Input-Output Stability Implies Exact Linearization

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Abstract

We study the recently introduced notion of input-output stability, which is a robust variant of the minimum phase property for general smooth nonlinear control descriptor systems. This paper develops the theory of input-output stability in the single-input, single-output setting. We knew that if a descriptor system has a uniform M vector relative degree and is detectable through the output and its derivatives up to some order, uniformly over all inputs that produce a given output, then it is input-output stable. In this paper we show that the converse is also true, thus arriving at a useful equivalent characterization of input-output stability. In this paper, exact linearization via state feedback and stabilization for affine descriptor systems are considered. Using a local diffeomorphism and a state feedback, the affine descriptor system is transformed to a linear normal form system. Choosing a appropriate feedback controller, the closed-loop system is asymptotically stabilized.

Index terms: descriptor control systems, uniform M vector relative degree, input-output stability, exact linearization

1. Introduction

Descriptor systems (also called singular systems or generalized state-space systems) have attracted much attention in recent years, because of their ability to capture the dynamical behaviour of many natural phenomena (S. kawaji and E. Z. Taha,1994). It is well-established that the DAE (differential algebraic equation) systems may have fundamentally different characteristics from the ordinary differential equation (ODE) system (L. R. Petzold,1982). A concept that is commonly used to provide a measure of the differences between DAE and ODE systems is that of the (differential) index (A. Kumar, P. Daoutidis,1999). Loosely speaking, the index of DAE system is the minimum number of differentiations required to obtain an equivalent ODE system. DAE systems with index more than one are referred to as high-index systems.

The zero dynamics of a control system are the dynamics describing the "internal" behavior of the system when input and initial conditions have been chosen in such a way as to constrain the output to remain identically zero. Some research on the zero dynamics of nonlinear control systems has been proceeded. The relationship between the zero dynamics of the system and the stability of the system has been found; The problems of reproducing the reference output have also been considered with the aid of the zero dynamics of the system (A. Isidori, 1995). Those results show that the zero dynamics of a system is an important property of the system. The concept of the zero dynamics studies special class of control systems (systems that are affine in control), therefore; many studies for the stability of the systems that nonlinear in general as input-output stability in (Hassan K. Khalil, 1996), and M. Vidyasagar input-output stability in (M. Vidyasagar, 1993) studied the relation between the input and the output of the systems. The notion of input-output-to-state stability was studied in (M. Krichman and E. D. Sontag, 1998) and (M. Krichman, E. D. Sontag, and Y. Wang, 2001). The new definition of input-output stability such a concept was proposed in the recent paper (D. Liberzon. A. S. Morse, and E. D. Sontag, 2002). Its definition requires the state and the input of the system to be bound by a suitable function of the output and derivatives of the output, modulo a decaying term depending on initial conditions. Input-output stability can be investigated with the help of the tools that have been developed over the years to study ISS (input-to-state stability), OSS (output-to-state stability), and related notions which introduced in (E. D. Sontag, 1989) and (E. D. Sontag and Y. Wang, 1997). In (D. Liberzon, 2004) and (Daniel Liberzon, 2005) Liberzon is continuing to study the input-output stability property for multi-input, multi-output (MIMO) systems. Liberzon shows the relevance of the nonlinear structure algorithm in establishing input-output stability.

For nonlinear descriptor control systems that are affine in controls, a major contribution of J. Wang, C. Chen (J. Wang and C. Chen, 2004), (J. Wang and C. Chen, 2001), (J. Su, C. Chen, 2004), and (G.H.Xu, C.Chen, J. Wang , and D.D.Li, 2004) was to use the ideas of differential geometric control theory to define *M* derivative and *M* bracket in order to study the index one nonlinear descriptor control systems and proposed a systematic state feedback linearization strategy to produce stabilization and adaptive stabilization control laws. They define the minimum-phase property in terms of the concept of zero dynamics. The zero dynamics are the internal dynamics of the descriptor system under the action of an input that holds the output constantly at zero. The descriptor system is called minimum-phase if the zero dynamics are (globally) asymptotically stable. In the SISO case, a unique input capable of producing the zero output is guaranteed to exist if the descriptor system has *M* relative degree, which is now defined to be the number of times one has to differentiate the output until the input appears (J. Wang and C. Chen, 2001). Extensions to MIMO systems are discussed in (J. Wang and C. Chen, 2004) and (J. Su, C. Chen, 2004). In (Wasen I. khalil, 2009) we introduced the notion of input-output stability for nonlinear control descriptor systems in general form. In view of the need to work with the zero dynamics, the definition of a minimum-phase nonlinear descriptor control system prompts one to look for a change of coordinates that transforms the descriptor system into a certain normal form. It has also been recognized that just asymptotic stability of the zero dynamics is sometimes insufficient for control design purposes, so that additional requirements need to be placed on the internal dynamics of the descriptor system.

