# DIRECT ESTIMATION FOR MULTIVARIATE POLYNOMIAL APPROXIMATION

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**Abstract**
The Bramble-Hilbert lemma is a fundamental result on multivariate polynomial approximation .It is frequently applied in the analysis of finite element methods used for numerical solutions. Our main result is to improve the following Bramble-Hilbert lemma to the case $0 < p < 1$ :let $\Omega \subset R^n$ be abounded convex domain and let $g \in W_p^m(\Omega)$, $m \in \mathrm{N}, 0 \le p \le \infty$,where $W_p^m(\Omega)$ is the Sobolev spaces ,then there exists a polynomial $P$ of degree m-1 for which

$$|g-P|_{k,p} \le \mathrm{c(n,m)(diam}\,\Omega)^{m-k} |\ \mathrm{g}\ |,$$

$k = 0,1,...,m$, where $|\ .\ |_{k,p} = \sum_{|\alpha|=k} \|\ D^{\alpha}\ \|_{lp(\Omega)}$ is the Sobolev semi norm of order $k$.

As a consequence we get that for $f \in L_{p(\Omega)}$, $p < 1$ .

$$E_{m-1}(f,\Omega)_p \le \mathrm{c}(m,n)\,\tau_m(f,t)_p,$$

where

$$E_{m-1}(f,\Omega)_p = \inf_{p \in \pi_{m-1}} \|f-p\|_{lp(\Omega)},$$

is the rate of polynomial approximation of degree $m-1$ , and $\tau_m(f,t)_p$ is the averaged modulus of smoothness, and $t > 0$.

## 1. Introduction

We begin by recalling classical smoothness measure over multivariate domains . Here and throughout the paper we assume that the domain $\Omega \subset R^n$ is compact with a nonempty interior . A first notion of smoothness uses the Sobolev spaces $W_p^m(\Omega)$ . These are spaces of functions $\mathrm{g} \in L_p(\Omega)$ which have all their distributional derivatives of order up to $m$ ,

$$D^{\alpha} g = \frac{\partial^k g}{\partial x_1^{\alpha 1} \ldots \partial x_n^{\alpha n}}\ ,\ \alpha = (\alpha 1, \alpha 2, ..., \alpha n),\ \alpha \in Z_+^n \quad , |\alpha| = \sum_{i=1}^{n} \alpha i = k, 0 \le k \le 1.$$

In $Lp(\Omega)$.the semi norm of $W_p^m(\Omega)$ is given by

$$|g|_{m,p} = \sum_{|\alpha|=m} \|D^{\alpha} g\|_{Lp(\Omega)} < \infty,$$

and may be regarded as a measure of the smoothness of order $m$ of a function in $W_p^m(\Omega)$ . $K$-functional of order $m$ of $f \in Lp(\Omega)$ [4,6] is defined by

$$K_m(f,t)_p = K(f,t,Lp(\Omega),W_p^m(\Omega)) = \inf_{g\in W_p^m(\Omega)} \left\{\|f-g\|_p + t|g|_{m,p}\right\}.$$

Since we assume $\Omega$ to be compact we may denote $K_m(f,\Omega)_p = K_m(f,d^m)_p$, where $d = diam(\Omega)$ the diameter of $\Omega$ .

For $f \in Lp(\Omega), 0 < p \le \infty, h \in R^n$ and $m \in N$ ,we recall the $m\,th$ order difference operator $\Delta_h^m(f,.):\Omega \to R$, $\Delta_h^m(f,x) = \Delta_h^m(f,\Omega,x) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} f(x+kh)$ *if* $[x,x+mh]\subset\Omega$, *0* otherwise, where $[x,y]$ denotes the line segment connecting any two points $x,y\in R^n$.

