Gosper's Algorithm and Hypergeometric Solutions

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Abstract

In this paper, we study Gosper's algorithm where we use Petkovšek's technique to give a derivation for Gosper's algorithm. We show that the least common multiplier can be used to give two simpler algebraically motivated approaches to find hypergeometric solutions of linear recurrences with the additional restriction that the leading and trailing coefficients are constant. In the second approach we use the universal denominator idea. The main result of these approaches that finding hypergeometric solutions reduces to finding polynomial solutions.

Keywords: Gosper's algorithm, least common multiplier, hypergeometric solution, rational solution, universal denominator.

1. Introduction

Let N be the set of nonnegative integers, K be a field of characteristic zero, $K(n)$ be the field of rational functions over K, $K[n]$ be the ring of polynomials over K, deg(p) denotes the polynomial degree in n of any $p \in K[n]$, lc ($p(n)$) be the leading coefficient of any $p \in K[n]$, E be the shift operator on $K[n]$, i.e. $(Ep)(n) = p(n+1)$ for any $p \in K[n]$, lcm be the least common multiplier for polynomials, gcd be the greatest common divisor for polynomials. As usual, we assume that subject to normalization the gcd of two polynomials always takes a value as a monic polynomial, namely, polynomials with the leading coefficient being 1.

A nonzero term t_n is called a hypergeometric term over K if the consecutive term ratio

$$\frac{t_{n+1}}{t_n} = r(n).$$

is a rational function. If $r(n)=a(n)/b(n)$, where $a(n),b(n)\in K[n]$ then the function $a(n)/b(n)$ is called a rational representation of the rational function $r(n)$.

Gosper's algorithm belongs to the standard methods implemented in most computer algebra systems. It has been extensively studied and widely used to verify hypergeometric identities, see [Chen et al (2005), Chen et al (2008), Gerhard(1998), Gosper (1978), Koepf(1998), Milekoric(2005), Koepf(1996), Lisoněk(1994), Qing-Hu Hou(2004)]. Now we give a historical survey of Gosper's algorithm and hypergeometric solutions. [Gosper (1978)] clarified that any rational function $r(n)$ can be written in the following form, called Gosper representation

$$r(n)=\frac{a(n)}{b(n)}\frac{c(n+1)}{c(n)}, \tag{1.1}$$

where a,b and c are polynomials over K and

$\gcd(a(n),b(n+h))=1$ for all nonnegative integer h. (1.2)

[Petkovšek (1992)] perceives that the Gosper representation becomes unique which is called the Gosper- Petkovšek representation or GP representation, for abbreviate, If b and c are monic polynomials such that

$$\gcd(a(n),c(n)) =1,$$

$$\gcd(b(n),c(n+1))=1.$$

[Petkovšek (1992)] derived the algorithm Hyper to find all hypergeometric solutions of the recurrence

$$\sum_{i=0}^{d} p_i(n)\cdot z_{n+i}=0,$$

where $p_0(n),p_1(n),\ldots,p_d(n)=p(n)\in K[n]$ are given polynomials. The algorithm Hyper can be used to find all hypergeometric solutions of the recurrence

$$\sum_{i=0}^{d} p_i(n)\cdot z_{n+i}=t_n, \tag{1.3}$$

where t_n is given hypergeometric term such that $p_0(n)$ *and* $p_d(n)$ are constants. [Petkovšek (1994)] gave a derivation for Gosper's algorithm and a derivation to find all hypergeometric solutions of the recurrence (1.3) with the additional restriction that $p_0(n)$ *and* $p_d(n)$ are constants. [Chen and Saad(2005)] presented a simplified version for Gosper's algorithm by using GP representation which is similar to the version of Paule and Strehl. They found all hypergeometric solutions of (1.3) such that $p_0(n)$ *and* $p_d(n)$ are constants.

