Available online at: www.basra-science-journal.org

ISSN –1817 –2695

Simultaneous Approximation for a generalization Mixed Szãsz –Beta Type Operators

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Received 11-3-2012,Accepted 18-7-2012

Abstract

In this paper, we introduce a generalization of mixed Szãsz-Bata of Phillips type operators. First, we study the convergence theorem by applying the Korovkin's theorem, then we define and study the *m-th* order moment for this operators. Next, we prove the Voronovskaja-type asymptotic formula for our operators.

1. Introduction

In [1] Maheshwar proposed mixed Szãsz-Bata type operators to approximate functions integrable on $[0,\infty)$ as:

$$L_n(f;x)=\sum_{v=0}^{\infty}p_{n,v}(x)\int_0^{\infty}b_{n,v}(t)f(t)dt,\quad x\in[0,\infty).$$

Where

$p_{n,v}(x)=\frac{e^{-nx}(nx)^v}{v!}$, $b_{n,k}(t)=\frac{1}{B(n,v+1)}t^v(1+t)^{-n-v-1}$,

and $B(n,v+1)$being the beta function given by $\frac{v!(n-1)!}{(n+v)!}$.

The Phillips Szãsz-Bata operators of the operators $L_n(f,x)$ was defined as:

$$B_n(f;x)=\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}b_{n,k-1}(t)f(t)dt+f(0)p_{n,0}(x),\quad x\in[0,\infty).$$

Suppose that $C_\alpha[0,\infty)=\{f(t)\in C[0,\infty):|f(t)|\leq\ M(1+t)^\alpha, for\ some\ M,\alpha>0\}$

For $f\in C_\alpha[0,\infty)$, we defined a family of generalization of the operators $B_n(f;x)$ as follows:

$$B_{n,p}(f;x)=\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}b_{n,k+p-1}(t)f(t)dt+f(0)e^{-nx},\quad p\in N.\qquad(1.1)$$

Clearly $B_{n,0}(f;x)=B_n(f;x)$.

In this paper, we study the convergence of operator $B_{n,p}(f;x)$. First, we prove that these operators are convergent to *f(x)* as $n\to\infty$ in the space $C_\alpha[0,\infty)$.

Then, we define the *m-th* order moment of $B_{n,p}$ and drive the recurrence relation for this moment. Also, we study the convergence theorem of the operators $B_{n,p}$ in simultaneous approximation (approximation of derivatives of a function by the corresponding order derivatives of operators), and introduce a Voronovskaja-type asymptotic formula. Finally, we give an error estimate in terms of modulus of continuity of the function being approximation.

Here, we give this Lemma which we needed in our paper.

1.1 Lemma: For $x \in [0,\infty)$ we have(see [2], [3]and [4]):

1) $\sum_{k=0}^{\infty} p_{n,k}(x) = 1$; 2) $\sum_{k=0}^{\infty} kp_{n,k}(x) = nx$;

3) $\sum_{k=0}^{\infty} k^2 p_{n,k}(x) = n^2x^2 + nx$; 4) $xp'_{n,k}(x) = (k - nx)p_{n,k}(x)$;

5)suppose that $\Phi_{n,m}(x) = \sum_{k=0}^{\infty} k^m P_{n,k}(x); then\ \Phi_{n,m+1}(x) = x\acute{\Phi}_{n,m}(x) + nx\Phi_{n,m}(x)$, and $\Phi_{n,m}(x) = (nx)^m + \frac{m(m-1)}{2}(nx)^{m-1} + term\ in\ lower\ powers\ of\ f(nx)$.

6) $x(x+1)b'_{n,k}(x) = (k - nx - x)b_{n,k}(x)$; 7) $\int_0^{\infty} b_{n,k}(x)t^m = \frac{(m+k)!\,(n-m-1)!}{k!\,(n-1)!}$.

The next Lemma shows that the Korovkin's conditions in the space $C_\alpha[0,\infty)$ are hold for the operators $B_{n,p}$.