The relative degree of a continuous time linear single-input–single-output (SISO), which equals the difference between the degrees of the denominator and the numerator of the transfer function. Namely, if a linear system of relative degree r is minimum-phase, then the "inverse" system, driven by the r^{th} derivative of the output of the original system, is stable (A. Isidori, 1989). For affine control in the SISO case, a unique input capable of producing the zero output is guaranteed to exist if the system has a relative degree, which is defined to be the number of times one has to differentiate the output until the input appears. For control systems nonlinear in general form, in (D. Liberzon. A. S. Morse, and E. D. Sontag, 2002) they define uniform relative degree. The above concept of relative degree is generalization to index one descriptor control system that affine in controls (J. Wang and C. Chen, 2004). For general nonlinear descriptor control systems we define the uniform *M* vector relative degree to index one by using the generalization of *M* derivative and *M* bracket in (Wasen I. khalil, 2009). It follows from our definition that if a system has a uniform *M* relative degree (in an appropriate sense) and is detectable through the output and its derivatives up to some order, uniformly over all inputs that produce a given output, then it is input-output stable.

In this paper is a continuation of the work reported in (Wasen I. khalil, 2009). For SISO descriptor systems that are real analytic in controls, we will show that the converse is also true, thus arriving at a useful equivalent characterization of input-output stability (Theorem 1). By using this characterization we arrive to exact linearization for affine descriptor control systems.

Before providing necessary definitions in sec. 2 arriving at a useful equivalent characterization of input-output stability in sec. 3, we prove and discuss our main result for affine descriptor systems in sec. 4. Brief conclusions in sec. 5.

2 . Background

Consider the descriptor system

$$\dot{x} = f(x,z,u)$$
$$0 = \sigma_i(x,z) \quad \forall \ i = 1,...,m$$
$$y = h(x,z)$$

(1) where the dynamic state $x \in \Re^n$, the algebraic state $z \in \Re^m$, the input $u \in \Re$, and the output $y \in \Re$. Whenever possible and convenient, these equations will be rewritten in the more condensed form

$$\dot{x} = f(x,z,u)$$
$$0 = \sigma(x,z)$$
$$y = h(x,z)$$

where $\sigma(x,z) = (\sigma_1(x,z),...,\sigma_m(x,z))^T$. We assume that f, σ, and h are all smooth mapping, and the Jacobian matrix of $\sigma(x,z)$ with respect to z is nonsingular at all $x \in \Re^n$ and $z \in \Re^m$, we restrict admissible input (or "control") signals to be at least continuous, and for every initial condition $(x(0), z(0))$ and every input $u(.)$, there is a solution $(x(.), z(.))$ of (1) defined on a maximal interval $[0, T_{\max})$ and the corresponding output $y(.)$. We write C^k for the space of *k* times continuously differentiable functions $u:[0,\infty) \to \Re^p$, where *k* is some nonnegative integer. Whenever the input *u* is in C^k, the derivatives $\dot{y}, \ddot{y},..., y^{(k+1)}$ exist and are continuous; they are given by

$$y^{(i)}(t) = H_i(x(t), z(t), u(t),..., u^{(i-1)}(t)) \ , \ i = 1,..., k+1, \quad t \in [0, T_{\max})$$

where for $i = 0,1,...$ the functions $H_i : \Re^{n+m} \times \Re^{pi} \to \Re^l$ are defined recursively via $H_0 := h(x,z)$ and

$$H_i(x,z,u,...,u^{(i-1)}) = M_{f(x,z,u)} H_{i-1} + \sum_{j=0}^{i-2} \frac{\partial H_{i-1}}{\partial u^{(j)}} u^{(j+1)}$$

where the arguments of H_i are $(u, \dot{u},..., u^{(i-1)}) \in \Re^i$ and $(x,z) \in \Re^{n+m}$.

We will let $\|.\|_{[a,b]}$ denote the supremum norm of a signal restricted to an interval $[a,b]$,i.e.,

$$\|z\|_{[a,b]} := \sup\{|z(s)| : a \le s \le b\} \ ,$$

where $|.|$ is the standard Euclidean norm.

According to definition 2 of (Wasen I. Khalil, 2009), the system (1) is called input-output stable if there exist a positive integer N, a class KL function[1] β , and a class K_∞ function γ such that for every initial state $(x(0), z(0))$ and every $N-1$ times continuously differentiable input u the inequality

$$\left\|\begin{pmatrix} u(t) \\ x(t) \\ z(t) \end{pmatrix}\right\| \le \beta\left(\left\|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right\|, t\right) + \gamma\left(\left\|Y^N\right\|_{[0,t]}\right) \qquad (2)$$

holds for all t in the domain of the corresponding solution of (1), where $Y^N = (y, \dot{y}, \ldots, y^{(N)})^T$. (The assumption that u belongs to C^{N-1} is made to guarantee that $y^{(N)}$ is well defined, and can be weakened if the function H_N is independent of $u^{(N-1)}$).

