Monoidal (k, q+m; *f*) – arcs of type (m, q+m) in PG(2, q)

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Abstract

In this paper we discuss a **Monoidal (k, q+m; f)-arcs** *of type (m, q+m) in the projective plane of order q and we proved that the points of weight 0 form a* $(q^2-(m-1)q\ ,\ q)$ *– arc in which* $|\text{q-secants}| = q-(m-1)$, $|\text{q-(m-1)-secants}| = q^2$, $|\text{0-secants}| = m$ *and* $|i-\text{secants}| = 0$, $i \neq q$, $q-(m-1)$, 0. *Also we proved that the points of weight 1 form a (mq, q) – arc with* $|\text{q-secants}| = m$, $|\text{m-secants}| = q^2$, $|\text{0-secants}| = q-(m-1)$ *and* $|i-\text{secants}| = 0$, $i \neq q, m, 0$, *and we gave some examples when q=3,5,7 .*

Introduction

A (k , n)-arc **k** in PG(2,q) is a set of k points such that some line of the plane meets **k** in n points but such that no line meets **k** in more than n points , where $n \geq 2$. A line ℓ in PG(2,q) is an i-secant of a (k , n)-arc **k** if $|\ell \cap k| = i$. Let τ_i denote the total number of i-secants to **k** in π = PG(2, q), then we have the following lemma :

Lemma [J.W.P. Hirschfeld, 1979]

For a (k , n)-arc **k**, the following equations hold :

(1) $\sum_{i=0}^{n} \tau_i = q^2+q+1$; (2) $\sum_{i=1}^{n} i\tau_i = k(q+1)$;

(3) $$\sum_{i=2}^{n} \frac{i(i-1)}{2}\tau_i = \frac{k(k-1)}{2} \quad ;$$

Def . A (k, n) – arc is complete if there is no (k+1, n) – arc containing it .

Let $\boldsymbol{K}$ be the set of k points in a finite projective plane of order q. Assume that $\boldsymbol{K}$ is partitioned into subsets $W_i (i = 0,1,2,...,g)$ and that to each point of W_i has given a weight $w_i > 0$. The set $\boldsymbol{K}$ is defined to be a $(k,n;\{w_i\})-arc\ \ (n \geq q)$ if n is the maximum value of $\sum w_i p_i^{(s)}$, where

s is any line of the plane and $p_i^{(s)}$ denote the number of points of W_i which belong to the line s . Note that if g = 1 and $w_i = 1$

We obtained the definition of a (k , n) – arc .

If $\boldsymbol{K}$ is a $(k, n; \{w_i\}) - arc$ and $p \in K$, the weight of p will be denoted by $w(p)$. A line s of the plane such that $\sum w_i p_i^{(s)} = m_j$ will be called an $m_j -$secant .If $m_j = 1$, then s is called a tangent. If s does not intersect the set $\boldsymbol{K}$, will be called an external line. Let $\{m_1, m_2, ..., m_{i-1}, n\}$, $(m_1 < m_2 < ... < m_{i-1} < n)$ be the set of values taken by the integers $\sum w_i p_i^{(s)}$, where s describes the set of the lines in the plane . Then $\boldsymbol{K}$ is said to be of type $(m_1, m_2, \ldots, m_{i-1}, n)$.

Barnabei in 1979, has proved to be a necessary condition for the existence of a $(k, n; \{w_i\}) - arc$ of type (m , n) $0 < m < n$ is that

$q \equiv 0 \bmod (n - m)$ . . . (i)

and $g \le n - m$ . . . (ii)

The case of $m = n - 2$ was discussed by (D'Agostini, 1979) . The case of $m = n - 3$ was discussed by (B. J. Wilson, 1986) and (F.K.Hameed, 1989) . The case of $m = n - 5$ was discussed by(Raida D. Mahmood,1990) .

Monoidal arcs

In this section we consider $(k, q+m; \{w_i\}) - arc$ of type (m, q+m) in which only one point p whose weight m and $1 < m \le q$, while all other points of the arc have weight one. Such arcs will be called *monoidal arcs*.

