Solution of non-linear optimal control problems using non-classical variational approach

حل مسائل أنظمة السيطرة الدينامية باستخدام الصياغة التغايرية غير التقليدية

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Abstract
The aim of this paper, modify a new approach based on variational techniques to solve some optimal control problems including linear and nonlinear optimal control problems with equality and inequality constraints. This approach has its bases on using Magri's approach for every operator, the results are established using direct Ritz method as well as optimization method to solve these problems numerically.

الخلاصة
الهدف من هذا البحث هو لايجاد تقنية جديدة مستندة على تقنياتِ الصياغة التغايريةِ لحل بَعْض مسائل أنظمة السيطرةِ الامثلية اللاخطّيةِ والخطيّةِ بقيودِ عدمَ المساواة والمساواةَ قواعدها الأساسية هي استخدام Magri's approach ونستعمل طريقة direct Ritz method لتحقيق الامثلية لحل هذه المسائل بشكل عددي.

1.Introduction

in this paper, we proposed and demonstrate the effectiveness of the non-classical variational technique for non-linear optimal control problems with equality-inequality constraint. The use of this technique has been demonstrated by solving some optimal control problems, and taken following optimal problem, let A(t) , B(t) and C(t) are continuous matrices of dimensions n×n, n×m and n×1, respectively:

$$\min_{u\in U}\int_0^T f_0(x(t),u(t),t)dt+\Phi(x(T)) \qquad (1)$$

subject to:

$$x'(t)=A(t)\,x(t)+B(t)u(t)+C(t) \qquad (2)$$

$$x(0)=x_0 \qquad (3)$$

$$x(T)=x_1 \qquad (4)$$

$$x \in X = \{x(t) \mid x(.): Co^2[0,T] \to R^n\} \quad (5)$$

$$u \in U = \{u(t) \mid u(.): (0,T) \to R^m, \text{and } u(t) \text{ is a piecewise continuous function}\} \quad (6)$$

where $Co^2[0,T]$ stands for the class of functions having continuous partial derivatives up to the second order, including the second order on time interval (0,T).

2.Problems formulations

The aim of this paper is to solve problem (1)-(6) numerically using the non-classical variational technique.

let A(t) , B(t) and C(t) are continuous matrices of dimensions n×n, n×m and n×1, respectively. Assumed that , the final time T is given , the function $f_0(x(t), u(t),t)$ is twice continuously differentiable in x and u is continuous in t, the function Φ(x(T)) is twice continuously differentiable in x. The general solution of equation (1) can be represented as the sum of a particular solution and a homogeneous solution.

$$x'(t) = A(t)\,x(t) + B(t)u(t) + C(t) \quad ; x \in R^n, u \in R^m$$

then:

$$\left[\frac{d}{dt}. - A(t).\right] x(t) = B(t)u(t) + C(t)$$

then, we have:

$$L_1 x(t)=f(t), \quad (7)$$

where :

$$f(t)=B(t)u(t)+C(t)$$

and

$$L_1 = \left[\frac{d}{dt}. - A(t).\right],$$ L_1 is a linear operator.

Now, the problem (1)-(6) can be written as:

$$\min \int_0^T f_0(x(t), u(t), t)dt + \Phi(x(T)), \quad (8)$$

subject to :

$$L_1 x(t)=f(t), \quad (9)$$

$$x(0)=x_0 , \quad (10)$$

$$x(T)=x_1 \; , \; x \in X \text{ and } u \in U, \quad (11)$$

Remark (2.1)

1. L_1 is not symmetric operator relative to the classical bilinear form (inner product), since it has the derivative with respect to time t (d/dt).
2. To find a variational formulation , we use the idea of the inverse problem of calculus of variation [3] equation (2).

Consider an arbitrary non degenerate bilinear form $(\tilde{u},\tilde{v})$. For example :

$$(\tilde{u},\tilde{v}) = \int_0^T \tilde{u}(t)\tilde{v}(t)dt, \qquad (12)$$

sine L_1 is not symmetric relative to the bilinear form equation (12) , one can define a new bilinear form as following :

$$\begin{aligned} <\tilde{u},\tilde{v}> &= (\tilde{u}, L_1\tilde{v}) \\ &= \int_0^T \tilde{u}(t) L_1 \tilde{v}(t)\, dt \\ &= \int_0^T \tilde{u}(t)\left[\frac{d}{dt}\tilde{v}(t) - A(t)\tilde{v}(t)\right]dt \quad , \end{aligned}$$

where L_1 as defined in equation (9).