It is perhaps best to interpret input-output stability as a combination of three separate properties of the descriptor control system. The first property is that if the output and its derivatives are small, then the dynamic state becomes small. This property is expressed by the following inequality

$$|x(t)| \le \beta\left(\left\|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right\|, t\right) + \gamma\left(\left\|Y^N\right\|_{[0,t]}\right) \qquad (3)$$

The second property is that if the output and its derivatives are small, then the algebraic state becomes small. This property is expressed by the following inequality

$$|z(t)| \le \beta\left(\left\|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right\|, t\right) + \gamma\left(\left\|Y^N\right\|_{[0,t]}\right) \qquad (4)$$

The results of (M. Krichman, E. D. Sontag, and Y. Wang, 2001) imply that the inequalities (3) and (4) hold with k if there exists a smooth, positive definite, radially unbounded function $V : \Re^{n+m} \to \Re$ that satisfies

$$\nabla V(x,z) F f(x,z,u) \le -\alpha\left(\left\|\begin{pmatrix} x \\ z \end{pmatrix}\right\|\right) + \chi\left(\left\|Y^N\right\|\right) \quad (5)$$

where $F(x,z) = \begin{bmatrix} I_n \\ -(\frac{\partial \sigma}{\partial z})^{-1} \frac{\partial \sigma}{\partial x} \end{bmatrix}$, for all $(x,z) \in \Re^{n+m}, u \in \Re^p$ for some functions $\alpha, \chi \in K_\infty$. The inequality (5) is equivalent to weak uniform 0-detectability of order N for equivalent control system $\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = F(x,z) f(x,z,u)$ according to [22] (is equivalent to holding the following inequality

$$\left\|\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right\| \le \beta_1\left(\left\|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right\|, t\right) + \gamma_1\left(\left\|Y^N\right\|_{[0,t]}\right) \qquad (6)$$

class KL function β , and a class K_∞ function γ) .

The third property is that if the output and its derivatives are small, then the input becomes small. This ingredient of the input-output stability property is described by the following inequality

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[1] Recall that a function $\alpha : [0,\infty) \to [0,\infty)$ is said to be of *class* K if it is continuous, strictly increasing, and $\alpha(0) = 0$. If it is unbounded, then it is said to be of *class* K_∞. A function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is said to be of *class KL* if $\beta(.,t)$ is of class K for each fixed $t \ge 0$ and $\beta(s,t)$ decreases to 0 as $t \to 0$ for each fixed $s \ge 0$.

$$|u(t)| \le \beta(\left\|\begin{pmatrix} x(0) \\ z(0) \end{pmatrix}\right\|, t) + \gamma(\|Y^N\|_{[0,t]}) \qquad (7)$$

Which says that the input should become small if the output and its derivative are small. Loosely speaking, this suggests that the system has a stable left inverse in the input-output sense. Unlike uniform detectability, this property does not seen to admit a Lyapunove-like characterization. In the SISO case it is closely related to the existence of a M uniform relative degree see (J. Wang and C. Chen, 2004), (J. Wang and C. Chen, 2001), (J. Su, C. Chen, 2004), and (G.H.Xu, C.Chen, J. Wang , and D.D.Li, 2004). In general, however, this third property needs be understood better, which is precisely the goal of the present paper. In the next section we formulate and study a useful result which implies that input-output stability is equivalent to the existence of a uniform M relative degree for SISO systems.

3. uniform M relative degree

Consider a general descriptor system (1). We recall from (Wasen I. khalil , 2009) that a positive integer r is the uniform M relative degree of descriptor system (1) if the following two conditions hold:

1- for each $k < r$, the function H_k is independent of $u, \dot{u}, ..., u^{(k-1)}$;

2- there exist two class K_∞ functions ρ_1 and ρ_2 such that

$$|u| \le \rho_1(\left\|\begin{pmatrix} x \\ z \end{pmatrix}\right\|) + \rho_2(|H_r(x,z,u)|) \qquad (8)$$ for

all $x \in \Re^n, z \in \Re^m$ and all $u \in \Re$.

The theorem (1) in (Wasen I. khalil , 2009) implies that for descriptor systems with uniform M relative degree, input-output stability is equivalent to weak uniform 0-detectability of order k for some $k \le N$, in the following we explain that for descriptor systems satisfying the additional assumptions on f and h stated in the following theorem, input-output stability is equivalent to the existence of a uniform M relative degree $r < N$ plus weak uniform 0-detectability of order k for some $k \le N$.

Theorem 1:

Suppose that the descriptor system (1) is input-output stable, that the function $f(x,z,.)$ is real analytic in u for each fixed (x,z), and $f(0,0,0)=0$, $h(0,0)=0$. Then (1) has uniform M relative degree $r \le N$.

Proof: Since the descriptor system (1) is input-output stable, we know in particular that for some positive integer N and for some class KL function β and class K_∞ function γ the inequality (7) holds along solution of (1) for all smooth inputs. The function H_k cannot be independent of $u, \dot{u}, ..., u^{(k-1)}$ for all $k = 1, ..., N-1$. Otherwise, letting $(x(0), z(0)) = (0,0)$ and applying an arbitrary constant input u , we would deduce from (7) that $|u| \le c := \gamma(|H_0(0,0);...;H_N(0,0)|)$, a contradiction. Thus, the integer

$$r := \max\{k < N : H_k \text{ is independent of } u,...,u^{(k-1)}\} + 1 \qquad (9)$$

is well defined. Condition 1 of Proposition (3) in (D. Liberzon. A. S. Morse, and E. D. Sontag, 2002) holds with this r.

