

New treatment of the solution of initial boundary value problems by using variational iteration method

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Abstract

In this paper, a new technique is applied to treatment of initial boundary value problems by using initial and boundary conditions together to obtain a new initial solution at every iteration using variational iteration method (VIM). The structure of new successive initial solutions can give a more accurate solution in a little number of steps.

1. Introduction

Several numerical and analytical techniques including the spectral methods, characteristics method, and Adomian's decomposition method have been developed for solving initial boundary value problems [Wazwaz A.M. (2000)]. For implementation of the Adomian decomposition method, one has to find the so-called the Adomian polynomial, which is itself a difficult problem. To overcome these difficulties and drawbacks, He J.H. [(1999, 2000, 2004, 2005, 2007)] developed variational iteration method for solving linear and nonlinear problems, which arise in various branches of pure and applied sciences. It is

worth mentioning that the origin of variational iteration method can be traced back to the reference [Inokuti et al.(1978)]. Furthermore, He J.H. [(2005,2004)] also introduced the homotopy perturbation method, which is developed by combining the standard homotopy and perturbation method.

Many researchers discussed the initial and boundary value problems, exact and numerical solutions for non-linear Burger's equation by variational iteration method was applied by [Biazar J. and Aminikhah H. (2009)]. Variational homotopy perturbation method for solving higher dimensional initial boundary value problems discussed by [Noor M.A. and

Mohyud-Din S.T. (2008)]. Solving higher dimensional initial boundary value problems by variational iteration decomposition method by [Noor M.A. and Mohyud-Din S.T. (2008)], a weighted algorithm based on the homotopy analysis method is applied to inverse heat conduction problems and discussed by [Shidfar A. and Molabahrani A. (2010)].

The boundary value problems was applied by [Niu Z. and Wang C. (2010)] to calculate a one step optimal homotopy analysis method for linear and nonlinear differential equations, and homotopy perturbation technique for solving two-point boundary value problems—compared it with other methods was discussed by [Chun C. and Sakthivel R. (2010)].

Although the problems of these researches contain initial boundary value problems , but the researchers discussed those problems by using either initial or boundary conditions . So we present a reliable framework by applying a new technique by using initial and boundary conditions together to obtain a new initial solution at every iteration using variational iteration method. The construct of new successive initial solutions can give a more accurate solution, some examples are given in this paper and compared with the standard VIM to illustrate the effectiveness and convenience of the new technique.

2. Variational iteration method

In this section, we introduce the basic idea underlying the variational iteration method for solving nonlinear equations. Consider the general nonlinear differential equation:

$$Lu + Nu = g(x, t). \quad (2.1)$$

where L is a linear differential operator, N is a nonlinear operator, and g is a given analytical function. The essence of the method is to construct a correction functional of the form [Biazar J. et.al (2010), Gomez C. A. and Salas A. H. (2010)]:

$$u_{n+1}(x, t) = u_n(x, t) + \int_0^t \lambda(t, s)(Lu_n(x, s) + N\tilde{u}_n(x, s) - g(x, s))ds.$$

Where λ is a Lagrange multiplier which can be identified optimally via the variational theory, u_n is the approximate solution and \tilde{u}_n denotes the restricted variation, i.e. $\delta\tilde{u}_n = 0$. After determining the Lagrange multiplier λ and selecting an appropriate initial function u_0 , the successive approximations u_n of the solution u can be readily obtained. Consequently, the solution of Eq. (2.1) is given by $u = \lim_{n \rightarrow \infty} u_n$.

3. New technique for solution of initial boundary value problems by variational iteration method

To convey the basic idea for treatment of initial and boundary conditions by variational

iteration method for solving initial boundary value problems, we consider the following general differential equation:

$$Lu(x_1, x_2, \dots, x_k, t) + Nu(x_1, x_2, \dots, x_k, t) = g(x_1, x_2, \dots, x_k, t),$$

$$0 < x_i < 1, \quad i = 1, 2, \dots, k, \quad t > 0$$

with initial conditions:

$$u(x_1, x_2, \dots, x_k, 0) = u_{(0)}(x_1, x_2, \dots, x_k),$$

$$u_t(x_1, x_2, \dots, x_k, 0) = u'_{(0)}(x_1, x_2, \dots, x_k),$$

and boundary conditions:

$$u(0, x_2, \dots, x_k, t) = g_{01}(x_2, x_3, \dots, x_k, t),$$

$$u(x_1, 0, \dots, x_k, t) = g_{02}(x_1, x_3, \dots, x_k, t),$$

$$\vdots$$

$$u(x_1, x_2, \dots, 0, t) = g_{0k}(x_1, x_2, \dots, x_{k-1}, t),$$

and

$$u(1, x_2, \dots, x_k, t) = g_{11}(x_2, x_3, \dots, x_k, t),$$

$$u(x_1, 1, \dots, x_k, t) = g_{12}(x_1, x_3, \dots, x_k, t),$$

$$\vdots$$

$$u(x_1, x_2, \dots, 1, t) = g_{1k}(x_1, x_2, \dots, x_{k-1}, t).$$

The initial solution can be written as :

$$u_0(x_1, x_2, \dots, x_k) = u_{(0)}(x_1, x_2, \dots, x_k) + tu'_{(0)}(x_1, x_2, \dots, x_k).$$

According to variational iteration method [Biazar J. et.al (2010)], we can construct a correct functional as follows:

$$u_{n+1}(x_1, x_2, \dots, x_k, t) = u_n(x_1, x_2, \dots, x_k, t) + \int_0^t \lambda(t, s)(Lu_n(x_1, x_2, \dots, x_k, s) + N\tilde{u}_n(x_1, x_2, \dots, x_k, s) - g(x_1, x_2, \dots, x_k, s))ds. \tag{3.1}$$

where λ is a Lagrange multiplier which can be identified optimally via the variational theory, u_n is the approximate solution and \tilde{u}_n denotes the restricted variation, i.e. $\delta\tilde{u}_n = 0$. The successive approximations u_n of the solution u

can be readily obtained, but must be satisfying the initial and boundary conditions together. So in this paper we construct a general new successive solutions u_n^* by applying a new technique:

$$\begin{aligned} u_n^*(x_1, x_2, \dots, x_k, t) &= u_n(x_1, x_2, \dots, x_k, t) + \\ &(1 - x_1)[g_{01}(x_2, x_3, \dots, x_k, t) - u_n(0, x_2, \dots, x_k, t)] \\ &+ x_1[g_{11}(x_2, x_3, \dots, x_k, t) - u_n(1, x_2, \dots, x_k, t)] \\ &\vdots \\ &(1 - x_k)[g_{0k}(x_1, x_2, \dots, x_{k-1}, t) - u_n(x_1, x_2, \dots, x_{k-1}, 0, t)] \\ &+ x_k[g_{1k}(x_1, x_2, \dots, x_{k-1}, t) - u_n(x_1, x_2, \dots, x_{k-1}, 1, t)]. \end{aligned} \tag{3.2}$$

where $n = 0, 1, 2, \dots$

Not that, in special case at one dimensional the iterations formula of initial solution u_n^* is:

$$u_n^*(x, t) = u_n(x, t) + (1 - x)[g_0(t) - u_n(0, t)] + x[g_1(t) - u_n(1, t)] \quad , n = 0, 1, 2, \dots \tag{3.3}$$

where, $g_0(t) = u_n(0, t)$, $g_1(t) = u_n(1, t)$.

It is clear that the new initial successive solutions u_n^* in Eq. (3.3) satisfying initial and boundary conditions such as:

if $x = 0$ then $u_n^*(0, t) = g_0(t)$,

if $x = 1$ then $u_n^*(1, t) = g_1(t)$,

if $t = 0$ then $u_n^*(x, 0) = u_n(x, 0)$.

Now; by using Eq.(3.2) we can be rewritten Eq. (3.1) in a new formulation to obtain the correct functional as:

$$u_{n+1}(x_1, x_2, \dots, x_k, t) = u_n^*(x_1, x_2, \dots, x_k, t) + \int_0^t \lambda(t, s) (Lu_n^*(x_1, x_2, \dots, x_k, s) + N\tilde{u}_n^*(x_1, x_2, \dots, x_k, s) - g(x_1, x_2, \dots, x_k, s)) ds. \tag{3.4}$$

such treatment is very effective as shown in this paper.

