like terms with the identical powers of p for $s = 1$ and $s = 2$.

(1) We putting $s = 1$ in Eq.(24) and obtain the equations corresponding to powers of p as follows:

$$\boldsymbol{p^0}: \frac{\partial^2 v_0}{\partial t^2} = \frac{\partial^2 u_0}{\partial t^2} \quad (25)$$

$$\boldsymbol{p^1}: \frac{\partial^2 v_1}{\partial t^2} = -\frac{\partial^2 u_0}{\partial t^2} - \frac{1}{\tau}\left(\frac{\partial v_0}{\partial t} + \delta v_0\frac{\partial v_0}{\partial x} - \frac{\partial^2 v_0}{\partial x^2}\right) \quad (26)$$

$$\boldsymbol{p^2}: \frac{\partial^2 v_2}{\partial t^2} = -\frac{1}{\tau}\left(\frac{\partial v_1}{\partial t} + \delta v_1\frac{\partial v_0}{\partial x} + \delta v_0\frac{\partial v_1}{\partial x} - \frac{\partial^2 v_1}{\partial x^2}\right) \quad (27)$$

$$\boldsymbol{p^3}: \frac{\partial^2 v_3}{\partial t^2} = -\frac{1}{\tau}\left(\frac{\partial v_2}{\partial t} + \delta v_0\frac{\partial v_2}{\partial x} + \delta v_1\frac{\partial v_1}{\partial x} + \delta v_2\frac{\partial v_0}{\partial x} - \frac{\partial^2 v_2}{\partial x^2}\right) \quad (28)$$

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$$p^j: \frac{\partial^2 v_j}{\partial t^2} = -\frac{1}{\tau}\left(\frac{\partial v_{j-1}}{\partial t} + \delta \sum_{k=0}^{j-1} v_j \frac{\partial v_{j-k-1}}{\partial x} - \frac{\partial^2 v_{j-1}}{\partial x^2}\right) \quad (29)$$

By integration of the both sides of the equations ,we obtain the following multiple solutions:

$$v_0 = u_0 \quad (30)$$

$$v_j = -\frac{1}{\tau}\int_0^t\int_0^t \left(\frac{\partial v_{j-1}}{\partial t} + \delta \sum_{k=0}^{j-1} v_k \frac{\partial v_{j-k-1}}{\partial x} - \frac{\partial^2 v_{j-1}}{\partial x^2}\right) dtdt \quad ,j = 1,2,3,\dots \quad (31)$$

(2) We putting $s = 2$ in Eq.(24) and obtain the equations corresponding to powers of p as follows:

$$p^0: \frac{\partial^2 v_0}{\partial t^2} = \frac{\partial^2 u_0}{\partial t^2} \quad (32)$$

$$p^1: \frac{\partial^2 v_1}{\partial t^2} = -\frac{\partial^2 u_0}{\partial t^2} - \frac{1}{\tau}\left(\frac{\partial v_0}{\partial t} + \delta {v_0}^2 \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_0}{\partial x^2}\right) \quad (33)$$

$$p^2: \frac{\partial^2 v_2}{\partial t^2} = -\frac{1}{\tau}\left(\frac{\partial v_1}{\partial t} + \delta {v_0}^2 \frac{\partial v_1}{\partial x} + 2\delta v_0 v_1 \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_1}{\partial x^2}\right) \quad (34)$$

$$p^3: \frac{\partial^2 v_3}{\partial t^2} = -\frac{1}{\tau}\left(\frac{\partial v_2}{\partial t} + \delta {v_0}^2 \frac{\partial v_2}{\partial x} + 2\delta v_0 v_1 \frac{\partial v_1}{\partial x} + 2\delta v_0 v_2 \frac{\partial v_0}{\partial x} + \delta {v_1}^2 \frac{\partial v_0}{\partial x} - \frac{\partial^2 v_2}{\partial x^2}\right) \quad (35)$$

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$$p^j: \frac{\partial^2 v_j}{\partial t^2} = -\frac{1}{\tau}\left(\frac{\partial v_{j-1}}{\partial t} + \delta \sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} v_i v_k \frac{\partial v_{j-k-i-1}}{\partial x} - \frac{\partial^2 v_{j-1}}{\partial x^2}\right) \quad (36)$$