Many approaches, even ours, for generalizing Gosper's algorithm can be reduced to find rational solutions. So in this paper we need to mention to the rational solutions $y(n)$ for the linear difference equation

$$\sum_{i=0}^{d} p_i(n)y(n+i) = p(n), \qquad (1.4)$$

where $p_0(n), p_1(n),..., p_d(n), p(n) \in K[n]$ are given polynomials such that $p_0(n)$ and $p_d(n) \neq 0$. A polynomial $g(n) \in K[n]$ is called universal denominator for (1.4) if for every solution $y(n) \in K(n)$ for (1.4) there exists $f(n) \in K[n]$ such that $y(n) = f(n)/g(n)$. In order to find rational solutions of (1.4), first we need to find the universal denominator for the equation (1.4) and then find the polynomial solution of the resulting equation.

[Chen et al (2008)] found a convergence property for the gcd of the raising factorial and falling factorial. Based on this property, they presented an approach to compute the universal denominator that appears in Gosper's algorithm. They found all hypergeometric solutions of the recurrence (1.3),[Paule(1995)].

The dispersion $\text{dis}(a(n), b(n))$ of the polynomials $a(n), b(n) \in K[n]$ is the greatest nonnegative integer k (if it exists) such that $a(n)$ and $b(n+k)$ have nontrivial common divisor, i.e.,

$$\text{dis}(a,b) = \max \{k \in N : \deg \gcd(a(n), b(n+k)) \geq 1\}.$$

If k does not exist then we set $\text{dis}(a,b) = -1$,[Petkovšek et al (1996), Paule et al(1995),].

2. A derivation of Gosper's Algorithm

[Petkovšek (1994)] presented a derivation to find hypergeometric solutions of the linear recurrence (1.3) with the additional restriction that the leading and trailing coefficients are constant. In this section, we use Petkovšek's technique to give a derivation for Gosper's algorithm.

Given a hypergeometric term t_n and suppose that there exists a hypergeometric solution z_n satisfying

$$z_{n+1} - z_n = t_n. \qquad (2.1)$$

Let $y(n) = z_n / t_n$. By using (2.1) we get

$$y(n) = \frac{z_n}{t_n} = \frac{z_n}{z_{n+1} - z_n} = \frac{1}{\frac{z_{n+1}}{z_n} - 1}.$$

Hence $y(n)$ is unknown rational function of n. Substitute $y(n)t_n$ instead of z_n in (2.1), we obtain

$$r(n)y(n+1) - y(n) = 1, \qquad (2.2)$$

where $r(n) = t_{n+1}/t_n$ is a rational function of n. Thus the problem of finding hypergeometric solution of (2.1) is reduced to the problem of finding rational solution of (2.2). Let us express $r(n)$ in terms of their Gosper representation as in (1.1). Let

$$k_\circ = \operatorname{dis}(a(n-1), b(n)) = \max\{k \in \mathrm{N} : \deg\gcd(a(n-1), b(n+k)) \geq 1\}, \quad (2.3)$$

and

$$y(n) = \frac{f(n)}{g(n)\,c(n)},$$

such that $\gcd(f(n), g(n)) = 1$. Substitute this together with (1.1) into (2.2), we get

$$a(n)\,g(n)\,f(n+1) - b(n)\,f(n)\,g(n+1) = b(n)\,c(n)\,g(n)\,g(n+1).$$

From this equation we get

$$g(n) \mid b(n)g(n+1) \quad \text{and} \quad g(n) \mid a(n-1)g(n-1).$$

Using these two relations repeatedly, we obtain

$$g(n) \mid b(n)b(n+1)\ldots b(n+k-1)\,g(n+k),$$
$$g(n) \mid a(n-1)\,a(n-2)\ldots a(n-k)\,g(n-k),$$

for large enough k. Since K has characteristic zero then

$$\gcd(g(n), g(n+k)) = \gcd(g(n), g(n-k)) = 1,$$

for all large enough k. It follows that

$$g(n) \mid b(n)b(n+1)\ldots b(n+k-1),$$
$$g(n) \mid a(n-1)\,a(n-2)\ldots a(n-k),$$

for all large enough k. Therefore

$$g(n) \mid \gcd(a(n-1)a(n-2)\ldots a(n-k), b(n)b(n+1)\ldots b(n+k-1))$$

When k goes to infinity we find

$$g(n) \mid \gcd(a(n-1)a(n-2)\ldots a(n-k_\circ), b(n)b(n+1)\ldots b(n+k_\circ)).$$