For every input u we have

$$\dot{y} = H_1(x,z)$$

$$\dots$$

$$y^{(r-1)} = H_{r-1}(x,z)$$

$$y^{(r)} = (\frac{\partial H_{r-1}}{\partial x} - \frac{\partial H_{r-1}}{\partial z}(\frac{\partial \sigma}{\partial z})^{-1}\frac{\partial \sigma}{\partial x}) f(x,z,u)$$

$$= M_{f(x,z,u)} H_{r-1} = H_r(x,z,u)$$

$$y^{(r+1)} = (\frac{\partial H_r}{\partial x} - \frac{\partial H_r}{\partial z}(\frac{\partial \sigma}{\partial z})^{-1}\frac{\partial \sigma}{\partial x}) f(x,z,u) + \frac{\partial H_r}{\partial u}\dot{u}$$

$$=: G_1(x,z,u) + \frac{\partial H_r}{\partial u}\dot{u}$$

$$y^{(r+2)} =: G_2(x,z,u,\dot{u}) + \frac{\partial H_r}{\partial u}\ddot{u}$$

$$\dots$$

$$y^{(N)} =: G_{N-r}(x,z,u,...,u^{(N-r-1)}) + \frac{\partial H_r}{\partial u}u^{(N-r)} \qquad 10)$$

The following fact will be useful.

Lemma 1. If (7) holds and r is defined by (9), then there cannot exist a bounded sequence $\{x_{j,}, z_j\}$ in $\Re^{n+m}$, a sequence $\{w_j\}$ in $\Re$ with $\lim_{j\to\infty}|w_j| = \infty$, and a positive constant K such that for all j we have $|H_r(x_j, z_j, w_j)| < K$ and $(\frac{\partial H_r}{\partial u})(x_j, z_j, w_j) \ne 0$.

Proof: Suppose that there exist sequences $\{x_{j,}, z_j\}$ and $\{w_j\}$ and a positive constant K with the properties indicated in the statement of the lemma. Fix an arbitrary positive integer j. Consider the initial state $(x(0), z(0)) = (x_j, z_j)$, and pick a smooth (e.g., polynomial) input function $u_j(.)$ with $u_j(0) = w_j$ whose derivatives at $t = 0$ are specified recursively by the equations

$$u_j^{(i)}(0) = -\frac{G_i(x_j, z_j, u_j(0),...,u_j^{(i-1)}(0))}{\frac{\partial H_r}{\partial u_0}(x_j, z_j, u_j(0))}, \; i = 1,..., N-r . \qquad (11)$$

In view of (10), we will then have

$$\dot{y}(0) = H_1(x_j, z_j),..., y^{(r-1)}(0) = H_{r-1}(x_j, z_j)$$

$$|y^{(r)}(0)| = |H_r(x_j, z_j, w_j)| < K ,$$

$$y^{(r+1)}(0) = ... = y^{(N)}(0) = 0$$

Therefore, if such an input u_j is applied and if ε is an arbitrary fixed positive number, then there exists a sufficiently small time T_j such that for all $t \in [0, T_j]$ the following inequalities hold:

$$|(x(t),z(t))| < |(x_j,z_j)| + \varepsilon \qquad |\dot{y}(t)| < |H_1(x_j,z_j)| + \varepsilon \ , \ldots$$
$$|y^{(r-1)}(t)| < |H_{r-1}(x_j,z_j)| + \varepsilon \qquad |y^{(r)}(t)| < K + \varepsilon$$
$$|y^{(r+1)}(t)| < \varepsilon \ ,\ldots \qquad |y^{(N)}(t)| < \varepsilon \,.$$

Repeating this construction for all j, we obtain a sequence of trajectories of (1) along which $|(x(t),z(t))|,\ |\dot{y}(t)|,\ \ldots,\ |y^{(N)}(t)|$ are uniformly bounded for small t, whereas $|u_j(t)|$ is unbounded for small t and large j. We arrive at a contradiction with (7), and the proof of the lemma is complete.

Let us denote by Θ the set of all $(x,z)\in\Re^{n+m}$ such that $H_r(x,z,.)$ is a constant function. The set Θ is closed [because if for a sequence $\{x_j,z_j\}$ converging to some (x,z) we have $(\frac{\partial H_r}{\partial u})(x_j,z_j,u)=0$ for all u and all j, then $(\frac{\partial H_r}{\partial u})(x,z,u)=0$ for all u].

Lemma 2. Suppose that $f(x,z,.)$ is real analytic in u for each fixed (x,z), that (7) holds, and that r is defined by (9). If $(\bar{x},\bar{z})\in\Re^{n+m}$ is such that for some $K>0$ we have $|H_r(\bar{x},\bar{z},u)|<K$ for all u, then $(\bar{x},\bar{z})$ is in the interior of Θ.