1.2 Lemma

For $x \in [0,\infty)$, if the following conditions hold:

1) $B_{n,p}(1;x) = 1$;

2) $B_{n,p}(t;x) = \frac{1}{n-1}\{nx + p(1 - e^{-nx})\} \to x$ as $n \to \infty$;

3) $B_{n,p}(t^2;x) = \frac{1}{(n-1)(n-2)}\{n^2x^2 + 2n(p+1)x + (p^2+p)(1 - e^{-nx})\} \to x^2$ as $n \to \infty$.

Therefore, $B_{n,p}(f(t);x) \to f(x) as\ n \to \infty$.

Proof:

By using 1.1 Lemma, we have:

$$1)B_{n,p}(1;x) = \sum_{k=1}^{\infty} p_{n,k}(x)\frac{(k+p-1)!\,(n-1)!}{(k+p-1)!\,(n-1)!} + p_{n,0}(x) = 1$$

$$2)B_{n,p}(t;x) = \sum_{k=1}^{\infty} p_{n,k}(x)\frac{(k+p)!\,(n-2)!}{(k+p-1)!\,(n-1)!} + 0$$

$$= \frac{1}{n-1}\left\{\sum_{k=1}^{\infty} kp_{n,k}(x) + p\sum_{k=1}^{\infty} p_{n,k}(x)\right\} = \frac{1}{n-1}\{nx + p(1 - e^{-nx})\}.$$

$$3)B_{n,p}(t^2;x) = \sum_{k=1}^{\infty} p_{n,k}(x)\frac{(k+p+1)!\,(n-3)!}{(k+p-1)!\,(n-1)!} + 0$$

$$= \frac{1}{(n-1)(n-2)}\left\{\sum_{k=1}^{\infty} k^2p_{n,k}(x) + (2p+1)\sum_{k=1}^{\infty} kp_{n,k}(x) + (p^2 + p)\left(\sum_{k=0}^{\infty} p_{n,k} - p_{n,0}(x)\right)\right\}$$

$$= \frac{1}{(n-1)(n-2)}\{n^2x^2 + 2n(p+1)x + (p^2+p)(1-e^{-nx})\}$$

Therefore, $B_{n,p}(f(t);x) \to f(x) as\ n \to \infty$ (see[2]).

2. Auxiliary Results

For $m \in N^0$, the *m-th* order moment for the operators $M_n(f(t);x)$ be defined by:

$$T_{n,m}(x) = B_{n,p}((t-x)^m;x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k+p-1}(t)(t-x)^m dt + (-x)^m e^{-nx},$$
$$\coloneqq Y_{n,m}(x) + (-x)^m e^{-nx}.$$

2.1 Lemma

For the function $Y_{n,m}(x)$ which is defined above, we have:

$Y_{0,m}(x) = 1 - e^{-nx}$,

$Y_{1,m}(x) = \dfrac{x(1+(n-1)e^{-nx}) + p(1+e^{-nx})}{n-1}$.

And, we have the following recurrence relation for $Y_{n,m}(x)$.

$$(n-m-1)Y_{n,m+1}(x) = xY'_{n,m}(x) + m(2x+x^2)Y_{n,m-1}(x) + \{(m+1)(1+2x) + (p-1-x)\}Y_{n,m}(x) \qquad ,m+1<n$$

Proof:

by direct computation, we can easily get the values of $Y_{n,0}\ (x)$, $Y_{n,1}(x)$ and $Y_{n,2}(x)$.
Now,

$$Y'_{n,m}(x) = \sum_{k=1}^{\infty} p'_{n,k}(x) \int_0^{\infty} b_{n,k+p-1}(t)(t-x)^m dt - mY_{n,m-1}(x).$$

By using 1.1 Lemma, we have:

$$xY'_{n,m}(x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} (k-nx) b_{n,k+p-1}(t)(t-x)^m dt - xmY_{n,m-1}(x).$$

Then,

$$x\{Y'_{n,m}(x) + mY_{n,m-1}(x)\}$$
$$= \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} t(t+1)b'_{n,k+p-1}(t-x)^m dt + (n+1) \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k+p-1}(t-x)^{m+1} dt + (p-1-x) \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k+p-1}(t-x)^m dt.$$

By using the integration by parts, we have:

$$x\{Y'_{n,m}(x) + mY_{n,m-1}(x)\}$$
$$= -m(x+x^2)Y_{n,m-1}(x) - (m+1)(1+2x)Y_{n,m}(x) - (m-2)Y_{n,m+1}(x) - (p-1-x)Y_{n,m}(x).$$

Form which the recurrence relation for $Y_{n,m}(x)$ is immediate.