Since $\gcd(a(n), b(n+h)) = 1 \quad \forall h \in N$, it follows that $g(n)$ is a constant. Then we can write $y(n) = q(n)/c(n)$, where $q(n)$ is unknown polynomial. Substitute (1.1) and $y(n) = q(n)/c(n)$ into (2.2) gives

$$a(n)\,q(n+1) = b(n)(q(n) + c(n)). \quad (2.4)$$

This shows that $b(n)$ divides $q(n+1)$. Let $x(n) = q(n)/b(n-1)$ and substitute this in (2.4) to obtain

$$a(n)\,x(n+1) - b(n-1)\,x(n) = c(n). \quad (2.5)$$

Therefore finding hypergeometric solutions of (2.1) is equivalent to finding polynomial solutions of (2.5). The correspondence between them is that if $x(n)$ is a nonzero polynomial solution of (2.5) then

$$z_n = \frac{b(n-1)\,x(n)}{c(n)}\,t_n, \quad (2.6)$$

is a hypergeometric solution of (2.1), and vice versa.

Example 2.1. *Let* $t_n = n\frac{\left(n-\frac{1}{2}\right)!}{(n+1)!}$. *Note that*

$$r(n) = \frac{t_{n+1}}{t_n} = \frac{\left(n+\frac{1}{2}\right)}{(n+2)}\frac{(n+1)}{n} = \frac{a(n)}{b(n)}\frac{c(n+1)}{c(n)}.$$

Hence $a(n) = n+\frac{1}{2},\ b(n) = n+2,\ c(n) = n$.

From equation (2.5) *we find*

$$(n+\frac{1}{2})x(n+1)-(n+1)x(n) = n.$$

The polynomial $x(n) = 2(n+1)$ *is a solution to this equation. Therefore, by* (2.6) *we obtain*

$$z_n = 2(n+1)\frac{\left(n-\frac{1}{2}\right)!}{n!}.$$

3. Hypergeometric Solutions for Recurrences of Arbitrary Order

In this section we give an approach to find the hypergeometric solutions of the linear recurrence (1.3) with the additional restriction that the leading and trailing coefficients are constant. Given a hypergeometric term t_n and the polynomials $p_\circ(n), p_1(n),..., p_d(n)$ with the additional constraints that $p_0(n)$ and $p_d(n)$ are constant. Suppose that there exists a hypergeometric solution z_n of (1.3). Let $y(n) = z_n / t_n$, by using (1.3) we get

$$y(n) = \frac{z_n}{t_n} = \frac{z_n}{\sum_{i=0}^{d} p_i(n)z_{n+i}} = \frac{1}{\sum_{i=0}^{d} p_i(n)\frac{z_{n+i}}{z_n}} = \frac{1}{\sum_{i=0}^{d} p_i(n)\prod_{j=0}^{i-1}\frac{z_{n+j+1}}{z_{n+j}}}$$

Since $\frac{z_{n+j+1}}{z_{n+j}}$ is a rational function $\forall\, j = 0,1,...,i-1$, it follows that $y(n)$ is unknown rational function of n. Substituting $y(n)t_n$ for z_n in (1.3), we get

$$\sum_{i=0}^{d} p_i(n)y(n+i)\prod_{j=0}^{i-1} r(n+j) = 1, \quad (3.1)$$

where $r(n) = t_{n+1} / t_n$ is a rational function of n, then the problem of finding hypergeometric solution of (1.3) is reduced to the problem of finding rational solutions of (3.1). Let us express $r(n)$ in terms of their Gosper representation as in (1.1). Let $y(n) = \frac{f(n)}{g(n)c(n)}$ such that $\gcd\left(f(n), g(n)\right) = 1$. Substitute this together with (1.1) into (3.1), we obtain

$$\sum_{i=0}^{d} p_i(n)\frac{f(n+i)}{g(n+i)c(n+i)}\prod_{j=0}^{i-1}\frac{a(n+j)c(n+j+1)}{b(n+j)c(n+j)} = 1$$

.