Proof: Take an arbitrary sequence $\{v_j\}$ in $\Re$ with $\lim_{j\to\infty}|v_j|=\infty$. By hypothesis, $|H_r(\bar{x},\bar{z},v_j)|<K$ for all j. By continuity, for each j there exist a neighborhood $B_j(\bar{x},\bar{z})$ of $(\bar{x},\bar{z})$ in $\Re^{n+m}$ and a positive number δ_j such that $|H_r(x,z,u)|<K$ for all $(x,z)\in B_j(\bar{x},\bar{z})$ and all $u\in(v_j-\delta_j, v_j+\delta_j)$. Moreover, the neighborhoods $B_j(\bar{x},\bar{z}),\ j=1,2,\ldots$ can be chosen to be nested, i.e., $B_i(\bar{x},\bar{z})\subset B_j(\bar{x},\bar{z})$ whenever $i>j$, and the sequence $\{\delta_j\}$ can be chosen to be nonincreasing. Now, suppose that $(\bar{x},\bar{z})$ is not in the interior of Θ. Fix an arbitrary j. We have $B_j(\bar{x},\bar{z})\not\subset\Theta$. Take an arbitrary $(x_j,z_j)\in B_j(\bar{x},\bar{z})/\Theta$. Then $(\frac{\partial H_r}{\partial u})(x_j,z_j,.)$ cannot be identically zero on the interval $(v_j-\delta_j, v_j+\delta_j)$, by virtue of real analyticity of $H_r(x_j,z_j,.)$ which follows from that of $f(x_j,z_j,.)$. Thus, we can find a $w_j\in(v_j-\delta_j, v_j+\delta_j)$ such that $(\frac{\partial H_r}{\partial u})(x_j,z_j,w_j)\neq 0$. This construction can be carried out for all j. Since $B_i(\bar{x},\bar{z})\subset B_j(\bar{x},\bar{z})$ when $i>j$, the points (x_j,z_j), $j=1,2,\ldots$ can be chosen in such a way that $|(x_j,z_j)|$ is uniformly bounded for all j. Moreover, we have $|w_j|\ge|v_j|-|\delta_j|\ge|v_j|-|\delta_1|\to\infty$ as $j\to\infty$. In view of lemma (1), we arrive at a contradiction with (7), which proves the lemma.

Corollary 1. Suppose that $f(x,z,.)$ is real analytic in u for each fixed (x,z). If (7) holds and r is defined by (9), then the set Θ is open.

Proof: By definition of Θ, every $(\bar{x},\bar{z}) \in \Theta$ satisfies the condition in the statement of lemma 2 , hence it lies in the interior of Θ. □

Corollary 2. Suppose that $f(x,z,.)$ is real analytic in u for each fixed (x,z). If (7) holds and r is defined by (9), then the set Θ is empty.

Proof: We know that Θ is closed. We also know that Θ is open (Corollary 1). Moreover $\Theta \neq \Re^{n+m}$, by virtue of (9). Being a closed and open proper subset of $\Re^{n+m}$, the set must be empty.

Our goal is to show that Conditions 2 and 3 of proposition (3) in (D. Liberzon. A. S. Morse, and E. D. Sontag, 2002) hold, which would imply that r is the relative degree of (1).We break this into two separate statements.

Lemma 3. Suppose that $f(x,z,.)$ is real analytic in u for each fixed (x,z). If (7) holds and r is defined by (9), then for each compact set $\chi \subset \Re^{n+m}$ and each positive constant K there exists a number M such that $|H_r(x,z,u)| \geq K$ whenever $(x,z) \in \chi$ and $|u| \geq M$.

Proof: Suppose that, contrary to the statement of the lemma, there exist a compact subset $\chi \subset \Re^{n+m}$, a positive constant K, a sequence $\{x_j, z_j\}$ in χ, and a sequence $\{v_j\}$ in $\Re$ with $\lim_{j\to\infty}|v_j| = \infty$ such that $|H_r(x_j,z_j,v_j)| < K$ for all j. By continuity, we can find a nonincreasing sequence of positive numbers $\{\delta_j\}$ such that $|H_r(x_j,z_j,u)| < K$ for all j and all $u \in (v_j - \delta_j, v_j + \delta_j)$. Fix an arbitrary j. Since Θ is empty by corollary (2) and $H_r(x_j,z_j,.)$ is real analytic, $\frac{\partial H_r}{\partial u}(x_j,z_j,.)$ cannot vanish identically on the interval $(v_j - \delta_j, v_j + \delta_j)$. Thus, we can find a $w_j \in (v_j - \delta_j, v_j + \delta_j)$ such that $\frac{\partial H_r}{\partial u}(x_j,z_j,w_j) \neq 0$. Repeat this construction for all j . lemma (1) applies again, yielding a contradiction with (7), and the proof of the lemma is complete. □

Lemma 4. Suppose that $f(x,z,.)$ is real analytic in u for each fixed (x,z) and that we have $f(0,0,0)=0$ and $h(0,0)=0$. If (7) holds and r is defined by (9), then $H_r(0,0,u) \neq 0$ for all $u \neq 0$.