Note that, deleting the point p from our monoidal arc, we obtain an ordinary $(k, n) - arc$. Let t_i be the number of lines of weight i in PG(2, q), then in the monoidal arc of type (m, q+m) which have minimal weight $W=m(q+1)$, we get :

Since there are only m – secants and (q+m) – secants in the monoidal arc then

$$t_m + t_{q+m} = q^2 + q + 1$$

$$mt_m + (q+m)t_{q+m} = W(q+1) = m(q+1)^2 = mq^2 + 2mq + m$$

$$mt_m + mt_{q+m} = mq^2 + mq + m$$, then we get

$$qt_{q+m} = mq \Rightarrow t_{q+m} = m \Rightarrow t_m = q^2 + q - (m-1)$$

Let ***K*** be a $(k,n;\{w_i\})-arc$ of type (m,n) , $m > 0$ and let v_m^s and v_n^s respectively the number of lines of weight *m* and the number of lines of weight n passing through a point of weight s . Then

$$(n-m)v_m^s = (n-s)(q+1) - (W-s)\ ;$$

$$(n-m)v_n^s = (W-s) - (m-s)(q+1)\ .$$

From the above and Im *f* = { 0, 1, *m* }, we deduce the following :

$$\left.\begin{array}{ll} v_m^0 = q+1 & v_{q+m}^0 = 0 \\ v_m^1 = q & v_{q+m}^1 = 1 \\ v_m^m = q-(m-1) & v_{q+m}^m = m \end{array}\right\} \qquad (*)$$

Since the number of points of weight *m* in the monoidal arc is one, then on (q+m) – secant one point of weight *m* and since there is no point of weight 0 on it then the number of points of weight 1 on it are q . Suppose that on m – secant of the monoidal arc are *u* points of weight 0, *v* points of weight 1 and *w* points of weight *m*, then by counting the points on m – secant we get :

$$u + v + w = q+1 \qquad (1.a)$$

and by counting the weights of its points we get :

$$v + m.w = m \qquad (1.b)$$

From (1.a), (1.b) and the above results we obtain :

Table (1)

Type of lines	The number of points of weight *m*	The number of points of weight 1	The number of points of weight 0	Total number
(q+m) – secant	1	q	0	*m*
m – secant Through ***p***	1	0	q	q – (m – 1)
m – secant Through Q ≠ ***p***	0	*m*	q – (m – 1)	q^2

Lemma(1) no q+1 points of weight zero can be collinear.

Proof: From Table (1) .

Let l_j be the number of points of weight j . Then we have

$$l_m = 1 \Rightarrow l_1 v^1_{q+m} = qt_{q+m} \Rightarrow l_1 = mq \Rightarrow l_0 + l_1 + l_m = q^2 + q + 1 \Rightarrow l_0 = q^2 - (m-1)q$$

Then we get the following theorem .

Theorem(1)

There is a monoidal (mq+1, q+m; f) – arc of type (m, q+m) in PG(2,q) with Im f = { 0, 1, m } and the points of weight 0 form a ($q^2 - (m-1)q$, q) – arc in which $|\text{q - secants}|$ = q – (m – 1), $|\text{q - (m-1) - secants}| = q^2$, $|\text{0 - secants}|$ = m and $|i - \text{secants}| = 0$, $i \neq$ q , q – (m – 1), 0.

Corollary : The points of weight 1 form a (mq, q) – arc with $|\text{q - secants}|$ = m, $|\text{m - secants}| = q^2$, $|\text{0 - secants}|$ = q – (m – 1) and $|i - \text{secants}| = 0$, $i \neq$ q , m, 0.

Examples of (k, q+2, f) – arcs of type (2, q+2) in PG(2, q) , q = 3, 5

(1) Now, when *q= 3*, then from (i) we must have $q = 3^h$, $h \geq 1$ and then (ii) requires that $g \leq 3$. In order to have an arc which is not simply a $(k,n) - arc$ we must have that

$$2(q+1) \leq W \leq q(q+1) + 2$$

It may easily be shown that from theorem(1) the following result :

The points of weight zero form (6, 3)-arc of type (2, 9, 0, 2) in PG(2, 3) **.**

For $W = 2(q+1)$ and from (*) we get **:**

$V^0_2 = 4$ $V^0_5 = 0$

$V^1_2 = 3$ $V^1_5 = 1$

$V^2_2 = 2$ $V^2_5 = 2$

proposition

For the existence of (7,5; f) – arc ***K*** of type (2, 5) , and the point of weight 0 form (6,3) – arc in $PG(2,3)$ we must have the following:

The number *of* 5 – secants *of K is* 2.

The number *of* 2 – secants *of K is* 11.

The number of points *of weight* 2 *is* 1.

The number of points *of weight*1 *is* 6.

From the above we get the following example in PG(2, 3)

Let p = (1,0,0) be the point of weight 2, the points of weight 1 are

(0,1,0), (1,1,0), (1,2,0), (1,0,1), (1,0,2), (0,0,1) and the points of weight 0 are

(1,2,2), (0,1,1), (0,1,2), (1,2,1), (1,1,2), (1,1,1) .

(2) No six points of weight zero in PG(2,5) can be collinear.