4. Examples:

Example 1: Consider the two-dimensional initial boundary value problem [Noor M.A. and Mohyud-Din S.T.(2008)]:

$$u_{tt} = \frac{1}{2} y^2 u_{xx} + \frac{1}{2} x^2 u_{yy}, \quad 0 < x, y < 1, \quad t > 0.$$

with boundary conditions

$$\begin{aligned} u(0, y, t) &= y^2 e^{-t}, & u(1, y, t) &= (1 + y^2) e^{-t}, \\ u(x, 0, t) &= x^2 e^{-t}, & u(x, 1, t) &= (1 + x^2) e^{-t}, \end{aligned}$$

and the initial conditions

$$u(x, y, 0) = x^2 + y^2, \quad u_t(x, y, 0) = -(x^2 + y^2).$$

Which has the exact solution $u(x, y, t) = (x^2 + y^2) e^{-t}$.

The solution by formula(3. 1):

According to the variational iteration method, we have a correction functional as follows:

$$u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^t \lambda(t, s) \left(\frac{\partial^2 u_n}{\partial s^2} - \frac{1}{2} y^2 \frac{\partial^2 \tilde{u}_n}{\partial x^2} - \frac{1}{2} x^2 \frac{\partial^2 \tilde{u}_n}{\partial y^2} \right) ds.$$

where \tilde{u}_n is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as

$\lambda = s - t$, which yields the following iteration formula:

$$u_{n+1}(x, y, t) = u_n(x, y, t) + \int_0^t (s - t) \left(\frac{\partial^2 u_n}{\partial s^2} - \frac{1}{2} y^2 \frac{\partial^2 u_n}{\partial x^2} - \frac{1}{2} x^2 \frac{\partial^2 u_n}{\partial y^2} \right) ds. \tag{4.1}$$

Now, we begin with the initial approximation:

$$u_0(x, y, t) = (x^2 + y^2) - (x^2 + y^2)t.$$

By the variational iteration formula (3.1), we have:

$$\begin{aligned}
 u_1(x, y, t) &= (x^2 + y^2) - (x^2 + y^2)t + (x^2 + y^2)\frac{t^2}{2!} - (x^2 + y^2)\frac{t^3}{3!}, \\
 u_2(x, y, t) &= (x^2 + y^2) - (x^2 + y^2)t + (x^2 + y^2)\frac{t^2}{2!} - (x^2 + y^2)\frac{t^3}{3!} + (x^2 + y^2)\frac{t^4}{4!} \\
 &\quad - (x^2 + y^2)\frac{t^5}{5!}, \\
 u_3(x, y, t) &= (x^2 + y^2) - (x^2 + y^2)t + (x^2 + y^2)\frac{t^2}{2!} - (x^2 + y^2)\frac{t^3}{3!} + (x^2 + y^2)\frac{t^4}{4!} \\
 &\quad - (x^2 + y^2)\frac{t^5}{5!} + (x^2 + y^2)\frac{t^6}{6!} - (x^2 + y^2)\frac{t^7}{7!}, \\
 u_4(x, y, t) &= (x^2 + y^2) - (x^2 + y^2)t + (x^2 + y^2)\frac{t^2}{2!} - (x^2 + y^2)\frac{t^3}{3!} + (x^2 + y^2)\frac{t^4}{4!} \\
 &\quad - (x^2 + y^2)\frac{t^5}{5!} + (x^2 + y^2)\frac{t^6}{6!} - (x^2 + y^2)\frac{t^7}{7!} + (x^2 + y^2)\frac{t^8}{8!} - (x^2 + y^2)\frac{t^9}{9!}, \\
 u_5(x, y, t) &= (x^2 + y^2) - (x^2 + y^2)t + (x^2 + y^2)\frac{t^2}{2!} - (x^2 + y^2)\frac{t^3}{3!} + (x^2 + y^2)\frac{t^4}{4!} - (x^2 + y^2)\frac{t^5}{5!} \\
 &\quad + (x^2 + y^2)\frac{t^6}{6!} - (x^2 + y^2)\frac{t^7}{7!} + (x^2 + y^2)\frac{t^8}{8!} - (x^2 + y^2)\frac{t^9}{9!} + (x^2 + y^2)\frac{t^{10}}{10!} - (x^2 + y^2)\frac{t^{11}}{11!}. \\
 &\vdots
 \end{aligned}$$

The series solution is given by:

$$u(x, y, t) = (x^2 + y^2)\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \frac{t^6}{6!} - \frac{t^7}{7!} + \dots\right).$$

and in a closed form by:

$$u(x, y, t) = (x^2 + y^2)e^{-t}.$$

which is in full agreement with [Wazwaz A.M. (2000)].