Proof: Suppose that $H_r(0,0,\bar{u}) = 0$ for some $\bar{u} \neq 0$. We know from corollary (2) that the set Θ is empty. Thus, by real analyticity $\frac{\partial H_r}{\partial u}(0,0,.)$ cannot vanish identically on any open neighborhood of $\bar{u}$. This implies that there exists a sequence $\{v_j\}$ converging to $\bar{u}$ such that $\frac{\partial H_r}{\partial u}(0,0,v_j) \neq 0$ $\forall\ j$ [if $\frac{\partial H_r}{\partial u}(0,0,\bar{u}) \neq 0$, simply let $v_j \equiv \bar{u}$]. Choose an arbitrary j. Take the initial state to be $(x(0), z(0)) = 0$. Pick a smooth (e.g., polynomial) input function $u_j(.)$ such that $u_j(0) = v_j$ and (11) holds with 0 in place of (x_j, z_j). From (10), we immediately see that $y^{(r+1)}(0) = ... = y^{(N)}(0) = 0$. Since $h(0,0) = 0$, we have $y(0) = 0$. We also know that $f(0,0,0) = 0$, which implies that $H_1(0) = \cdots = H_{r-1}(0) = 0$. It follows that $\dot{y}(0) = \cdots = y^{(r-1)}(0) = 0$. We conclude that if the input u_j is applied, then for every $\varepsilon > 0$ there exists a sufficiently small time T_j such that for all $t \in [0, T_j]$ the following inequalities hold:

$$\left|(x(t), z(t))\right| < \varepsilon \quad \left|\dot{y}(t)\right| < \varepsilon, \ldots, \left|y^{(r-1)}(t)\right| < \varepsilon$$

$$\left|y^{(r)}(t)\right| < \left|H_r(0,0,v_j)\right| + \varepsilon$$

$$\left|y^{(r+1)}(t)\right| < \varepsilon \ , \ldots, \left|y^{(N)}(t)\right| < \varepsilon$$

Carrying out the above construction for all j and noting that $\lim_{j\to\infty} H_r(0,0,v_j) = H_r(0,0,\bar{u}) = 0$, we see that $y(t), \dot{y}(t), ..., y^{(N)}(t)$ become arbitrarily small for small t as $j \to \infty$. On the other hand, $u_j(0) \to \bar{u} \neq 0$, so $u_j(t)$ is bounded away from 0 for small t and large j. This is a contradiction with (7), which proves the lemma. □

We have shown that the integer r defined by (9) satisfies all three conditions of proposition (3) in (D. Liberzon. A. S. Morse, and E. D. Sontag, 2002), thus r is the relative degree of the system (1). This proves Theorem 1. □

As an illustration, consider the affine descriptor systems in the following form

$$\begin{aligned} \dot{x} &= f(x,z) + g(x,z)u \\ 0 &= \sigma(x,z) \\ y &= h(x,z) \end{aligned} \qquad (12)$$

with $f(0,0) = 0$. Its right-hand side is obviously real analytic in u. Reconstructing the above proof for this case, we find that r is the smallest integer for which $M_g M_f^{r-1} h(x,z)$ is not identically zero on $\Re^{n+m}$, and $\Theta = \{(x,z) : M_g M_f^{r-1} h(x,z) = 0\}$. If the descriptor systems is input-output stable, then corollary (2) implies that Θ must be empty, which means that r is the uniform M relative degree (see lemma 1 in (Wasen I. khalil , 2009)). Since the hypothesis $h(0,0) = 0$ is only used in lemma (4), it is not needed in this case.

4. Exact Linearization for Affine Descriptor Control System

We will now establish an important feature of theorem (1), namely, that for SISO descriptor systems it reduces exact linearization via state feedback and stabilization for affine descriptor control systems are considered. Using a local differemorphism and a state feedback, the affine descriptor system is transformed to a linear normal system. Choosing a appropriate feedback controller, the closed-loop system is asymptotically stabilization.

We are now ready to state and prove a characterization of input-output stability for affine descriptor systems.

Theorem 2: Suppose that the affine descriptor (12) is input-output stable, $f(0,0)=0$, $h(0,0)=0$, and the dimension of dynamic state $n=\max\{k<N: H_k \, is independent of \, u,\dot{u},\ldots,u^{(k-1)}\}+1$.