Multiplying this equation by $\prod_{j=i}^{d-1}\frac{b(n+j)}{b(n+j)}$, we obtain

$$\sum_{i=0}^{d} p_i(n)\frac{f(n+i)}{g(n+i)}\prod_{j=0}^{i-1} a(n+j)\prod_{j=i}^{d-1} b(n+j) = c(n)\prod_{j=0}^{d-1} b(n+j) \qquad (3.2)$$

Letting

$$l(n)=\text{lcm}\big(g(n+1),g(n+2),\ldots,g(n+d)\big), \qquad (3.3)$$

and multiplying equation (3.2) by $l(n)\,g(n)$, we obtain

$$\sum_{i=0}^{d} p_i(n)\, f(n+i)\frac{l(n)}{g(n+i)}\prod_{j=0}^{i-1} a(n+j)\prod_{j=i}^{d-1} b(n+j)\; g(n) = c(n)\, l(n)\, g(n)\prod_{j=0}^{d-1} b(n+j).$$

Form (3.3), we have the following divisibility conditions

$g(n+i)\mid l(n)$ for $i=1,2,\ldots,d$.

Thus $l(n)/g(n+i)$ are polynomial for $i=1,2,\ldots,d$. From the above equation we obtain

$$g(n)\mid p_\circ(n)\prod_{j=0}^{d-1} b(n+j)\,\text{lcm}\big(g(n+1), g(n+2),\ldots, g(n+d)\big). \qquad (3.4)$$

Similarly, multiplying equation (3.2) by $l(n-1)\,g(n+d)$ and then substituting $n-d$ for n, we obtain that

$$g(n)\mid p_d(n-d)\prod_{j=1}^{d} a(n-j)\,\text{lcm}\big(g(n-1),g(n-2),\ldots,g(n-d)\big), \qquad (3.5)$$

shifting n by 1 in (3.4) yields

$$g(n+1)\mid p_\circ(n+1)\prod_{j=1}^{d} b(n+j)\,\text{lcm}\big(g(n+2), g(n+3),\ldots, g(n+d+1)\big),$$

substituting this equation in (3.4) we see that $g(n)$ divides

$$p_0(n)\prod_{j=0}^{d-1} b(n+j)\,\text{lcm}(p_\circ(n+1)\prod_{j=1}^{d} b(n+j)\,\text{lcm}\big(g(n+2), g(n+3),\ldots, g(n+d+1)\big)$$
$$, g(n+2),\ldots, g(n+d))$$

So we can write

$$g(n)\mid p_\circ(n)\,p_\circ(n+1)\prod_{j=0}^{d-1} b(n+j)\prod_{j=1}^{d} b(n+j)\,\text{lcm}\big(g(n+2), g(n+3),\ldots, g(n+d+1)\big).$$

By induction we may get for $k\ge 1$

$$g(n)\,|\prod_{j=0}^{k-1}p_0(n+j)\prod_{j=0}^{d-1}b(n+j)\prod_{j=1}^{d}b(n+j)\ldots\prod_{j=k-1}^{k+d-2}b(n+j)\,\mathrm{lcm}(g(n+k),g(n+k+1)$$
$$,\ldots,g(n+k+d-1)).$$

It follows that

$$g(n)\,|\prod_{j=0}^{k-1}p_0(n+j)\prod_{i=0}^{k-1}\left(\prod_{j=i}^{d+i-1}b(n+j)\right)g(n+k)\cdot g(n+k+1)\ldots g(n+k+d-1)\,.$$