2.2 Lemma

For the m-th order moment $Y_{n,m}(x)$ for the operators $B_{n,p}(f,x)$, we have:

$$T_{n,0}(x) = 1, T_{n,1}(x) = \frac{x}{n-1} + \frac{p(1-e^{-nx})}{n-1}$$

and

$$T_{n,2}(x)=\frac{n+2}{(n-1)(n-2)}x^2+\frac{2(n+2p)+4pe^{-nx}(n-1)}{(n-1)(n-2)}x+\frac{(p+p^2)(1-e^{-nx})}{(n-1)(n-2)}.$$

Also, we have the following recurrence relation for $T_{n,m}(x)$ whenever $m\geq n+1$:

$$\begin{aligned}(n-m-1)&T_{n,m+1}(x)\\ &=xT'_{n,m}(x)+m(2x+x^2)T_{n,m-1}(x)\\ &+\{(m+1)(1+2x)+(p-1-x)\}T_{n,m}(x)-(p+2mx)(-x)^m e^{-nx}.\end{aligned}$$

Proof:

By direct computation we can easily get the values of $T_{n,0}(x)$, $T_{n,1}(x)$ and $T_{n,2}(x)$.

Now, using 2.1 Lemma, we have

$$\begin{aligned}(n-m-1)T_{n,m+1}(x)&=(n-m-1)\{Y_{n,m+1}(x)+(-x)^{m+1}e^{-nx}\}\\ &=xY'_{n,m}(x)+m(2x+x^2)Y_{n,m-1}(x)\\ &+\{(m+1)(1+2x)+(p-1-x)\}Y_{n,m}(x)+(n-m-1)e^{-nx}\end{aligned}$$

$$\begin{aligned}(n-m-1)&T_{n,m+1}(x)\\ &=xT'_{n,m}(x)+m(2x+x^2)T_{n,m-1}(x)\\ &+\{(m+1)(1+2x)+(p-1-x)\}T_{n,m}(x)-(p+2mx)(-x)^m e^{-nx}.\end{aligned}$$

Consequently for each $x\in[0,\infty)$, we can determine $T_{n,m}(x)\ as\ O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$, where $\left[\frac{m+1}{2}\right]$ denotes the integer part of the value $\frac{m+1}{2}$.

3. Simultaneous Approximation

3.1 Lemma [5]: Let δ be a positive real number. Then, for every $m>0$ and $x\in[0,\infty)$ there exists a positive constant $C_{m,x}$ such that:

$$\int_{|t-x|\geq\delta} W_{n,p}(t,x)(1+t)^{-\alpha}dt\leq C_{m,x,p}n^{-m}, m=0,1,2,\ldots$$

where $C_{m,x,p}$ depending on m, x, p and $W_{n,p}(t,x)=\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}b_{n,k+p-1}(t)f(t)dy+\delta(t)e^{-nx}$, $x\in[0,\infty)$.

The δ(t) begins the Dirac-delta function. Making use of Taylors expansion for the function $(1+t)^{\alpha}$ about $x\in(0,\infty)$, Schwarz inequality for integration and then for summation and Lemma 2.2, the proof of the Lemma easily follows.