The modulus of smoothness [5,7] is defined by

$$\omega_m(f,t)_p = \sup_{|h|\le t}\left\|\Delta_h^m(f,\Omega,.)\right\|_{Lp(\Omega)}, t>0 \qquad (1.1)$$

for $h\in R^n$, $|h|$ denotes the norm of $h$ . We also denote

$$\omega_m(f,\Omega)_p = \sup_{h\in R^n}\left\|\Delta_h^m(f,\Omega,.)\right\|_{Lp(\Omega)} .$$

It is known that the $K-$functional of order $m$ of $f\in Lp(\Omega), 1\le p\le\infty$ and the modulus of smoothness in (1.1) are equivalent [8] . That is ,there exists $c_1,c_2>0$ , such that for any $t>0$,

$$c(\Omega)K_m(f,t^m)_p \le \omega_m(f,t)_p \le c(m)K_m(f,t^m)_p . \qquad (1.2)$$

The so called $\tau-$modulus (or Sendov-Popov modulus) , an averaged modulus of smoothness , defined for bounded measurable functions by

$$\tau_m(f,t,\Omega)_p = \|\omega_m(f,.,t)\|_{Lp(\Omega)} \qquad (1.3)$$

where

$$\omega_m(f,x,t) = \sup\left\{\left|\Delta_h^m(f,\Omega,y)\right| : y \pm \frac{mh}{2} \in \left[x-\frac{rt}{2}, x+\frac{rt}{2}\right]\cap\Omega\right\},$$

where $t>0, \Omega\subset R^n, h\in R^n, x,y\in R^n$ .

However , while it is easy to prove the following result

***Lemma1.4*** Let $\Omega\subset R^n$, and $\Omega_i$ be a bounded measurable disjoint subsets of $\Omega$ satisfying $\bigcup_{i=1}^{n}\Omega_i = \Omega$ , and $x^i\in\Omega_i$. Using measure of $\Omega_i = \delta_i$ we have

$$\left(\sum_{i=1}^{n}\omega_m(f,x^i,t)^p\delta_i\right)^{1/p} \le c(p)\left(\sum_{i=1}^{n}\int_{\Omega_i}\omega_m(f,x^i,t)^p\right)^{1/p} \le c(p)\tau_m(f,t)_p ,$$

(1.5)

where $c(p)$ denote constants which depend on $p$ only , and are not necessarily the same even when they occur on the same line .

Let $\prod_{m-1} = \prod_{m-1}(R^n)$ denote the multivariate polynomials of total degree *m-1* in $n$ variables . Given a nontrivial multivariate domain , our goal is to estimate the degree of approximation of a function $f\in Lp(\Omega), 0<p<1$ ,

$$E_{m-1}(f,\Omega)_p = \inf_{p\in\Pi_{m-1}}\|f-p\|_{L_p(\Omega)} .$$

Before we introduce the Bramble-Hilbert lemma we require the following definitions :

A domain $\Omega$ is star-shaped with respect to a ball $B \subseteq \Omega$, if for each point $x \in \Omega$ , the closed convex-hull of $\{x\} \bigcup B$ is contained in $\Omega$ . Let

$$\rho_{\max} = \max \{\rho : \Omega \text{ is star- shaped with respect to a ball } B \subseteq \Omega \text{ of radius } \rho \},$$

the chunkiness parameter of $\Omega$ is defined by

$$\gamma = \frac{d}{\rho_{\max}} \ , ((d = diam\Omega) \ .$$

This leads to the following formulation of the Bramble – Hilbert lemma [2]. Let $\Omega$ be star shaped with respect to some ball B and let $g \in W_p^m(\Omega), 1 \le p \le \infty, m \in N$ ,then there exists a polynomial $P \in \prod_{m-1}$ for which

$$|g - p|_{k,p} \le c(n,m,\gamma) d^{m-k} |g|_{m,p}, k = 0,...,m.$$

Earlier , Dechevski and Quak improved the Bramble lemma in some cases their result applies to the larger class of domains that are star-shaped with respect to a point . A domain $\Omega$ is aster-shaped with respect to a point $x_0 \in \Omega$ if for any point $x \in \Omega$ the line segment [ $x_0, x$ ] is contained in $\Omega$ .
The following is a modified version of their result :

***Proposition 1.6*** [3] let $\Omega$ be a Lipshitz domain which is star-shaped with respect to a point $x_0 \in \Omega$ , and let $g \in W_2^m(\Omega)$ .then for $m \in N$ and $2 \le n < p \le \infty$ , there exists a polynomial $P \in \prod_{m-1}$ for which

$$|g - p|_{k,p} \le c(n,m,p) d^{m-k} |g|_{m,p}, k = 0,1,...,m \ .$$

Our approach differs from previous work in the case $0 < p < 1$. Our main result is