Since K has characteristic zero, then for large enough k

$$\gcd\bigl(g(n),g(n+j)\bigr)=1 \quad \text{for } j\geq k\,.$$

It follow that

$$g(n)\,|\prod_{j=0}^{k-1}p_0(n+j)\prod_{i=0}^{k-1}\left(\prod_{j=i}^{d+i-1}b(n+j)\right),$$

for large enough k. Analogously form (3.5) we can get

$$g(n)\,|\prod_{j=0}^{k-1}p_d(n-d-j)\prod_{i=0}^{k-1}\left(\prod_{j=i+1}^{d+i}a(n-j)\right),$$

for large enough k. By assumption $p_0(n)$ and $p_d(n)$ are constant. Therefore

$$g(n)\,|\,\gcd\left(\prod_{i=0}^{k-1}\left(\prod_{j=i+1}^{d+i}a(n-j)\right),\prod_{i=0}^{k-1}\left(\prod_{j=i}^{d+i-1}b(n+j)\right)\right).$$

When k goes to infinity, we obtain

$$g(n)\,|\,\gcd\left(\prod_{i=0}^{k_\circ}\left(\prod_{j=i+1}^{d+i}a(n+j)\right),\prod_{i=0}^{k_\circ}\left(\prod_{j=i}^{d+i-1}b(n+j)\right)\right).$$

Since $\gcd\bigl(a(n),b(n+h)\bigr)=1\ \forall h\in N$, it follows that $g(n)$ is a constant. Then we can write

$$y(n)=q(n)/c(n)\,, \qquad (3.6)$$

where $q(n)$ is unknown polynomial. Inserting (1.1) and (3.6) into (3.1) gives

$$\sum_{i=0}^{d}p_i(n)q(n+i)\prod_{j=0}^{i-1}a(n+j)\prod_{j=i}^{d-1}b(n+j)=c(n)\prod_{j=0}^{d-1}b(n+j)\,.$$

Again $q(n)$ is divisible by $b(n-1)$ therefore we finally get

$$y(n)=\frac{b(n-1)\,x(n)}{c(n)},$$

where $x(n)$ is an unknown polynomial satisfying

$$\sum_{i=0}^{d} p_i(n)\,x(n+i)\prod_{j=0}^{i-1}a(n+j)\prod_{j=i-1}^{d-1}b(n+j)=c(n)\prod_{j=0}^{d-1}b(n+j). \qquad (3.7)$$

Therefore finding hypergeometric solutions of (1.3) is equivalent to finding polynomial solutions of (3.7). The correspondence between them is that if $x(n)$ is a nonzero polynomial solution of (3.7) then

$$z_n=\frac{b(n-1)\,x(n)}{c(n)}t_n, \qquad (3.8)$$

is a hypergeometric solution of (1.3), and vice versa.

Example 3.1. *Find all Hypergeometric Solutions of*

$$z_{n+2}-z_{n+1}+z_n=t_n, \qquad (3.9)$$

where $t_n=n+1$. *Then*

$$r(n)=\frac{t_{n+1}}{t_n}=\frac{\mathrm{n}+2}{\mathrm{n}+1}=\frac{a(n)}{b(n)}\frac{c(n+1)}{c(n)},$$

where $a(n)=1=b(n), c(n)=n+1$. *From* (3.7), $x(n)$ *is a polynomial satisfies*

$$x(n+2)-x(n+1)+x(n)=n+1,$$

The polynomial solution of this equation is $x(n)=n$. *By* (3.8), *we have*

$$z_n=\frac{b(n-1)\,x(n)}{c(n)}t_n=n$$

is a hypergeometric solution of (1.3)

4. Another Approach for Hypergeometric Solutions for Recurrences of Arbitrary Order

In this section we give another approach to find hypergeometric solutions of the linear recurrence (1.3) with the additional restriction that the leading and trailing coefficients are constant.

Lemma 4.1. *Let*

$$L = \max\left\{ k \in N : \deg \gcd\left(c(n)\prod_{j=0}^{d-1} a(n-d+j), c(n+k)\prod_{j=0}^{d-1} b(n+j+k) \right) \geq 1 \right\},$$

and let $k_\circ$ *be defined as in* (2.3). *Then* $L = k_\circ$.