Theorem:

If the linear equation $L_1x=f$, let $<u, v>$ be a certain bilinear form, if the operators L_1 is symmetric and non degenerate with respect to the chosen bilinear form $<u, v>$, then the critical points of the functional:

$F[x, u]= 1/2 <L_1x,x>-<f, x>$

are solution of this linear equation.

By using above theorem, the operator L_1 is now symmetric with respect to $<u, v>$, we can applied the theorem, the solution of equation (2) are the critical point of the functional:

$$
\begin{aligned}
F[x,u] &= \tfrac{1}{2} < L_1 x, x > - < f(t), x > \\
&= \tfrac{1}{2} (L_1 x, L_1 x) - (f(t), L_1 x) \\
&= \tfrac{1}{2} \int_0^T \left((L_1 x)^T (L_1 x) - f(t) L_1 x \right) dt \\
&= \tfrac{1}{2} \int_0^T \left[\frac{dx(t)}{dt} - A(t)x(t) \right]^T \left[\frac{dx(t)}{dt} - A(t)x(t) \right] - \\
&\quad f(t) \left[\frac{dx(t)}{dt} - A(t)x(t) \right]. \qquad (13)
\end{aligned}
$$

To solve the system of differential equation can be solve by interpreting the equation as Euler-Lagrange equation of some functional and finding the critical points of the functional.

Remark (2.2):

The following remark is necessary for computational procedure. consider the problem (1)-(6). Let:

$$\left[\frac{d}{dt} . - A(t) \right] x(t) = B(t)u(t) + C(t)$$

$L_1 x(t) = B(t)u(t)+C(t)$

$= f(t)$, for all $u(t) \in R^n$ and B(t) , C(t) are matrices.

Since for $u \in U$ for simplicity one can also represent $L_1\, x(t) = f(t)$ for every $u \in U$ provided that the critical points of $J(x) = 1/2<L_1 x, x>-<f(t), x>$,be found for each selected arbitrary function u(t) that makes the performance index minimum equation (1), i.e., $\min_{u(t)} J$ in this case one can also solve the optimal control problem via a non-classical variational approach for every selected $u=u(.) \in U$. so we have to solve :

$$\min_{u(t)} J(u(.)) , \qquad (14)$$

subject to:

$$L_1 x(t) = f(t) , \; f(t) = B(t)u(t)+C(t) , \qquad (15)$$

$x \in X$, $u \in U$ .Thus, a non-classical variational approach for equation (15) can applied many times iteratively (not once) unit the objective function (14) satisfies its minimum. Hence, the numerical procedure for solving even non linear optimal control problem with equality and inequality constraint based on this fact can be modified and proposed easily.

3.Generalized Ritz method

Ritz method is very important procedure of the so called direct variational method, the essence of the method is to express the unknown variable of the given initial boundary value problem as a linear combination of the elements of functions which are completely relative to the class of the feasible functions, the methods is called generalized Ritz method. Towards this level, let $\{G_i(t)\},\{H_i(t)\}$ be a two complete sequences of functions relative to the class of admissible function. Let:

$$x(t)=W(t)+p(a, G(t)), \quad (16)$$

$$u(t)=M(t)+q(b, H(t)), \quad (17)$$

where $W(t)\in R^n$, $M(t)\in R^m$, which are chosen function of indicated variables satisfies the given (if possible) the non-homogeneous boundary and initial conditions, i.e.:

$$W(t)= x_0+\frac{t}{T}(x_1-x_0), \quad (18)$$

Where $x_0 \in R^n$, i.e., $(x_{01}, x_{02}, \ldots, x_{0n})^T$,$x_1 \in R^n$, i.e., $(x_{11}, x_{12}, \ldots, x_{1n})^T$ and $p(a, G(t)) \in R^n$ as follows:

$$p(a,G(t))=\begin{bmatrix} p_1(a,G(t)) \\ p_2(a,G(t)) \\ \vdots \\ p_n(a,G(t)) \end{bmatrix},$$