Then the nonlinear transformation

$$\begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} \Phi_1(x,z) \\ \sigma(x,z) \end{pmatrix} =: \Phi(x,z) \qquad (13)$$

is a differomrphism, where

$$\Phi_1(x,z) = \begin{Bmatrix} H_0(x,z) \\ H_1(x,z) \\ \vdots \\ H_{n-1}(x,z) \end{Bmatrix}$$

Proof: It is only need to prove that the Jacobi matrix

$$\left.\begin{bmatrix} \dfrac{\partial \Phi_1}{\partial x} & \dfrac{\partial \Phi_1}{\partial z} \\ \dfrac{\partial \sigma}{\partial x} & \dfrac{\partial \sigma}{\partial z} \end{bmatrix}\right|_{(x,z)=(0,0)} \qquad (14)$$

of $\Phi(x,z)$ a (0,0) is nonsingular. From assumption that the Jacobin matrix of $\sigma(x,z)$ with respect to z is nonsingular at all $x \in \Re^n$ and $z \in \Re^m$, we can see the matrix (14) is similar to

$$\left.\begin{bmatrix} \Psi & \dfrac{\partial \Phi_1}{\partial z} \\ 0 & \dfrac{\partial \sigma}{\partial z} \end{bmatrix}\right|_{(x,z)=(0,0)} \qquad (15)$$

where

$$\Psi = \left.\left(\frac{\partial \Phi_1}{\partial x} - \frac{\partial \Phi_1}{\partial z}\left(\frac{\partial \sigma}{\partial z}\right)^{-1}\frac{\partial \sigma}{\partial x}\right)\right|_{(x,z)=(0,0)}.$$

From Implicit Function Theorem, the algebraic equation $0=\sigma(x,z)$ determine a unique smooth mapping $p(x):\Re^n \to \Re^m$ defined on a neighborhood of the origin, such that $z=p(x)$. If we can get the analytic expression of $p(x)$, then substituting $p(x)$ into (12) results in

$$\dot{x} = \bar{f}(x) + \bar{g}(x)u \qquad (16)$$
$$y = \bar{h}(x)$$

where $\bar{f}(x) = f(x, p(x))$, $\bar{g}(x) = g(x, p(x))$, and $\bar{h}(x) = h(x, p(x))$.

Let $\overline{\Phi}_1(x) = \Phi_1(x, p(x))$. From (12) is real analytic in *u* and theorem (1) in sec. 3 we can see that (12) has uniform *M* relative degree *n* this implies that (12) has M relative degree in the sense of (J. Wang and C. Chen, 2004), (J. Wang and C. Chen, 2001), (J. Su, C. Chen, 2004), (G.H.Xu, C.Chen, J. Wang , and D.D.Li, 2004) according to lemma 3.1 in (Wasen I. khalil , 2009). From lemma (1) in

(A. Isidori,1995), we can see (16) has the relative degree *n*. From lemma (1.2) in (Z. Jiandong and C. Zhaolin, 2002), it is easy to see the matrix $\frac{\partial \overline{\Phi}_1}{\partial x}(0)$ is nonsingular. Via simple computation, we can see that

$$\frac{\partial \overline{\Phi}_1}{\partial x} = \frac{\partial \Phi_1}{\partial x} - \frac{\partial \Phi_1}{\partial z}\left(\frac{\partial \sigma}{\partial z}\right)^{-1}\frac{\partial \sigma}{\partial x}\bigg|_{z=p(x)} \qquad (17)$$

Let $x=0$, then

$$\frac{\partial \overline{\Phi}_1}{\partial x}(0) = \Psi . \qquad (18)$$

From (15) and (17), we can see that (14) is a nonsingular matrix , i.e., the nonlinear transformation (13) is differomorphism. □

Base on the above discussion, we give the main result.

Theorem (3): If the affine descriptor (12) is input-output stable, $f(0,0)=0$, $h(0,0)=0$, and the dimension of dynamic state

$$n = \max\{k < N : H_k \, is independent of \, u, \dot{u}, \ldots, u^{(k-1)}\} + 1$$

,

then the exact linearization of the affine descriptor (12) can be realized via the local diffeomorpism (13) and a state feedback.

Proof: let the i^{th} component of the vector $\overline{x}$ is $\overline{x}_i$, i.e., let $\overline{x}_i = M_f^{i-1} h(x,z),\ i = 1,2,\ldots,n.$

Take derivative in both sides of the algebraic equation, we get

$$\dot{z} = -\left(\frac{\partial \sigma}{\partial z}\right)^{-1}\frac{\partial \sigma}{\partial x}\dot{x} \qquad (19)$$

Using (19) and theorem (2), we obtain

$$\dot{\overline{x}}_1 = \frac{\partial h}{\partial x}\dot{x} + \frac{\partial h}{\partial z}\dot{z}$$

$$= \left(\frac{\partial h}{\partial x_1} - \frac{\partial h}{\partial z}\left(\frac{\partial \sigma}{\partial z}\right)^{-1}\frac{\partial \sigma}{\partial x}\right)(f_1 + gu)$$

$$= M_{f_1} h(x,z) + M_g h(x,z) u$$

$$= M_{f_1} h(x,z) = \overline{x}_2 .$$

With the same way, we can get

$$\dot{\overline{x}}_2 = M_{f_1}^2 h(x,z) = \overline{x}_3$$

$$\vdots$$

$$\dot{\overline{x}}_{n-2} = M_{f_1}^{n-2} h(x,z) = \overline{x}_{n-1}$$

$$\dot{\overline{x}}_{n-1} = M_{f_1}^{n-1} h(x,z) = \overline{x}_n$$

$$\dot{\overline{x}}_n = M_{f_1}^{n} h(x,z) + M_g M_{f_1}^{n-1} h(x,z) u$$

Let

$$\alpha(x,z) = M_{f_1}^{n} h(x,z),$$

$$\beta(x,z) = M_g M_{f_1}^{n-1} h(x,z),$$

$$\overline{\alpha}(x,z) = \alpha(\Phi^{-1}(\overline{x},\overline{z})),$$

$$\overline{\beta}(x,z) = \beta(\Phi^{-1}(\overline{x},\overline{z}))$$

then

$$\dot{\overline{x}}_n = \alpha(x,z) + \beta(x,z)u$$
$$= \overline{\alpha}(\overline{x},\overline{z}) + \overline{\beta}(\overline{x},\overline{z})u$$