Proof. Let

$$g_1(k) = \gcd\big(c(n)\,a(n-1), c(n+k)\,b(n+k)\big),$$

and

$$g_2(k) = \gcd\left(c(n)\prod_{j=0}^{d-1} a(n-d+j), c(n+k)\prod_{j=0}^{d-1} b(n+k+j) \right).$$

For any k we have $g_1(k) \mid g_2(k)$ hence $L \geq k_\circ$.

Suppose that for an irreducible polynomial $t(n)$, $t(n) \mid g_2(L)$, i.e.

$$t(n) \mid c(n)\,a(n-d+i) \quad \text{for some } i \in \{0,1,\ldots,d-1\},$$

and

$$t(n) \mid c(n+L)b(n+j+L) \quad \text{for some} \quad j \in \{0,1,\ldots,d-1\}.$$

Hence

$$t(n+d-i-1) \mid c(n+d-i-1)\,a(n-1),$$

and

$$t(n+d-i-1) \mid c(n+d-i-1+L)b(n+d-i-1+j+L).$$

It follows that

$$t(n+d-i-1) \mid \gcd\big(c(n+d-i-1)\,a(n-1), c(n+d-i-1+L)b(n+d-i-1+j+L)\big).$$

Then we have

$$d-i-1+j+L \leq k_\circ,$$

which implies that $L \leq k_\circ$. Hence $L = k_\circ$. □

Given a hypergeometric term t_n and suppose that there exists a hypergeometric solution z_n satisfying equation (1.3). We immediately start with equation (3.1) because there is no difference at the beginning. We look for a rational function $y(n)$ satisfying

$$\sum_{i=0}^{d} p_i(n)y(n+i)\prod_{j=0}^{i-1} r(n+j)=1,$$

where $r(n)=t_{n+1}/t_n$ is rational function of n, then the problem of finding hypergeometric solution of (1.3) is reduced to the problem of finding rational solutions of (3.1). Let us express $r(n)$ in term of their Gosper representation as in (1.1) and let $y(n)=f(n)/g(n)$ such that $\gcd(f(n),g(n))=1$. Substitute this together with (1.1) into (3.1), we obtain

$$\sum_{i=0}^{d} p_i(n)c(n+i)\frac{f(n+i)}{g(n+i)}\prod_{j=0}^{i-1} a(n+j)\prod_{j=i}^{d-1} b(n+j) = c(n)\prod_{j=0}^{d-1} b(n+j). \quad (4.1)$$

Letting

$$l(n)=\operatorname{lcm}\big(g(n+1),g(n+2),...,g(n+d)\big), \quad (4.2)$$

and multiplying equation (4.1) by $l(n)\,g(n)$, we obtain

$$\sum_{i=0}^{d} p_i(n)c(n+i)f(n+i)\frac{l(n)}{g(n+i)}\prod_{j=0}^{i-1} a(n+j)\prod_{j=i}^{d-1} b(n+j)\ g(n) = c(n)l(n)g(n)\prod_{j=0}^{d-1} b(n+j). \quad (4.3)$$

From (4.2), we have the following divisibility conditions

$g(n+i)\,|\,l(n)$ for $i=1,2,...,d$.

Thus $l(n)/g(n+i)$ are polynomial for $i=1,2,...,d$. Form (4.3) we obtain

$$g(n)\,|\,p_\circ(n)c(n)\prod_{j=0}^{d-1} b(n+j)\operatorname{lcm}\big(g(n+1),g(n+2),...,g(n+d)\big). \quad (4.4)$$

Similarly, multiplying equation (4.1) by $l(n-1)\,g(n+d)$ and then substituting $n-d$ for n, we obtain that

$$g(n)\,|\,p_d(n-d)c(n)\prod_{j=1}^{d} a(n-j)\operatorname{lcm}\big(g(n-1),g(n-2),...,g(n-d)\big). \quad (4.5)$$

Shifting n by 1 in (4.4) yields

$$g(n+1)\,|\,p_\circ(n+1)c(n+1)\prod_{j=1}^{d} b(n+j)\ \operatorname{lcm}\big(g(n+2),g(n+3),...,g(n+d+1)\big). \quad (4.6)$$