By integration of the both sides of the equations ,we obtain the following multiple solutions:

$$v_0 = u_0 \quad (37)$$

$$v_j = -\frac{1}{\tau}\int_0^t\int_0^t \left(\frac{\partial v_{j-1}}{\partial t} + \delta \sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} v_i v_k \frac{\partial v_{j-k-i-1}}{\partial x} - \frac{\partial^2 v_{j-1}}{\partial x^2}\right) dtdt \quad ,j = 1,2,3,\dots \quad (38)$$

We obtain the approximate solution of equation(3)by setting $p = 1$ as follows:

$$u = v_0 + v_1 + v_2 + \cdots$$

The equation (24) is defined for all values of $\tau \neq 0$.

So that, we will study the homotopy perturbation method for time-delay Burgers equation for $\tau = 0$ but in this case we take $L = \frac{\partial}{\partial t}$,so that we have Burgers equation without time delay in the following form :

$$u_t = u_{xx} - \delta u^s u_x \quad (39)$$

A homotopy can be constructed to eq.(39) as follows:

$$(1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial v}{\partial t} + \delta v^s \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2}\right) = 0 \quad (40)$$

By substituting (10) into (40) and equating the coefficients of like terms with the identical powers of p and take for $s = 1$ and $s = 2$.

(1) We putting $s = 1$ in Eq.(40) and obtain the equations corresponding to powers of p as follows:

$$p^0: \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t} \quad (41)$$

$$p^1: \frac{\partial v_1}{\partial t} = -\frac{\partial u_0}{\partial t} - \delta v_0 \frac{\partial v_0}{\partial x} + \frac{\partial^2 v_0}{\partial x^2} \quad (42)$$

$$p^2: \frac{\partial v_2}{\partial t} = -\delta v_0 \frac{\partial v_1}{\partial x} - \delta v_1 \frac{\partial v_0}{\partial x} + \frac{\partial^2 v_1}{\partial x^2} \quad (43)$$

$$\boldsymbol{p^3}: \frac{\partial v_3}{\partial t} = -\delta v_0 \frac{\partial v_2}{\partial x} - \delta v_1 \frac{\partial v_1}{\partial x} - \delta v_2 \frac{\partial v_0}{\partial x} + \frac{\partial^2 v_1}{\partial x^2} \qquad (44)$$

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$$\boldsymbol{p^j}: \frac{\partial v_j}{\partial t} = -\delta \sum_{k=0}^{j-1} v_k \frac{\partial v_{j-k-1}}{\partial x} + \frac{\partial^2 v_{j-1}}{\partial x^2} \qquad (45)$$

By integration of the both sides of the equations ,we obtain the following multiple solutions:

$$v_0 = u_0 \qquad (46)$$

$$v_j = \int_0^t \left(\frac{\partial^2 v_{j-1}}{\partial x^2} - \delta \sum_{k=0}^{j-1} v_k \frac{\partial v_{j-k-1}}{\partial x} \right) dt \quad , \ j = 1,2,3,\ldots \qquad (47)$$

(2) We putting $s = 2$ in Eq.(40) and obtain the equations corresponding to powers of p as follows:

$$\boldsymbol{p^0}: \frac{\partial v_0}{\partial t} = \frac{\partial u_0}{\partial t} \qquad (48)$$

$$\boldsymbol{p^1}: \frac{\partial v_1}{\partial t} = -\frac{\partial u_0}{\partial t} - \delta {v_0}^2 \frac{\partial v_0}{\partial x} + \frac{\partial^2 v_0}{\partial x^2} \qquad (49)$$

$$\boldsymbol{p^2}: \frac{\partial v_2}{\partial t} = -\delta {v_0}^2 \frac{\partial v_1}{\partial x} - 2\delta v_0 v_1 \frac{\partial v_0}{\partial x} + \frac{\partial^2 v_1}{\partial x^2} \qquad (50)$$

$$\boldsymbol{p^3}: \frac{\partial v_3}{\partial t} = -\delta {v_0}^2 \frac{\partial v_2}{\partial x} - 2\delta v_0 v_1 \frac{\partial v_1}{\partial x} - 2\delta v_0 v_2 \frac{\partial v_0}{\partial x} - \delta {v_1}^2 \frac{\partial v_0}{\partial x} + \frac{\partial^2 v_1}{\partial x^2} \qquad (51)$$