When $q = 5$, then from (i) we must have $q = 5^h$, $h \geq 1$ and then (ii) requires that $g \leq 5$. In order to have an arc which is not simply $a \quad (k, n) - arc$ we must have that

$$2(q+1) \leq W \leq q(q+1) + 2$$

It may easily be shown that from theorem (1) the following result :

The points of weight zero form (20, 5)-arc of type (4, 25, 0, 0, 0, 2) in PG(2,5) **.**

From (*) we get **:**

$$V_2^0 = 6 \qquad V_7^0 = 0$$

$$V_2^1 = 5 \qquad V_7^1 = 1$$

$$V_2^2 = 4 \qquad V_7^2 = 2$$

Lemma(2)

Through a point of weight 2 there are four 5-secant zero 4-secant and two 0-secant.

Proof

Since $V_7^2 = 2$ *and* $V_2^2 = 4$ suppose p is a point of weight two and pass through p zero 5-secant, five 4-secant and one 0-secant of the (20,5) –arc, since the i -secant $(i = 4,5)$ of the (20,5)-arc are 2- secant of K and the 0-secant of a(20,5) –arc which have point of weight 2 are 7-secant of K. therefore through p there pass 5 2-secant and one 7-secant which is contradiction .

Hence the points of type zero 5-secant, five 4-secant and one 0-secant are not point of weight 2.

Suppose Q is a point and through Q there pass four 5-secant, zero 4-secant and two 0-secant (which are 2-secant of K) and two 0-secant of (20,5)-arc which are either 7-secant or 2-secant with respect to K according to Q is a point of weight one respectively. Hence Q is possibly a point of weight 2.

Theorem(2)

For the existence of $(11,7;f) -$ arc ***K*** of type (2, 7) , and the point of weight 0 form (20,5) – arc in $PG(2,5)$ we must have the following:

The number *of* $7-$secants *of K is* 2.

The number *of* $2-$secants *of K is* 29.

The number of points *of weight* 2 *is* 1.

The number of points *of weight* 1 *is* 10.

Remarks If $q = 5$, the points of weight one are

(0,1,0) , (0,0,1) , (1,0,1) , (1,3,0) , (1,0,3) , (1,2,0) , (1,0,4) , (1,1,0) ,
(1,0,2) , (1,4,0), the point of weight 2 is **(1,0,0)** and the remaining points of $PG(2,5)$ are assigned weight zero .

Monoidal (29, 11; *f*) – arcs of type (4, 11) in PG(2, 7)

In this section we discuss a monoidal arc with Im f = { 0, 1, 4 } in PG(2, 7) . From theorem(1) we get the following result :

The points of weight zero form (28, 7)-arc of type (4, 49, 0, 0, 0, 0, 0, 4) in PG(2, 7) **.**

Then from (*) we get **:**

$$V_4^0 = 8 \qquad V_{11}^0 = 0$$

$$V_4^1 = 7 \qquad V_{11}^1 = 1$$

$$V_4^4 = 4 \qquad V_{11}^4 = 4$$

Lemma(3)

For the existence of (29, 11; f)-arc ***K*** of type (4, 11), and the point of weight 0 form (28,7)-arc in $PG(2,7)$ we must have the following:

1) The number of 11−secants $of\ K\ is$ 4.
2) The number of 4−secants $of\ K\ is$ 53.
3) The number of points $of weight 4\ is$ 1.
4) The number of points $of weight 1\ is$ 28.

From the above we get the following example in PG(2, 7)

Let p = (1,0,0) be the point of weight 4, the points of weight 1 are the points of the equations

$$x_2 = 0, \quad x_3 = 0, \quad x_2 + x_3 = 0, \quad x_2 + 2\, x_3 = 0$$

and the remaining points of $PG(2,7)$ are having weight zero .

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الخلاصة:

في هذا البحث برهنا على وجود الأقواس الموزونة (k, n) الأحادية من النوع (m, q+m) التي يكون فيها عدد النقاط من الوزن صفر تشكل $(q^2-(m-1)q, q)$ – arc حيث عدد q-secants يساوي $q-(m-1)$ و $q-(m-1$)-secants هو q^2 و عدد 0-secants هو m و عدد i-secants يساوي 0 حيث $i \neq q, q-(m-1), 0$ و يكون عدد النقاط التي لها وزن واحد تمثل (mq, q) – arc حيث عدد q-secants يساوي m و عدد m-secants هو q^2 و عدد 0-secants هو $q-(m-1)$ و عدد i-secants يساوي 0 حيث $i \neq q, m, 0$ و قد حصلنا على مثال لهذا القوس في المستوي الاسقاطي من الرتبة 7,5,3 .