Substituting (4.6) in (4.4) we see that $g(n)$ d

Ivides

$$p_0(n)c(n)\prod_{j=0}^{d-1}b(n+j)\ \mathrm{lcm}(p_\circ(n+1)c(n+1)\prod_{j=1}^{d}b(n+j)\,\mathrm{lcm}(g(n+2),g(n+3),\ldots,g(n+d+1)),$$
$$g(n+2),\ldots,g(n+d)).$$

So we can write

$$g(n)\,|\,p_\circ(n)p_\circ(n+1)c(n)c(n+1)\prod_{j=0}^{d-1}b(n+j)\prod_{j=1}^{d}b(n+j)\,\mathrm{lcm}\big(g(n+2),g(n+3),\ldots,g(n+d+1)\big).$$

By induction we may derive for $k\geq 1$

$$g(n)\,|\prod_{j=0}^{k-1}p_0(n+j)\prod_{j=0}^{k-1}c(n+j)\prod_{j=0}^{d-1}b(n+j)\prod_{j=1}^{d}b(n+j)\ldots\prod_{j=k-1}^{k+d-2}b(n+j)\,\mathrm{lcm}(g(n+k),g(n+k+1)$$
$$,\ldots,g(n+k+d-1)).$$

It follows that

$$g(n)\,|\prod_{j=0}^{k-1}p_0(n+j)\prod_{j=0}^{k-1}c(n+j)\prod_{j=0}^{k-1}\left(\prod_{j=i}^{d+i-1}b(n+j)\right)g(n+k)\cdot g(n+k+1)\cdots g(n+k+d-1)$$

Since K has characteristic zero, then for large enough k we get

$$\mathrm{gcd}\big(g(n),g(n+j)\big)=1,\quad \text{for } j\geq k.$$

It follows that

$$g(n)\,|\prod_{j=0}^{k-1}p_0(n+j)\prod_{j=0}^{k-1}c(n+j)\prod_{i=0}^{k-1}\left(\prod_{j=i}^{d+i-1}b(n+j)\right),$$

for large enough k . Analogously form (4.5) we get

$$g(n)\,|\prod_{j=0}^{k-1}p_d(n-d-j)\prod_{j=0}^{k-1}c(n-j)\prod_{i=0}^{k-1}\left(\prod_{j=i+1}^{d+i}a(n+j)\right),$$

for large enough k . By assumption $p_0(n)$ and $p_d(n)$ are constant. Therefore

$$g(n)\,|\,\mathrm{gcd}\left(\prod_{j=0}^{k-1}c(n-j)\prod_{i=0}^{k-1}\left(\prod_{j=i+1}^{d+i}a(n-j)\right),\prod_{j=0}^{k-1}c(n+j)\prod_{i=0}^{k-1}\left(\prod_{j=i}^{d+i-1}b(n+j)\right)\right),$$

i.e.,

$$g(n)\,|\,c(n)\gcd\left(\prod_{j=1}^{k-1}c(n-j)\prod_{i=0}^{k-1}\left(\prod_{j=i+1}^{d+i}a(n-j)\right),\prod_{j=1}^{k-1}c(n+j)\prod_{i=0}^{k-1}\left(\prod_{j=i}^{d+i-1}b(n+j)\right)\right).$$

When k goes to infinity, we obtain

$$g(n)\,|\,c(n)\gcd\left(\prod_{j=1}^{L}c(n-j)\prod_{i=0}^{L}\left(\prod_{j=i+1}^{d+i}a(n-j)\right),\prod_{j=1}^{L}c(n+j)\prod_{i=0}^{L}\left(\prod_{j=i}^{d+i-1}b(n+j)\right)\right).$$

From equation (4.1) we get

$$\sum_{i=0}^{d}p_i(n)c(n+i)f(n+i)\prod_{\substack{j=0\\ j\neq i}}^{k}g_j(n+i)\prod_{j=0}^{i-1}a(n+j)\prod_{j=i}^{d-1}b(n+j) = c(n)\prod_{j=0}^{k}g_j(n+i)\prod_{j=0}^{d-1}b(n+j). \tag{4.7}$$