The solution by formula (3.4):

By applying a new approximations u_n^* in Eq. (3.2) we obtain:

$$u_n^*(x, y, t) = u_n(x, y, t) + (1-x)[y^2 e^{-t} - u_n(0, y, t)] + x[(1+y^2)e^{-t} - u_n(1, y, t)] \\ + (1-y)[x^2 e^{-t} - u_n(x, 0, t)] + y[(1+x^2)e^{-t} - u_n(x, 1, t)].$$

Then from Eq. (4.1) we have a correction functional as follows:

$$u_{n+1}(x, y, t) = u_n^*(x, y, t) + \int_0^t (s-t) \left(\frac{\partial^2 u_n^*}{\partial s^2} - \frac{1}{2} y^2 \frac{\partial^2 u_n^*}{\partial x^2} - \frac{1}{2} x^2 \frac{\partial^2 u_n^*}{\partial y^2} \right) ds. \quad (4.2)$$

Now, we begin with a new initial approximation u_0^* :

$$u_0^*(x, y, t) = (x^2 + y^2) - (x^2 + y^2)t + (1-x)[y^2 e^{-t} - y^2 + y^2 t] + x[(1+y^2)e^{-t} \\ - (1+y^2) + (1+y^2)t] + (1-y)[x^2 e^{-t} - x^2 + x^2 t] + y[(x^2 + 1)e^{-t} - (x^2 + 1) + (x^2 + 1)t]. \\ = -x - y + x e^{-t} + y e^{-t} + x^2 e^{-t} + y^2 e^{-t} + (x+y)t.$$

By using the variational iteration formula (4.2), we obtain the exact solution as soon as:

$$u_1(x, y, t) = (x^2 + y^2)e^{-t}.$$

i.e. we obtain the exact solution from the first iteration.

Example 2: Consider the two-dimensional nonlinear inhomogeneous initial boundary value problem [Noor M.A. and Mohyud-Din S.T. (2008)]:

$$u_{tt} = 2x^2 + 2y^2 + \frac{15}{2}(xu_{xx}^2 + yu_{yy}^2), \quad 0 < x, y < 1, \quad t > 0,$$

with boundary conditions:

$$u(0, y, t) = y^2 t^2 + y t^6, \quad u(1, y, t) = (1+y^2)t^2 + (1+y)t^6, \\ u(x, 0, t) = x^2 t^2 + x t^6, \quad u(x, 1, t) = (1+x^2)t^2 + (1+x)t^6,$$

and the initial conditions:

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = 0.$$

Which enjoys exact solution $u(x, y, t) = (x^2 + y^2)t^2 + (x+y)t^6$.

The solution by formula (3.1):

The correct functional is given as:

$$u_{n+1}(x, y, t) = \int_0^t \lambda(t, s) \left(\frac{\partial^2 u_n}{\partial s^2} - \frac{15}{2} \left(x \frac{\partial^2 u_n^2}{\partial x^2} + y \frac{\partial^2 u_n^2}{\partial y^2} \right) - 2(x^2 + y^2) \right) ds.$$

where \tilde{u}_n is considered as a restricted variation. Making the above functional stationary, the

Lagrange multiplier can be determined as $\lambda = s - t$, which yields the following iteration formula:

$$u_{n+1}(x, y, t) = \int_0^t (s - t) \left(\frac{\partial^2 u_n}{\partial s^2} - \frac{15}{2} \left(x \frac{\partial^2 u_n^2}{\partial x^2} + y \frac{\partial^2 u_n^2}{\partial y^2} \right) - 2(x^2 + y^2) \right) ds. \quad (4.3)$$

Now, we begin with the initial approximation $u_0(x, y, t) = 0$. By variational iteration formula (3.1), we have:

$$u_1(x, y, t) = (x^2 + y^2)t^2,$$

$$u_2(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6.$$

The solution is obtained as $u(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6$, which is in full agreement with [Wazwaz A.M. (2000)].