***Theorem I***: Let $\Omega \subset R^n$ be convex , and let $g \in W_p^m(\Omega)$ , $m \in N, 0 < p < 1$ .
Then there exists a polynomial $P \in \prod_{m-1}$ for which

$$|g - p|_{W^k_{p(\Omega)}} \le c(n,m,p,\Omega) |g|_{W^k_{o(\Omega)}} .$$

A direct consequence from the proof of theorem I is the following :

***Corollary II***: For all convex domains $\Omega \subset R^n$ and a function $g \in Lp(\Omega), 0 < p < 1$ . there exists a polynomial $P \in \prod_{m-1}$ for which

$$\|g - P\|_p \le c \tau_m (g,t)_p \ ,$$

where $c$ is a constant depending on $p, m, n, \Omega$ .

***Corollary III***: for all convex domains $\Omega \subset R^n$ and function $f \in Lp(\Omega), p < 1,$

$$E_{m-1}(f,\Omega)_p = K_{m-1}(f)_p .$$

## 2. The averaged Taylor polynomial [3]

We recall some basic definitions of multivariate polynomials , differentials and Taylor series thought this section we use the notation of section 2 in [3] . For multi index $\alpha \in Z_+^n$ $\alpha! = \prod_{i=1}^{n} \alpha_i!$ and denoted by $x^\alpha = \prod_{i=1}^{n} x_i^{\alpha - i}$ , the multivariate monomial of total degree $|\alpha|$ . Denote the set of all multivariate polynomials of total degree $m-1$ by

$$\pi_{m-1}\left(R^n\right) = \sum_{|\alpha| \le m-1} c_\alpha x^\alpha \ .$$

The classical Taylor polynomial of order *m* (degree *m-1* ) of a function $g \in C^m(\Omega)$ at $x \in \Omega$ , about the point $y \in \Omega$ , is given by

$$T_y^m(g(x)) = \sum_{|\alpha|<m} \frac{D^\alpha g(y)}{\alpha!}(x-y)^\alpha \ .$$

The Taylor remainder of order *m* of a function $g \in C^m(\Omega)$ at $x \in \Omega$ about the point $y \in \Omega$ , is given by

$$TR_y^m g(x) = m \sum_{|\alpha|=m} \frac{(x-y)^\alpha}{\alpha!} \int_0^1 S^{m-1} D^\alpha g(x + s(y-x)) ds \ .$$

It is meaningful provided the segment $[y, x]$ is contained in $\Omega$ . Then we have

$$g(x) = T_y^m g(x) + TR_y^m g(x) \ .$$

Next we introduce the averaged Taylor polynomial . It can be shown that for a ball

$B(x_0, \rho) = \{z \in R^n : |z - x_0| \le \rho\}$ there exists a cut-off function $\phi_\beta$ with the following properties (i) $\int_{R^n} \phi_\beta(x)dx = 1$ (ii) $\sup p(\phi_\beta) = \beta$ (iii) $\phi_\beta \in C^\infty(R^n)$ (iv) $\|\phi_\beta\|_\infty \le \rho^{-n}$ .

Given $g \in C^m(\Omega)$ the averaged Taylor polynomial of order $m$ (degree $m-1$) (average over a ball $B \subseteq \Omega$) is defined by

$$Q^m g(x) = \int_B T_y^m g(x) \phi_\beta(y) dy \quad x \in \Omega \ .$$

We also define the averaged Taylor remainder , namely

$$R^m g(x) = g(x) - Q^m g(x) \quad .$$

The following lemma is a special case of the classical Bramble-Hilbert which estimate the simultaneous degree of approximation of the averaged Taylor polynomial in normalized setting .