The next step is simply to set

$$g(n)=c(n)\gcd\left(\prod_{j=1}^{L}c(n-j)\prod_{i=0}^{L}\left(\prod_{j=i+1}^{d+i}a(n-j)\right),\prod_{j=1}^{L}c(n+j)\prod_{i=0}^{L}\left(\prod_{j=i}^{d+i-1}b(n+j)\right)\right). \tag{4.8}$$

Finding hypergeometric solutions of (1.3) is therefore equivalent to finding polynomial solutions of (4.7). The correspondence between them is that if $f(n)$ is a nonzero polynomial solution of (4.7) then

$$z_n=\frac{f(n)}{g(n)}t_n, \tag{4.9}$$

is a hypergeometric solution of (1.3), and vice versa.

Example 4.1. *Find all Hypergeometric Solutions of*

$$z_{n+2}-z_{n+1}+z_n=t_n, \tag{4.10}$$

where, $t_n=n+1$. *Then*

$$r(n)=\frac{t_{n+1}}{t_n}=\frac{\mathrm{n}+1}{\mathrm{n}}=\frac{a(n)}{b(n)}\frac{c(n+1)}{c(n)}.$$

Hence $a(n)=1=b(n),\ \ c(n)=n+1$.

From **Lemma 4.1,** $L=0$. *From* (4.8), $g(n)=n+1$. *By* (4.7), $f(n)$ *is a polynomial satisfies*

$$f(n+2)-f(n+1)+f(n)=n+1.$$

The polynomial solution of this equation is $f(n)=n$. *By* (4.9), *we have*

$$z_n=\frac{f(n)}{g(n)}t_n = n$$

is the hypergeometric solution of (4.10).

References

A. Chen W.Y.C. and H.L. Saad, On the Gosper- Petkovšek representation of rational functions, *J. Symbolic Computation,* **40** (2005) 955-963.

B. Chen W.Y.C. , P. Paule and H.L. Saad, Converging to Gosper's Algorithm, *Advances in Applied Mathematics*, **41** (2008) 351–364.

D. Gosper , Decision procedure for infinite hypergeometric summation, *Proc. Natl. Acad. Sci. USA*, **75** (1978) 40-42.

E. Koepf, *Hypergeometric Summation*, Vieweg, Braunschweig/Wiesbaden, 1998.

F. Koepf, Summation in Maple. *Maple Tech. Newselett.*, **3** (1996) 26-32.

G. Lisoněk, Computer-assisted Studies in Algebraic Combinatorics, Doctor of Philosophy thesis, Angefertigt am Forschungsinstitut für Symbolishes Rechnen Technisch-Naturwissenschaftliche Fakultät Johannes Kepler Universität Linz, (1994).

H. Milekoric, K. J. Copton, Average case analysis of Gosper's algorithm for a Class of Urn Model Inputs, *Algorithmica*, 43 (2005) 211-244.

I. P. Paule Greatest factorial factorization and symbolic summation, *J. Symbolic Computation*, **20** (1995) 235-268.

C. Gerhard, High Degree Solutions of Low Degree Equations, *Proc. ISSAC*'**98**, ACM Press, (1998) 284-289.

J. P. Paule and V. Strehl, Symbolic summation - some recent developments, RISC Linz Report Series 95-11, Johannes Kepler University, Linz. *Computer Algebra in Science and Engineering - Algorithms, Systems, and Applications,* J. Fleischer, J. Grabmeier, F. Hehl, W. Küchlin (eds.), World Scientific, Singapore, 1995.

K. Petkovšek, A generalization of Gosper's algorithm, *Discrete Math.*, **134** (1994) 125-131.

L. Petkovšek, Hypergeometric solutions of linear recurrences with polynomial coefficients, *J. Symbolic Computation*, 14 (1992), 243-264.

M. Petkovšek, H. S . Wilf and D. Zeilberger, *A=B* , A.K. Peters, 1996.

N. Qing-Hu Hou, k-free recurrences of double hypergeometric terms, *Advances in Applied Mathematics*, **32** (2004) 468–484.