The solution by formula (3.4):

We applying a new approximations u_n^* in Eq.(3.2) so we obtain:

$$\begin{aligned}
u_n^*(x, y, t) = & u_n(x, y, t) + (1-x)[y^2t^2 + yt^6 - u_n(0, y, t)] + x[(1+y^2)t^2 \\
& + (1+y)t^6 - u_n(1, y, t)] + (1-y)[x^2t^2 + xt^6 - u_n(x, 0, t)] \\
& + y[(1+x^2)t^2 + (1+x)t^6 - u_n(x, 1, t)].
\end{aligned}$$

Then from Eq. (4.3) we have the following iteration formula:

$$u_{n+1}(x, y, t) = \int_0^t (s-t) \left(\frac{\partial^2 u_n^*}{\partial s^2} - \frac{15}{2} \left(x \frac{\partial^2 u_n^*}{\partial x^2} + y \frac{\partial^2 u_n^*}{\partial y^2} \right) - 2(x^2 + y^2) \right) ds. \quad (4.4)$$

Now, we begin with a new initial approximation u_0^* :

$$\begin{aligned}
u_0^*(x, y, t) = & (1-x)[y^2t^2 + yt^6] + x[(1+y^2)t^2 + (1+y)t^6] \\
& + (1-y)[x^2t^2 + xt^6] + y[(1+x^2)t^2 + (1+x)t^6].
\end{aligned}$$

By using the variational iteration formula (4.4), we have immediately exact solution:

$$u_1(x, y, t) = (x^2 + y^2)t^2 + (x + y)t^6.$$

i.e. we obtained the exact solution from a first iteration.

Example 3 :

Consider the one-dimensional convection-diffusion equation [Dehghan M.(2005)]:

$$u_t + au_x = \alpha u_{xx}, \quad 0 < x < 1, \quad 0 < t \leq T,$$

with boundary conditions

$$u(0, t) = g_0(t), \quad 0 < t \leq T,$$

$$u(1, t) = g_1(t), \quad 0 < t \leq T,$$

and the initial conditions

$$u(x, 0) = f(x), \quad 0 \leq x \leq 1,$$

where:

$$f(x) = \exp\left[-\frac{(x-2)^2}{8}\right],$$

$$\alpha = 0.1 \quad , \quad a = 0.8.$$

$$g_0(t) = \sqrt{\frac{20}{20+t}} \exp\left[-\frac{2(5+2t)^2}{5(t+20)}\right],$$

For which the exact solution is:

$$u(x,t) = \sqrt{\frac{20}{20+t}} \exp\left[-\frac{(x-2-0.8t)^2}{0.4(t+20)}\right].$$

$$g_1(t) = \sqrt{\frac{20}{20+t}} \exp\left[-\frac{(5+4t)^2}{10(t+20)}\right],$$

:

The solution by formula (3.1):

According to the variational iteration method we have a correction functional as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda(x,s) \left(\frac{\partial \tilde{u}_n}{\partial t} + 0.8 \frac{\partial u_n}{\partial s} - 0.1 \frac{\partial^2 \tilde{u}_n}{\partial s^2} \right) ds.$$

Where \tilde{u}_n is considered as a restricted variation. Making the above functional stationary, the Lagrange multiplier can be determined as

$\lambda = -1/a = -5/4$, which yields the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \frac{5}{4} \int_0^x \left(\frac{\partial u_n}{\partial t} + 0.8 \frac{\partial u_n}{\partial s} - 0.1 \frac{\partial^2 u_n}{\partial s^2} \right) ds. \tag{4.5}$$

Now, we begin with the initial approximation:

$$u(0,t) = \sqrt{\frac{20}{20+t}} \exp\left[-\frac{2(5+2t)^2}{5(t+20)}\right].$$

By the variational iteration formula (3.1), we have:

$$u_1 := \frac{1}{4} \frac{1}{(t+20)^3 \sqrt{\frac{1}{t+20}}} \left(\sqrt{5} e^{-\frac{2}{5} \frac{(5+2t)^2}{t+20}} (8t^2 + 320t + 3200 + 645xt + 1600x + 16xt^2) \right)$$

$$u_2 := \frac{1}{64} \frac{1}{(t+20)^5 \sqrt{\frac{1}{t+20}}} \left(\sqrt{5} e^{-\frac{2}{5} \frac{(5+2t)^2}{t+20}} (20480000 + 4096000t + 307200t^2 + 5152000xt + 540800xt^2 + 10240000x + 620000x^2 + 10240t^3 + 128t^4 + 20640x^2t^3 + 256x^2t^4 + 20560xt^3 + 256xt^4 + 1968000x^2t + 467275x^2t^2) \right)$$