3.2 Lemma [6]: There exist the polynomials $Q_{i,j,r}(x)$ independent on n,k such that

$$x^r p_{n,k}^{(r)}(x)=\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i(k-nx)^j Q_{i,j,r}(x)p_{n,k}(x).$$

3.3 Lemma [5]: For $x\in[0,\infty)$,

$$\begin{aligned}\sum_{k=1}^{\infty}p_{n,k}(x)(k-nx)^{2j}&=n^{2j}\left[\sum_{k=0}^{\infty}p_{n,k}(x)\left(\frac{k}{n}-x\right)^{2j}-e^{-nx}(-x)^{2j}\right]\\ &=n^{2j}\left[O\left(n^{-j}\right)+O(n^{-s})\right]=O(n^j)\text{ for any s}>0.\end{aligned}$$

3.4 Lemma: for $m\geq 1$, we have:

$$B_{n,p}(t^m;x)=\frac{(n-m-1)!}{(n-1)!}\{(nx)^m+m\big(p+(m-1)\big)(nx)^{m-1}+O(n^{-2})\}.$$

Proof:

By using Lemma 1.1, we get

$$B_{n,p}(t^m;x) = \sum_{k=1}^{\infty} P_{n,k}(x) \int_0^{\infty} b_{n,k+p-1}(t) t^m dt + 0$$
$$= \sum_{k=1}^{\infty} P_{n,k}(x) \frac{(m+k+p-1)!\,(n-m-1)!}{(k+p-1)!\,(n-1)!}$$

$$= \frac{(n-m-1)!}{(n-1)!} \sum_{k=1}^{\infty} P_{n,k}(x)(k+p+m-1)(k+p+m-2) \ldots\ldots (k+p)$$

$$= \frac{(n-m-1)!}{(n-1)!} \sum_{k=1}^{\infty} P_{n,k}(x) \left\{ k^m + \left(mp + \frac{m(m-1)}{2} \right) k^{m-1} + O(n^{-2}) \right\}$$
$$= \frac{(n-m-1)!}{(n-1)!} \left\{ \left\{ (nx)^m + m\big(p + (m-1)\big)(nx)^{m-1} + O(n^{-2}) \right\} \right\}$$

3.1 Theorem: Let $f \in C_\alpha[0,\infty), \alpha > 0$ and $f^{(r)}$ exists on $(0,\infty)$, then

$$\lim_{n\to\infty} B_{n,p}^{(r)}(f(t);x) = f^{(r)}(x). \qquad (3.1)$$

Further, if $f^{(r)}$ exists and is continuous on $(a-\eta, a+\eta) \subset (0,\infty), \eta > 0$, then the limit (3.1) holds uniformly on the interval $[a,b]$.

Proof: By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r} \frac{f^{(r)}(x)}{i!} (t-x)^i + \varepsilon(t,x)(t-x)^r,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$, hence

$$B_{n,p}^{(r)}(f(t);x) = \int_0^\infty W_n^{(r)}(t,x) f(t) dt$$
$$= \sum_{i=0}^{r} \frac{f^{(r)}(x)}{i!} \int_0^{\infty} W_n^{(r)}(t,x)\,(t-x)^i dt + \int_0^{\infty} W_n^{(r)}(t,x)\varepsilon(t,x)(t-x)^r dt$$
$$:= \Sigma_1 + \Sigma_2.$$

Using Lemma 3.4, we get that

$$\Sigma_1 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \frac{d^r}{dx^r} \left[\int_0^{\infty} W_n(t,x) t^j dt \right]$$
$$= \frac{f^{(r)}(x)}{r!} \left[\frac{(n-r-1)!}{n^{-r}(n-1)!} r! \right] = f^{(r)}(x) \left[\frac{(n-r-1)!}{n^{-r}(n-1)!} \right] \to f^{(r)}(x), \text{ as } n \to \infty.$$

Next, making use of Lemma 3.2, we have

$$\Sigma_2 = \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i \frac{Q_{i,j,r}(x)}{x^r} \sum_{k=1}^{\infty} (k-nx)^j p_{n,k}(x) \int_0^{\infty} b_{n,k+p-1}(x)\varepsilon(t,x)(t-x)^r dt$$
$$+ (-n)^r e^{-nx} \varepsilon(0,x)(-x)^r.$$