***Lemma 2.1*** [3] let $B(0,1) \subseteq \Omega \subseteq B(0,n)$ , be star shaped with respect to $B(0,1)$. Then for any $g \in C^\infty(\Omega), m \in N$ and $1 \le p \le \infty$ ,we have

$$\left| g - Q^m g \right|_{k,m} \le c(n,m) \left| g \right|_{m,p} , k = 0,1,..,m,$$

where $Q^m$ is averaged over $B(0,1)$

**3.John's theorem** [3]

In this section we also use the notation in section 3 of [3]. An ellipsoid *E* is the image of the closed unite ball in $R^n$ under a nonsingular affine map

$A(x) = Mx + b, m \in M_{n \times n}(R), b \in R^n$ ,the center of $E$ is $b = A(0)$ .Now let $c + n(E - c) = \{c + n(x - c) : x \in E\}$. Then we need the following result from [1],[3]

***Lemma 3.1*** *(John's theorem)* . let $\Omega \in R^n$ be convex .then there exists an ellipsoid $E \subseteq \Omega$ such that if $x_0$ is the center of $E$ then $E \subseteq \Omega \subseteq x_0 + n(E - x_0)$ . By the definition above and John's theorem implies that for each convex domain $\Omega$ we can find an affine map such that $B(0,1) \subseteq A^{-1}(\Omega) \subseteq B(0,n)$ .

For the proof of our main result we also need the following lemma from [3]

***Lemma 3.2*** let $\Omega \subseteq R^n$ , and let $A$ be a nonsingular affine map such that $B(0,1) \subseteq A^{-1}(\Omega)$. Then for $g \in C^m(\Omega)$ and $\alpha \in Z_+^n, |\alpha| = k, 1 \le k \le m-1$

$$D^{\alpha}[Q^{m}(g(A.))(A^{-1}x)] = Q^{m-k}((D^{\alpha}g)(A.))(A^{-1}x), \qquad (3.3)$$

where $Q^{m}$ is with respect to $B(0,1)$.

**4. The proof of the main results**

One may take $P(x) = Q^{m}(g(A.))(A^{-1}x)$, where $Q^{m}$ is the averaged Taylor polynomial over the ball $B(0,1) \subset R^{n}$, and $A$ is an affine transformation related to $\Omega$.

Using lemma 2.1. ,(1.2),(1.5) and (1.3) to obtain

$$\left|g - Q^{m}g\right| \le c(n,m)|g|_{m,\infty}$$
$$\le c(n,m,\Omega)K_{m}(g,t^{m})_{\infty}$$
$$\le c(n,m,\Omega)\omega_{m}(g,t)_{\infty}.$$

Then

$$\left\|g - Q^{m}g\right\|_{p} \le c(n,m,\Omega)(\int_{\Omega}\omega_{m}(g,t)_{\infty}^{p})^{1/p}$$
$$\le c(n,m,\Omega)(\sum_{i=1}^{n}\int_{\Omega_i}\omega_{m}(g,x_{i},t)^{p})^{1/p}, t > 0$$
$$\le c(p,n,m,\Omega)\tau_{m}(g,t)_{p} \quad \text{(corollary 1.2)}$$
$$\le c(p,n,m,\Omega)|g|$$
$$\le c(p,n,m,\Omega)\|g\|_{Lp(\Omega)} \quad .$$

For $1 \le k \le m-1$ take $\alpha \in Z_{+}^{n}$ , $|\alpha| = k, 1 \le k \le m-1$ and $h = D^{\alpha}g$ , then (3.3) yields

$$\left\|D^{\alpha}(g-p)\right\|_{Lp(\Omega)} = \left\|h(x) - Q^{m-k}(h(A.))(A^{-1}x)\right\|_{Lp(\Omega)} \quad .$$

By the case $k = 0$ proved above we have

$$\left\|h(x) - Q^{m-k}(h(A.))(A^{-1}x)\right\|_{Lp(\Omega)} \le c(p,n,m,\Omega)\|h\|_{p} \quad .$$

Then

$$\left\|D^{\alpha}(g-p)\right\|_{Lp(\Omega)} \le c(p,n,m,\Omega)\|h\|_{p} \le c(p,n,m,\Omega)\sum_{|\alpha|=m}\left\|D_{x}^{\alpha}g\right\|_{Lp(\Omega)} \quad .$$

Then by the definition of Sobolev semi norm we get

$$|g-p|_{W_{p}^{k}(\Omega)} \le c(n,m,p,\Omega)|g|_{W_{p}^{m}(\Omega)} \qquad ☺$$

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