⋮

The solution by formula (3.4):

By applying a new approximations u_n^* in Eq. (3.3) we obtain:

$$u_n^*(x, t) = u_n(x, t) + (1-x)[g_0(t) - u_n(0, t)] + x[g_1(t) - u_n(1, t)].$$

Then from Eq. (4.5) we have the following iteration formula:

$$u_{n+1}^*(x, t) = u_n^*(x, t) - \frac{5}{4} \int_0^x \left(\frac{\partial u_n^*}{\partial t} + 0.8 \frac{\partial u_n^*}{\partial s} - 0.1 \frac{\partial^2 u_n^*}{\partial s^2} \right) ds. \quad (4.6)$$

Now, we begin with a new initial approximation u_0^* :

$$u_0^*(x, t) = f(x) + (1-x)[g_0(t) - u_0(0, t)] + x[g_1(t) - u_0(1, t)]$$

$$= e^{-\frac{1}{8}(x-2)^2} + (1-x) \left(2\sqrt{5} \sqrt{\frac{1}{t+20}} e^{-\frac{2(5+2t)^2}{5t+100}} - e^{-\frac{1}{2}} \right) + x \left(2\sqrt{5} \sqrt{\frac{1}{t+20}} e^{-\frac{(5+4t)^2}{10t+200}} - e^{-\frac{1}{8}} \right)$$

by the variational iteration formula (4.6), we have:

$$u_1(x,t) =$$

$$\begin{aligned} & -\frac{1}{32} \frac{1}{(t+20)^3} \sqrt{\frac{1}{t+20}} \left(-2 e^{-\frac{1}{8}(x-2)^2} \sqrt{\frac{1}{t+20}} t^3 \right. \\ & \left. + e^{-\frac{1}{8}(x-2)^2} \sqrt{\frac{1}{t+20}} x t^3 + 2 e^{-\frac{1}{2}} \sqrt{\frac{1}{t+20}} t^3 \right. \\ & \left. + 120 e^{-\frac{1}{2}} \sqrt{\frac{1}{t+20}} t^2 \right. \\ & - 120 e^{-\frac{1}{8}(x-2)^2} \sqrt{\frac{1}{t+20}} t^2 - 64 x^2 \sqrt{5} e^{-\frac{1}{10} \frac{(5+4t)^2}{t+20}} t^2 \\ & \left. + 60 e^{-\frac{1}{8}(x-2)^2} \sqrt{\frac{1}{t+20}} x t^2 \right. \\ & - 128 \sqrt{5} e^{-\frac{2}{5} \frac{(5+2t)^2}{t+20}} x t^2 - 64 \sqrt{5} e^{-\frac{2}{5} \frac{(5+2t)^2}{t+20}} t^2 \\ & \left. + 64 \sqrt{5} e^{-\frac{2}{5} \frac{(5+2t)^2}{t+20}} x^2 t^2 \right. \\ & \left. + 1200 e^{-\frac{1}{8}(x-2)^2} \sqrt{\frac{1}{t+20}} x t + 2400 e^{-\frac{1}{2}} \sqrt{\frac{1}{t+20}} t \right. \\ & \left. - 5160 \sqrt{5} e^{-\frac{2}{5} \frac{(5+2t)^2}{t+20}} x t \right. \\ & - 2560 \sqrt{5} e^{-\frac{2}{5} \frac{(5+2t)^2}{t+20}} t - 2400 e^{-\frac{1}{8}(x-2)^2} \sqrt{\frac{1}{t+20}} t \\ & \left. - 2580 x^2 \sqrt{5} e^{-\frac{1}{10} \frac{(5+4t)^2}{t+20}} t + 2580 \sqrt{5} e^{-\frac{2}{5} \frac{(5+2t)^2}{t+20}} x^2 t \right) \end{aligned}$$