Then

$$|\Sigma_2| \le \sum_{\substack{2i+j\le r \\ i,j\ge 0}} n^i \frac{|Q_{i,j,r}(x)|}{x^r} \sum_{k=1}^{\infty} |k-nx|^j p_{n,k}(x) \int_0^{\infty} b_{n,k+p-1}(x)|\varepsilon(t,x)||(t-x)|^r dt$$
$$+ n^r e^{-nx} |\varepsilon(0,x)| x^r.$$
$$:= I_1 + I_2.$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$, for given $\varepsilon > 0, \exists \delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$, whenever $0 < |t-x| < \delta$. for $|t-x| \geq \delta, \exists$ a constant $M > 0$ such that $|\varepsilon(t,x)(t-x)^r| \leq M|t-x|^\alpha$ Hence

$$I_1 \leq \sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{|Q_{i,j,r}(x)|}{x^r} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n)^i \sum_{k=1}^{\infty} |k-nx|^j p_{n,k}(x) \left(\varepsilon \int_{|t-x|<\delta}^{\infty} b_{n,k+p-1}(x)|t-x|^r dt + \int_{|t-x|\geq\delta}^{\infty} b_{n,k+p-1}(x) M|t-x|^\alpha dt \right).$$

$$:= I_3 + I_4.$$

Let $\sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{|Q_{i,j,r}(x)|}{x^r} = C_1, x \in (0,\infty)$ be fixed. Now , applying Schwarz inequality for integration and then summation, we get:

$$I_3 = \varepsilon C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \sum_{k=1}^{\infty} |k-nx|^j p_{n,k}(x) \left[\int_{|t-x|<\delta}^{\infty} b_{n,k+p-1}(x) dt \right]^{1/2} \left[\int_{|t-x|<\delta}^{\infty} b_{n,k+p-1}(x)(t-x)^{2r} dt \right]^{1/2}.$$

$$\leq \varepsilon C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left[\sum_{k=1}^{\infty} (k-n)^{2j} p_{n,k}(x) \right]^{1/2} \left[\sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k+p-1}(x)(t-x)^{2r} dt \right]^{1/2}.$$

$\sum_{k=1}^{\infty} p_{n,k}(x) \int_{|t-x|\geq\delta}^{\infty} b_{n,k+p-1}(x)(t-x)^{2r} dt = T_{n,2r}(x) - (-x)^{2r} e^{-nx} = o(1)$ for any $s > 0$.

Then from 3.3 Lemma, we get

$$I_3 \leq \varepsilon C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n)^i O(n^{j/2}) O(n^{-s}) = \varepsilon O\left(n^{\frac{r}{2}-s}\right) = O(n^{-s}) \; for\; s > \frac{r}{2}.$$

Next again using Schwarz inequality for integration and then summation, we get

$$I_4 \leq C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq o}} n^i \sum_{k=1}^{\infty} p_{n,k}(x) |k-nx|^j \left[\int_{|t-x|\geq\delta} b_{n,k+p-1}(t) dt \right]^{1/2} \left[\int_{|t-x|\geq\delta} b_{n,k+p-1}(t)(t-x)^{2\alpha} dt \right]^{1/2}$$

$$\leq C_1 \sum_{\substack{2i+j\leq r\\ i,j\geq o}} n^i \left[\sum_{k=1}^{\infty} p_{n,k}(x)(k-nx)^{2j} \right]^{1/2} \left[\sum_{k=1}^{\infty} P_{n,k}(x) \int_0^{\infty} b_{n,k+p-1}(t)(t-x)^{2\alpha} dt \right]^{1/2}$$

By using 3.3 Lemma

$$\leq \sum_{\substack{2i+j\leq r\\ i,j\geq o}} n^i O(n^{\frac{j}{2}}) O(n^{-s}) = O\left(n^{\frac{r}{2}-s}\right) = o(1) \text{ for } s > \frac{r}{2}$$

Since $\varepsilon > 0$ arbitrary, it follows that $I_1 \to 0$ as $n \to \infty$ and hence $I_2 = o(1)$. Combining the estimates of I_1 and I_2, we get $lim_{n\to\infty} B_{n,p}^{(r)}(f(t);x) = f^{(r)}(x)$.