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$$\boldsymbol{p^j}: \frac{\partial v_j}{\partial t} = -\delta \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} v_i v_k \frac{\partial v_{j-k-i-1}}{\partial x} + \frac{\partial^2 v_{j-1}}{\partial x^2} \qquad (52)$$

By integration of the both sides of the equations ,we obtain the following multiple solutions:

$$v_0 = u_0 \qquad (53)$$

$$v_j = \int_0^t \left(\frac{\partial^2 v_{j-1}}{\partial x^2} - \delta \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} v_i v_k \frac{\partial v_{j-k-i-1}}{\partial x} \right) dt \quad , \ j = 1,2,3,\ldots \qquad (54)$$

5-Numerical examples

Example 5.1[2]:

By taking $s = 1, \delta = 0.1, \tau = 0.5, k = -0.05, \omega = 0.1, t = 0.1$ and apply the relation(31), we get:

$$u_0(x,t) = 1 + 0.00502513\, t\, \mathrm{sech}^2(0.05025x) - \tanh(0.05025x)$$

$$\begin{aligned} u_1(x,t) = {} & [4.229698 \times 10^{-8} sech^4(0.05025x) \tanh(0.05025x)]t^4 \\ & - 0.666667\big[0.000025\, sech^2(0.05025x)\, tanh^2(0.05025x) \\ & + 0.000252\, sech^2(0.05025x)\big(0.05025 - 0.05025\, tanh^2(0.05025x)\big) \\ & - 0.0000252(1 - \tanh(0.05025x))sech^2(0.05025x)tanh(0.05025x) \\ & + 0.000251\, sech^2(0.05025x)\big(-0.05025 + 0.05025\, tanh^2(0.05025x)\big)\big]t^3 \\ & - \big[0.005025\, sech^2(0.05025x)\big(-0.05025 + 0.05025\, tanh^2(0.05025x)\big) \\ & - 0.1005 \tanh(0.05025x)\, \big(0.05025 - 0.05025 tanh^2(0.05025x)\big)\big]t^2. \end{aligned}$$

$$u_2(x,t) = -0.33333[-3.41768 \times 10^{-10} sech^4(0.05025x) tanh^3(0.05025) + 5.95118 \times 10^{-9}\, sech^4(0.05025x) tanh(0.05025x)(0.05025 - 0.05025\, tanh^2(0.05025))]\, t^6 - 0.4(4.22968 \times 10^{-8} sech^4(0.05025x) tanh(0.05025x) - 4.27199 \times 10^{-8} sech^2(0.05025x) tanh^4(0.05025x) + 0.0000029 sech^2(0.05025x) tanh^2(0.05025x)(0.05025 - (0.05025x) tanh^2(0.05025x)) - \cdots$$

The other omponents of the decomposition series can be easily obtained and the general solution can be evaluated in the series form:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

Table (1): represents the exact solution $u_E(x,t)$, numerical solution $u_n(x,t)$ and the error between them where $s=1, \delta=0.1, \tau=0.5, k=-0.05, \omega=0.1, t=0.1$