$$\begin{aligned}
 &+ 8000 e^{-\frac{1}{8}(x-2)^2} \sqrt{\frac{1}{t+20}} x - 12800 \sqrt{5} e^{-\frac{2}{5} \frac{(5+2t)^2}{t+20}} x \\
 &+ 16000 e^{-\frac{1}{2} \sqrt{\frac{1}{t+20}}} - 3500 x^2 \sqrt{5} e^{-\frac{1}{10} \frac{(5+4t)^2}{t+20}} \\
 &- 16000 e^{-\frac{1}{8}(x-2)^2} \sqrt{\frac{1}{t+20}} - 25600 \sqrt{5} e^{-\frac{2}{5} \frac{(5+2t)^2}{t+20}} \\
 &+ 6400 \sqrt{5} e^{-\frac{2}{5} \frac{(5+2t)^2}{t+20}} x^2 \Big) \\
 &:
 \end{aligned}$$

The comparison between the absolute error of the exact solution with approximation solutions of standard VIM and another hand with a new technique which applying a new initial successive solutions u_n^* in formula (3.4) when $t = 1$ after 10 iterations will be given in Table 2. Also we have a

numerical results by finite difference by [Dehghan M.(2005)] in Table 1. From these Tables, the numerical results when we apply a new technique for one-dimensional convection-diffusion equation have the smallest absolute error, so we conclude the effectiveness and convenience of the new technique.

Table 1: Numerical results by finite difference for example (3) at $t = 1$ by [Dehghan M.(2005)]

x	Exact value	Absolute error of Second-order $\times 10^{-3}$	Absolute error of Third-order $\times 10^{-3}$	Absolute error of Fourth-order $\times 10^{-5}$
0.1	0.40973191	1.3	2.7	3.4
0.2	0.43641705	1.1	2.7	3.2
0.3	0.46373468	1.2	2.6	3.1
0.4	0.49159044	1.4	2.6	2.9
0.5	0.51988016	1.3	2.7	2.7
0.6	0.54849038	1.1	2.4	2.7
0.7	0.57729893	1.4	2.0	2.5
0.8	0.60617562	1.5	2.3	2.2
0.9	0.63498306	1.7	2.5	2.0

Table 2: Numerical results in this paper for example (3) at $t = 1$.

x	Exact value	Approximate solution of formula (3.1)	Approximate solution of new formula (3.4)	Absolute error of formula (3.1) $\times 10^{-4}$	Absolute error of new formula (3.4) $\times 10^{-6}$
0	0.38376366	0.38376366	0.38376366	0	0
0.1	0.40973191	0.40971813	0.40973658	0.1378	4.67
0.2	0.43641705	0.43638153	0.43642477	0.3552	7.72
0.3	0.46373468	0.46366669	0.46374378	0.6799	9.10
0.4	0.49159044	0.49147653	0.49159911	1.1391	8.67
0.5	0.51988016	0.51970452	0.51988673	1.7564	6.57
0.6	0.54849038	0.54823520	0.54849322	2.5518	2.84
0.7	0.57729893	0.57694544	0.57729671	3.5349	2.22
0.8	0.60617562	0.60570544	0.60616764	4.7018	7.98
0.9	0.63498306	0.63438026	0.63496966	6.0280	0.1340
1	0.66357768	0.66283180	0.66356094	7.4588	0.1674

Fig. 1. The surface solution by formula (3.1) of example (3)

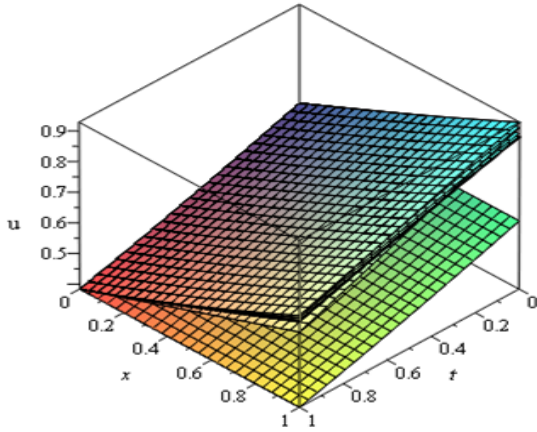


Fig. 2. The surface solution by formula (3.4) of example (3)