To prove the uniformly assertion is sufficient to remark that $\delta(\varepsilon)$ in the above proof can be chosen to be independent of $x \in [a,b]$ and also that the other estimates hold uniformly in $x \in [a,b]$.

3.2 Theorem: let $f \in C_\alpha[0,\infty), \alpha > 0$ and $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n[B_{n,p}^{(r)}(f(t);x) - f^{(r)}(x)]$$
$$= \frac{r(r+1)}{2} f^{(r)}(x) + \{x(1+r)+(p+r)\} f^{(r+1)}(x) + \frac{(x+x^2)}{2} f^{(r+2)}(x).$$

Further, if $f^{(r+2)}$ exists and is continuous on the interval $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$ then (3.1) holds uniformly on $[a,b]$.

Proof: by the Taylor's expansion of f, we get

$$f(t) = \sum_{i=0}^{r+2} f^{(i)}(x) \frac{(t-x)^i}{i!} + \varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Then

$$B_{n,p}^{(r)}(f(t);x) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(t;x)(t-x)^i dt + \int_0^\infty W_n^{(r)}(t;x)\, \varepsilon(t,x)(t-x)^{r+2} dt$$
$$:= \Sigma_1 + \Sigma_2 .$$

By using the same technique of 3.1 theorem, we get $\Sigma_2 \to 0 \;\; as\; n \to \infty$.

Then, by using 3.4 Lemma, we have

$$\Sigma_1 = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=r}^{i} \binom{i}{j} (-x)^{i-j} \int_0^\infty W_n^{(r)}(t;x) t^j\, dt$$

$$= \frac{f^{(r)}(x)}{r!}\left[\frac{(n-r-1)!n^r}{(n-1)!} r!\right] + \frac{f^{(r+1)}(x)}{(r+1)!}\left[(r+1)(-x)\int_0^\infty W_n^{(r)}(t,x)t^r dt + \int_0^\infty W_n^{(r)}(t,x)t^{r+1}dt\right] + \frac{f^{(r+2)}(x)}{(r+2)!}\left[\frac{(r+1)(r+2)}{2} x \int_0^\infty W_n^{(r)}(t,x)t^r dt + (r+2)(-x)\int_0^\infty W_n^{(r)}(t,x)t^{r+1}dt + \int_0^\infty W_n^{(r)}(t,x)t^{r+2}dt\right]$$

$$= f^{(r)}(x)\left[\frac{(n-r-1)!\,n^r}{(n-1)!}\right]$$
$$+ \frac{f^{(r+1)}(x)}{(r+1)!}\left[\frac{(n-r-1)!}{(n-1)!}(r+1)(-x)n^r r! + \frac{(n-r-2)!}{(n-1)!}[n^{r+1}(r+1)!\,x + r!\,n^r(r+1)(p+r)]\right]$$
$$+ \frac{f^{(r+2)}(x)}{(r+2)!}\left[\frac{(n-r-1)!}{(n-1)!}\frac{(r+1)(r+2)}{2} x^2 n^r r! + (r+2)(-x)\frac{(n-r-2)!}{(n-1)!}[(r+1)!\,n^{r+1}x + r!\,n^r(r+1)(p+r)] + \frac{(n-r-3)!}{(n-1)!}\left[\frac{(r+2)!}{2} n^{r+2}x^2 + (r+2)(p+r+1)(r+1)!\,n^{r+1}x\right]\right].$$

Hence,

$$\lim_{n\to\infty} n[B_{n,p}^{(r)}(f(t);x) - f^{(r)}(x)]$$
$$= \frac{r(r+1)}{2} f^{(r)}(x) + \{x(1+r) + (p+r)\} f^{(r+1)}(x) + \frac{(x+x^2)}{2} f^{(r+2)}(x).$$

3.3 Theorem:

Let $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$ and $r \le v \le (r+2)$. if $f^{(v)}$ exists and is continuous on $(a-\eta, b+\eta)$then for$n \to \infty$:

$$\left\| B_{n,p}^{(r)}(f(t);x) - f^{(r)}(x) \right\|_{C[a,b]}$$
$$\le A_1 n^{-1} \sum_{i=r}^{v} \|f^{(i)}\|_{C[a,b]} + A_2 n^{-\frac{1}{2}} \omega_{f^{(v)}}\left(n^{-\frac{1}{2}}; (a-\eta, b+\eta)\right) + O(n^{-2}),$$

where A_1, A_2 are constants independent of f and n. $\omega_f(\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$, and $\|.\|$ denotes the sup-norm on the interval $[a,b]$.