x_i	$u_E(x,t)$	$u_n(x,t)$	Error $= \lvert u_E(x,t) - u_n(x,t)\rvert$
-100	1.9999137406612	1.9999137189862	$2.16749195474\times10^{-8}$
-90	1.9997643591661	1.9997643058809	$5.32851721193\times10^{-8}$
-80	1.9993563662586	1.9993562376914	$1.29344458214\times10^{-7}$
-70	1.9982425876656	1.9982422788189	$3.08846732807\times10^{-7}$
-60	1.9952060876758	1.9952053666574	$7.21018379945\times10^{-7}$
-50	1.9869572934034	1.9869556654669	$1.62796479616\times10^{-6}$
-40	1.9647656060952	1.9647621271914	$3.47890383863\times10^{-6}$
-30	1.9065906431675	1.9065839309271	$6.71224039126\times10^{-6}$
-20	1.7639058667419	1.7638954030800	$1.04636618550\times10^{-5}$
-10	1.4644850500518	1.4644751973837	$9.85266808845\times10^{-6}$
0	1.0005025125205	1.0005025117424	$7.78066948196\times10^{-10}$
10	0.5363035124100	0.5363133730499	$9.86063986806\times10^{-6}$
20	0.2365129952255	0.2365234743102	$1.04790847483\times10^{-5}$
30	0.0935885083529	0.0935952325555	$6.72420266623\times10^{-6}$
40	0.0353040366091	0.0353075221569	$3.48554785096\times10^{-6}$
50	0.0130687779771	0.0130704091056	$1.63112850177\times10^{-6}$
60	0.0048035348530	0.0048042573002	$7.22447129949\times10^{-7}$
70	0.0017609452607	0.0017612547222	$3.09461544640\times10^{-7}$
80	0.0006449283610	0.0006450579635	$1.29602519452\times10^{-7}$
90	0.0002361149042	0.0002361682958	$5.33916202232\times10^{-8}$
100	0.0000864328913	0.0000864546096	$2.17182562951\times10^{-8}$

Example 5.2[2]:

By taking $s = 2, \delta = 1, \tau = 0.2, k = -0.1002, \omega = 0.1, t = 1$ and apply the relation(39), we get:

$$u_0(x,t) = \frac{0.000751503\ t\ \text{sech}^2(0.1002x)}{\sqrt{0.15 - 0.15\tanh(0.1002x)}} - \sqrt{0.15 - 0.15\tanh(0.1002x)}$$

$$u_1(x,t) = -\frac{1}{0.15 - 0.15\tanh(0.1002x)}[1.41189210^{-7}\ \text{sech}^4(0.1002x)$$

$$(-\frac{0.00015\text{sech}^2(0.1002x)\tanh(0.1002x)}{\sqrt{0.15 - 0.15\tanh(0.1002x)}}(-\frac{0.00015\text{sech}^2(0.1002x)\tanh(0.1002x)}{\sqrt{0.15 - 0.15\tanh(0.1002x)}}$$

$$-\frac{0.000376\text{sech}^2(0.1002x)\left(-0.015+0.015\tanh^2(0.1002x)\right)}{\left(0.15-0.15\tanh(0.1002x)\right)^{3/2}})\ t^5]$$

$$-1.25[0.0005\text{sech}^2(0.1002x)(-\frac{0.0005\text{sech}^2(0.1002x)\tanh(0.1002x)}{\sqrt{0.15-0.15\tanh(0.1002x)}}$$

$$-\frac{0.000375\text{sech}^2(0.1002x)\left(-0.015+0.015\tanh^2(0.1002x)\right)}{\left(0.15-0.15\tanh(0.1002x)\right)^{3/2}})$$

$$+\frac{9.4126\times10^{-8}\text{sech}^4(0.1002x)\left(-0.015+0.015\tanh^2(0.1002x)\right)}{\left(0.15-0.15\tanh(0.1002x)\right)^{3/2}}]\ t^4$$

$$-1.66667(-\frac{0.000282\text{sech}^2(0.1002x)\left(-0.015+0.015\tanh^2(0.1002x)\right)^2}{\left(0.15-0.15\tanh(0.1002x)\right)^{5/2}}$$

$$-\frac{0.000015\text{sech}^2(0.1002x)\tanh^2(0.1002x)}{\sqrt{0.15-0.15\tanh(0.1002x)}}$$

$$+0.5\left(0.15-0.15\tanh(0.1002x)\right)(-\frac{0.00015\text{sech}^2(0.1002x)\tanh(0.1002x)}{\sqrt{0.15-0.15\tanh(0.1002x)}}$$