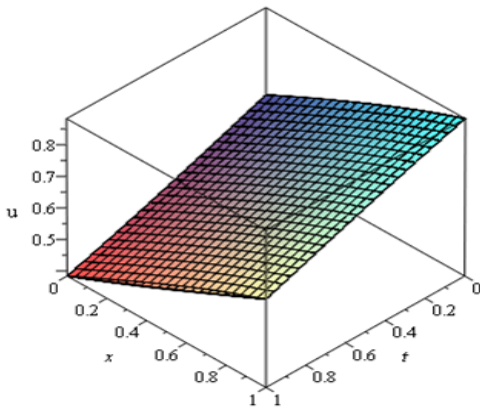


Fig. 3. the approximate solutions by formula (3.1) at t=1 of example (3)

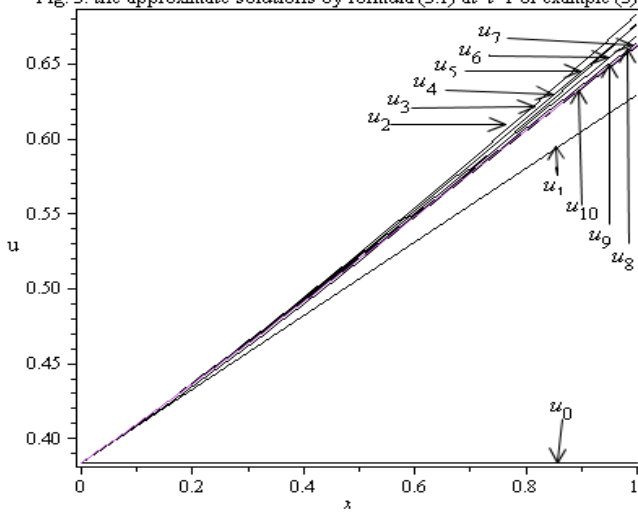
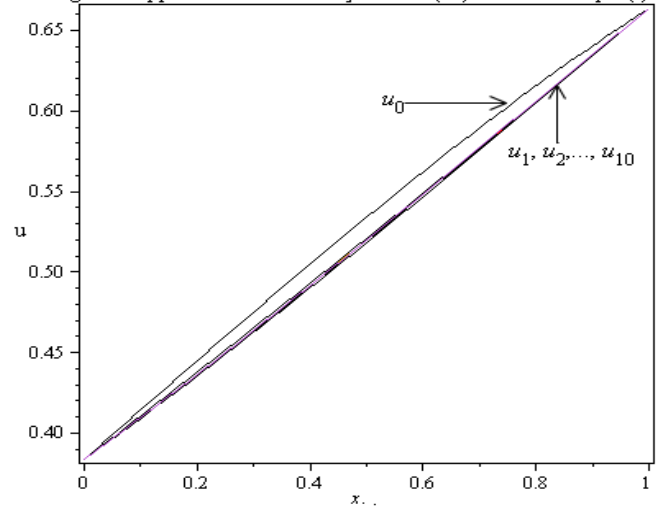


Fig.4. the approximate solutions by formula (3.4) at t=1 of example (3)



5. Conclusions

In this paper, a very simple but effective to construct a new initial successive solutions u_n^* by using initial and boundary conditions together which explained in formula (3.2) and used it to find successive approximations u_n of the solution u in a new correct functional which explained in Eq.(3.4) by applying variational iteration method for solving initial boundary value problems. The construct of new successive initial solutions can give a more accurate solution. Some examples are given in this paper and compared with the standard VIM to illustrate the effectiveness and convenience of the new technique. Also we show that the Fig. 2 and 4 explain the good approximate and convergence solutions for example(3) by using a new technique. In general the convergence of VIM is studied by Odibat Z.M. [2010]. Computations in this paper are performed using maple 13.

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معالجة جديدة للمسائل ذات القيم الابتدائية والحدودية باستخدام طريقة التغير التكراري

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المستخلص:

في هذا البحث، طبقنا اسلوب جديد يتمثل بمعالجة المسائل ذات القيم الابتدائية والحدودية وذلك بأضافة الشروط الابتدائية والحدودية معا لغرض الحصول على حل ابتدائي جديد عند كل خطوة تكرارية باستخدام طريقة التغير التكراري (VIM) ، ان تكوين متتابعة الحلول الابتدائية الجديدة يعطي حلا اكثر دقة في اقل عدد من الخطوات التكرارية.