Proof:

By Taylor`s expansion of f, we have

$$f(t) = \sum_{i=0}^{v} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(v)}(\xi) - f^{(v)}(\xi)}{v!}(t-x)^v \chi(t) + h(t,x)\big(1-\chi(t)\big),$$

where ξ lies between t, x and $\chi(t)$ is the characteristic function of the interval $(a-\eta, b+\eta)$.

For $t \in (a-\eta, b+\eta)$ and $x \in [a,b]$, we get:

$$f(t) = \sum_{i=0}^{v} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(v)}(\xi) - f^{(v)}(\xi)}{v!}(t-x)^v.$$

For $t \in [0,\infty) \setminus (a-\eta, b+\eta)$ and $x \in [a,b]$, we define

$$h(t,x) = f(t) - \sum_{i=0}^{v} \frac{f^{(i)}(x)}{i!}(t-x)^i.$$

Now,

$$B_{n,p}^{(r)}(f(t);x) - f^{(r)}(x)$$
$$= \left[\sum_{i=0}^{v} \frac{f^{(i)}(x)}{v!} B_{n,p}^{(r)}\big((t-x)^i;x\big) - f^{(r)}(x)\right]$$
$$+ B_{n,p}^{(r)}\left(\frac{f^{(v)}(\xi) - f^{(v)}(\xi)}{v!}(t-x)^v \chi(t);x\right) + B_{n,p}^{(r)}\big(h(t,x)\big(1-\chi(t)\big);x\big)$$
$$:= \Sigma_1 + \Sigma_2 + \Sigma_3.$$

By using 3.4 Lemma, we get:

$$\Sigma_1 = \sum_{i=0}^{v} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} B_{n,p}^{(r)}\big(t^j;x\big) - f^{(r)}(x)$$
$$= \sum_{i=0}^{v} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \frac{d^r}{dx^r}\left[\frac{(n-j-1)!}{(n-1)!}\big\{(nx)^j + r\big(p+(j-1)\big)(nx)^{j-1}\right.$$
$$\left. + O(n^{-2})\big\}\right] - f^{(r)}(x)$$

Consequently,

$\|\Sigma_1\|_{C[a,b]} \le A_1 n^{-1}\left[\sum_{i=r}^{v} \|f^{(i)}\|_{C[a,b]}\right] + O(n^{-2})$, uniformly on $[a,b]$.

To estimate Σ_2 we proceed as follows:

$|\Sigma_2| \le B_{n,p}^{(r)}\left(\frac{f^{(v)}(\xi) - f^{(v)}(\xi)}{v!}|t-x|^v \chi(t);x\right)$

$$\leq \frac{\omega_{f^{(v)}}(\delta;(a-\eta,b+\eta))}{v!} B_{n,p}^{(r)}\left(\left(1+\frac{|t-x|}{\delta}\right)|t-x|^{v};x\right)$$

$$\leq \frac{\omega_{f^{(v)}}(\delta;(a-\eta,b+\eta))}{v!}\Big[\sum_{k=1}^{\infty} P_{n,k}^{(r)}\quad(x)\int_0^{\infty} b_{n,k+p-1}(t)(|t-x|^{v}+\delta^{-1}|t-x|^{v+1})dt+$$
$$e^{-nx}(|x^{v}|+\delta^{-1}|x^{v+1}|)\Big];\ \delta>0.$$