$$-\frac{0.000375\text{sech}^2(0.1002x)\left(-0.015+0.015\tanh^2(0.1002x)\right)}{\left(0.15-0.15\tanh(0.1002x)\right)^{3/2}}$$

$$+\frac{0.000375\text{sech}^2(0.1002x)\left(-0.015+0.015\tanh^2(0.1002x)\right)}{\sqrt{0.15-0.15\tanh(0.1002x)}}$$

$$+\frac{0.000375\text{sech}^2(0.1002x)\left(0.1002-0.1002\tanh^2(0.1002x)\right)}{\sqrt{0.15-0.15\tanh(0.1002x)}}$$

$$-\frac{0.753\times10^{-4}10^{-5}\text{sech}^2(0.1002x)\tanh(0.1002x)\left(-0.015+0.015\tanh^2(0.1002x)\right)}{\left(0.15-0.15\tanh(0.1002x)\right)^{3/2}}]t^3$$

$$-2.5[\frac{0.000751\text{sech}^2(0.1002x)}{\sqrt{0.15-0.15\tanh(0.1002x)}}+0.5\sqrt{0.15-0.15\tanh(0.1002x)}.$$

$$\left(-0.015+0.015\tanh^2(0.1002x)\right)+\frac{0.25\left(-0.015+0.015\tanh^2(0.1002x)\right)^2}{\left(0.15-0.15\tanh(0.1002x)\right)^{3/2}}$$

$$-\frac{0.015\tanh(0.1002x)\left(0.1002-0.1002\tanh^2(0.1002x)\right)}{\sqrt{0.15-0.15\tanh(0.1002x)}}]t^2.$$

⋮

The other components of the decomposition series can be easily obtained and the general solution can be evaluated in the series form:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

Table (2): represents the exact solution $u_E(x,t)$, numerical solution $u_n(x,t)$ and the error between them where $s = 2, \delta = 1, \tau = 0.2, k = -0.1002, \omega = 0.1, t = 1$.

x_i	$u_E(x,t)$	$u_n(x,t)$	Error $= \lvert u_E(x,t) - u_n(x,t)\rvert$
-100	0.5477225576974	0.5477225576974	$1.79656420106\times10^{-13}$
-90	0.5477225535618	0.5477225535632	$1.36442147066\times10^{-12}$
-80	0.5477225282508	0.5477225282611	$1.03566944886\times10^{-11}$
-70	0.5477223404752	0.5477223405538	$7.85724595324\times10^{-11}$
-60	0.5477209474243	0.5477209480201	$5.95800380034\times10^{-10}$
-50	0.5477106130866	0.5477106176019	$4.51527680565\times10^{-9}$
-40	0.5476339636780	0.5476339978568	$3.41787895862\times10^{-8}$
-30	0.5470663254792	0.5470665827096	$2.57230350268\times10^{-7}$
-20	0.5429095762244	0.5429114372911	$1.86106673517\times10^{-6}$
-10	0.5147714859709	0.5147817043033	$1.02183323305\times10^{-5}$
0	0.3892338059738	0.3892441631525	$1.03571787131\times10^{-5}$
10	0.1904440774649	0.1904334432299	$1.06342350485\times10^{-5}$
20	0.0738914528172	0.0738849288700	$6.52394719524\times10^{-6}$
30	0.0273448743074	0.0273424068205	$2.46748694154\times10^{-6}$
40	0.0100503233630	0.0100494455049	$8.74831439645\times10^{-7}$
50	0.0036904479076	0.0036901349421	$3.06848578941\times10^{-7}$
60	0.0013549462816	0.0013548390326	$1.07249073793\times10^{-7}$

70	0.0004974602865	0.0004974229034	$3.73831383763\times10^{-8}$
80	0.0001826391003	0.0001826261073	$1.29930409479\times10^{-8}$
90	0.0000670546606	0.0000670501590	$4.50156830704\times10^{-9}$
100	0.0000246186458	0.0000246170917	$1.55405367367\times10^{-9}$

Example 5.3[2]:

By taking $s = 2, \delta = 0.1, \tau = 0, k = -0.1, \omega = 0.1, t = 1, \lambda = -1$ and apply the relation(54), we get:

$$u_0(x,t) = \sqrt{1.5 - 1.5\tanh(0.1x)}$$

$$u_1(x,t) = [-\frac{\left(-0.075 + 0.075\, tanh^2(0.1x)\right)^2}{(1.5 - 1.5\tanh(0.1x))^{3/2}} + \frac{0.015\tan(0.1x) - 0.015\tan^2(0.1x)}{\sqrt{1.5 - 1.5\tanh(0.1x)}}$$

$$-\sqrt{1.5 - 1.5\tanh(0.1x)}(-0.0075 + 0.0075 tanh^2(0.1x))]\ t\ .$$

⋮

The other components of the decomposition series can be easily obtained and the general solution can be evaluated in the series form:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

x_i	$u_E(x,t)$	$u_n(x,t)$	Error $= \lvert u_E(x,t) - u_n(x,t)\rvert$
-100	1.7320508058192	1.7320508058203	$1.07815872563\times10^{-12}$
-90	1.7320507946405	1.7320507946485	$7.96657476529\times10^{-12}$
-80	1.7320507120404	1.7320507120992	$5.88654382687\times10^{-11}$
-70	1.7320501017037	1.7320501021387	$4.34958410191\times10^{-10}$
-60	1.7320455919122	1.7320455951260	$3.21384389372\times10^{-9}$
-50	1.7320122769901	1.7320122936436	$2.37424578689\times10^{-8}$
-40	1.7317661112843	1.7317662864560	$1.75171705695\times10^{-7}$
-30	1.7299504785319	1.7299517586419	$1.28010988736\times10^{-6}$
-20	1.7167093460982	1.7167180646277	$8.71852948998\times10^{-6}$
-10	1.6274652154109	1.6275008022122	$3.55868012626\times10^{-5}$
0	1.2308531594061	1.2308310943157	$2.20650903684\times10^{-5}$
10	0.6032880851427	0.6032651481658	$2.29369769126\times10^{-5}$
20	0.2345819091346	0.2345800544536	$1.85468099796\times10^{-6}$
30	0.0869904729399	0.0869903739572	$9.89827159169\times10^{-8}$
40	0.0320369630941	0.0320369581079	$4.98620485895\times10^{-9}$
50	0.0117874837021	0.0117874834493	$2.52778189808\times10^{-10}$
60	0.0043364597489	0.0043364597347	$1.41371051391\times10^{-11}$
70	0.0015952987123	0.0015952987110	$1.27403013924\times10^{-12}$
80	0.0005868778140	0.0005868778137	$2.73184412079\times10^{-13}$
90	0.0002159002930	0.0002159002929	$9.07652681263\times10^{-14}$
100	0.0000794252797	0.0000794252796	$3.29060629444\times10^{-14}$

Table (3): represents the exact solution $u_E(x,t)$, numerical solution $u_n(x,t)$ and the error between them where $s = 2, \delta = 0.1, \tau = 0, k = -0.1, \omega = 0.1, t = 1, \lambda = -1$

6- Conclusion

We solved the time-delay Burgers equation by homotopy perturbation method. The exact and numerical results are presented and compared each other in tables 1,2 and 3 for some values of x and for three values of t and τ in three examples. We notice from tables 1,2 and 3 that the HPM is very accurate method since the error is very small and afew terms are required to obtain an accurate solution in comparison with the results of VIM and ADM, see[2]. So that the HPM is remarkably effective for solve the time delay Burgers equation. In our work, we use the Maple13 to calculate the results which is obtained from the iteration method HPM.

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تطبيق طريقة الاضطراب الهوموتوبي في حل معادلة بيركر التباطؤية

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الملخص

في هذا البحث، سنستعمل طريقة الاضطراب الهوموتوبي لإيجاد الحل التقريبي لمعادلة بيركر التباطؤية. في ثلاثة أمثلة لمعادلة بيركر التباطؤية ، سنقوم بحساب النتائج باستخدام طريقة اضطراب الهوموتوبي ونقارنها مع نتائج الحل التام لبيان دقة تلك الطريقة.