Via the differomorphism (13), the system (12) is equivalently transformed into

$$\dot{\overline{x}}_1 = \overline{x}_2 ,$$

$$\dot{\overline{x}}_2 = \overline{x}_3 ,$$

$$\vdots$$

$$\dot{\overline{x}}_{n-1} = \overline{x}_n ,$$

$$\dot{\overline{x}}_n = \overline{\alpha}(\overline{x},\overline{z}) + \overline{\beta}(\overline{x},\overline{z})u ,$$

$$0 = \overline{z} ,$$

$$y = \overline{x}_1$$

From assumption that the affine descriptor (12) is input-output stable, $f(0,0)=0$,

$h(0,0)=0$, and the dimension of dynamic state

$n=\max\{k<N: H_k \text{ is independent of } u,\dot{u},\ldots,u^{(k-1)}\}+1$,

and theorem (2) above , we know that $\beta(x,z)\neq 0$ and $\overline{\beta}(\overline{x},\overline{z})\neq 0$ in a neighborhood of the origin. Take the state feedback

$$u=-\frac{1}{\overline{\beta}(\overline{x},\overline{z})}[\overline{\alpha}(\overline{x},\overline{z})+K\overline{x}] \qquad (20)$$

then the closed-loop system is

$$\dot{\overline{x}}=\begin{bmatrix} 0 & 1 & \cdots & 0 \\ & \ddots & & \\ 0 & 0 & \cdots & 1 \\ -k_0 & -k_1 & & -k_{n-1} \end{bmatrix}\overline{x}, \qquad (21)$$

$$0=\overline{z}$$

$$y=[1,0,\ldots,0]\overline{x},$$

where $K=[k_0,k_1,\ldots,k_{n-1}]$. In the new coordinates, the closed-loop system (21) is a linear normal state space system. Appropriately choose K , such that $s^n+k_{n-1}s^{n-1}+\ldots+k_1s+k_0$ is Hulwize polynominal, then the closed-loop system (21) is asymptotically stable. The state feedback (20) is expressed by the state (x,z) of the original system as

$$u=-\frac{1}{\beta(x,z)}[\alpha(x,z)+K\Phi_1(x,z)] \qquad (22)$$

Under the state feedback (22), the closed-loop of the original system (12) is

$$\dot{x}=f(x,z)-g(x,z)\frac{1}{\beta(x,z)}[\alpha(x,z)+K\Phi_1(x,z)]$$

$$0=\sigma(x,z)$$

$$y=h(x,z) \qquad (23)$$

It is easy to see that the nonlinear transformation (13) is a local differomorphism between the system (23) and the system (21). So the closed-loop system (23) is asymptotically stable.

□

5. Conclusion

We provided characterization of input-output stability in terms of suitably defined notions of detectability and uniform M relative degree, the latter of which was proposed here and is of independent interest. By using this characterization and state feedback, the nonlinear descriptor control system is transformed to a linear state space system. Choosing a appropriate feedback controller, the closed- loop system is asymptotically stabilized.

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أستقرارية المدخلات والمخرجات تؤدي إلى شكل خطي مضبوط

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الخلاصة

في هذا البحث ندرس المصطلح المقدم حديثاً أستقرارية المدخلات والمخرجات، والتي تمثل خاصية مختلفة وأعم من خاصية ال "minimum- phase"ولكل الأنظمة الوصفية غير الخطية بصورة عامة. هذا البحث يهتم بتطوير نظرية أستقرارية المدخلات والمخرجات في حالة النظام مفرد المدخلات والمخرجات. نحن نعلم أذا كان النظام الوصفي يمتلك " uniform M vector relative degree" ويمتلك خاصية"detectable" بالنسبة الى مشتقات المخرجات لرتبة معينة، فانه تكون مدخلاته ومخرجاته مستقرة. في هذا البحث سوف نوضح العكس صحيح، الذي يمثل بمبرهنة التكافؤ لأستقرارية المدخلات والمخرجات. بالنسبة للأنظمة في حالة كون متغير السيطرة خطي وجدنا الشكل الخطي المضبوط عن طريق متغير السيطرة الارتدادي والمستقر. باستخدام تحويل محلي ومتغير السيطرة الارتدادي نقلنا النظام إلى شكله المعياري من ثم أيجاد متغير السيطرة الذي يجعل النظام المغلق مستقر من نوع "asymptotically ".