Now, for $s=0,1,2,....$ we have:

$$\sum_{k=1}^{\infty} P_{n,k}(x)|k-nx|^{j}\int_0^{\infty} b_{n,k+p-1}(t)|t-x|^{s}dt \qquad (3.2)$$

$$\leq \sum_{k=1}^{\infty} P_{n,k}(x)|k-nx|^{j}\left[\left(\int_0^{\infty} b_{n,k+p-1}(t)dt\right)^{\frac{1}{2}}\left(\int_0^{\infty} b_{n,k+p-1}(t)(t-x)^{2s}dt\right)^{\frac{1}{2}}\right]$$

$$\leq \left[\sum_{k=1}^{\infty} P_{n,k}(x)(k-nx)^{2j}\right]^{\frac{1}{2}}\left[\sum_{k=1}^{\infty} P_{n,k}(x)\int_0^{\infty} b_{n,k+p-1}(t)(t-x)^{2s}dt\right]^{\frac{1}{2}}$$

$$=O\left(n^{\frac{j}{2}}\right)O\left(n^{\frac{-s}{2}}\right)=O\left(n^{\frac{(j-s)}{2}}\right),\text{ uniformly on } [a,b].$$

Therefore, by using 3.2 Lemma and (3.2), we get:

$$\sum_{k=1}^{\infty}\left|P_{n,k}{}^{(r)}(x)\right|\int_0^{\infty} b_{n,k+p-1}(t)|t-x|^{s}dt \qquad (3.3)$$

$$\leq \sum_{k=1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i}|k-nx|^{j}\frac{|Q_{i,j,r}(x)|}{x^{r}}P_{n,k}(x)\int_0^{\infty} b_{n,k+P-1}(t)|t-x|^{s}dt$$

$$\leq \sup_{\substack{2i+j\leq r\\ i,j\geq 0}}\ \sup_{x\in[a,b]}\frac{|Q_{i,j,r}(x)|}{x^{r}}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i}\left[\sum_{k=1}^{\infty} P_{n,k}(x)\,|k-nx|^{j}\int_0^{\infty} b_{n,k+P-1}(t)|t-x|^{s}dt\right]$$

$$=A\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i}O\left(n^{\frac{j-s}{2}}\right)=O\left(n^{\frac{r-s}{2}}\right),\text{ uniformly on [a, b].}$$

since $sup_{\substack{2i+j\leq r\\ i,j\geq 0}}\ sup_{x\in[a,b]}\frac{|Q_{i,j,r}(x)|}{x^{r}}=A$ be fixed

Choosing $\delta=n^{\frac{-1}{2}}$ and applying (3.3), we led to

$$\|\Sigma_2\|\leq \frac{\omega_{f^{(v)}}\left(n^{\frac{-1}{2}};(a-\eta,b+\eta)\right)}{v!}\left[O\left(n^{\frac{r-v}{2}}\right)+n^{\frac{1}{2}}O\left(n^{\frac{r-v-1}{2}}\right)+O(n^{-m})\right],\text{ for any } m>0$$

$$\leq A_2 n^{\frac{-(r-v)}{2}}\ \omega_{f^{(v)}}\left(n^{\frac{-1}{2}};(a-\eta,b+\eta)\right).$$

Since $t\in[0,\infty)\setminus(a-\eta,b+\eta)$, we can choose$\delta>0$ in such a way that $|t-x|\geq\delta$ for all $x\in[a,b]$.

Thus, using Lemma (3.2), we obtain

$$\|\Sigma_3\|\leq \sum_{k=1}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i}|k-nx|^{j}\frac{|Q_{i,j,r}(x)|}{x^{r}}P_{n,k}(x)\int_0^{\infty} b_{n,k+P-1}(t)|h(t,x)|dt+e^{-nx}|h(0,x)|.$$

For $|t-x| \geq \delta$, we can find a constant $C > 0$ such that $|h(t,x)| \leq C|t-x|^{\alpha}$. Finally using Schwarz inequality for integration and then for summation, we get

$|\Sigma_3| = O(n^{-s}), s > 0$ uniformly on $[a,b]$. Combining the estimates of $\Sigma_1, \Sigma_2, \Sigma_3$ we see the required result is immediate.

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