where:

$$p_1(a,G(t))=\sum_{i=0}^{n_1} a_i G_i(t),\ p_2(a,G(t))=\sum_{i=n_1+1}^{n_2} a_i G_i(t),\cdots,\ p_n(a,G(t))=\sum_{i=n_{n-1}+1}^{n_n} a_i G_i(t), \text{ and}$$

$$a=(a_1,a_2, \ldots, a_{n_1}, a_{n_1}+1, \ldots, a_{n_2}, \ldots, \ldots, a_{n_{n-1}}+1, \ldots, a_{n_n}),$$

$$G=(G_1, G_2, \ldots, \mathbf{G}_{n_1}, G_{n_1}+1, \ldots, G_{n_2}, \ldots, \ldots, G_{n_{n-1}}+1, \ldots, \mathbf{G}_{n_n}).$$

Also, $q(b, H(t))\in R^m$ as follows:

$$q(b,(H(t))=\begin{bmatrix} q_1(b,H(t)) \\ q_2(b,H(t)) \\ \vdots \\ q_m(b,H(t)) \end{bmatrix},$$

where:

$$q_1(b,H(t))=\sum_{i=0}^{m_1} b_i H_i(t),\ q_2(b,H(t))=\sum_{i=m_1+1}^{m_2} b_i H(t),\cdots,\ q_m(b,H(t))=\sum_{i=m_{m-1}+1}^{m_m} b_i H(t),$$

and

$$b=(b_1,b_2,\ldots,b_{m_1},b_{m_1+1},\ldots,b_{m_2},\ldots,\ldots,b_{m_{m-1}+1},\ldots b_{m_m})$$

$$H=(H_1,H_2,\ldots,H_{m_1},H_{m_1+1},\ldots,H_{m_2},\ldots,\ldots,H_{m_{m-1}+1},\ldots,H_{m_m})$$

for suitable bases of functions H and G and their numbers. The constant a and b will be determined so that the desired (optimum) response and control are obtained.

For simplicity, the basic functions $\{G_i(t)\}$ and $\{H_i(t)\}$ are taken to be polynomials of the independent variable t.

Towards this ends, the non-classical variational formulation, is made as follows:

$$F[a,b]=\frac{1}{2}\int_0^T\Big\{\Big[\frac{d}{dt}[w(t)+p(a,G(t))]-A(t)[W(t)+p(a,G(t))]\Big]^T$$
$$\Big[\frac{d}{dt}[w(t)+p(a,G(t))]-A(t)[W(t)+p(a,G(t))]\Big]-$$
$$f(M(t)+q(b,H(t)))^T\Big[\frac{d}{dt}[w(t)+p(a,G(t))]-$$
$$A(t)[W(t)+p(a,G(t))]\Big]\Big\}dt$$
$$=\frac{1}{2}\int_0^T\Big\{\Big[\Big[\frac{dW(t)}{dt}+\frac{dp(a,G(t))}{dt}\Big]-[A(t)W(t)+A(t)p(a,G(t))]\Big]^T$$
$$\Big[\Big[\frac{dW(t)}{dt}+\frac{dp(a,G(t))}{dt}\Big]-[A(t)W(t)+A(t)p(a,G(t))]\Big]-$$
$$f(M(t)+q(b,H(t)))^T\Big[\Big[\frac{dw(t)}{dt}+\frac{dp(a,G(t)}{dt}\Big]-$$
$$[A(t)W(t)+A(t)p(a,G(t))]\Big]\Big\}dt \qquad (19)$$

as discussed in the previous paper , one can find the critical points of the above functional as follows:

$$\frac{\partial F[a,b]}{\partial a}=0,\ \text{for all vector}\ b,\ \text{or}$$
$$\frac{\partial F}{\partial a_i}=0,\ i=1,2,\ldots,n_2,n_1+1,\ldots,n_2,\ldots,n_n,\ \text{for all vectors of}\ b_k,$$
$$k=1,2,\ldots,m_1,m_1+1,\ldots,m_2,\ldots,m_m.$$

hence, the problem (1)- (6), becomes:

$$\min_b\int_0^T f_0(W(t)+p(a,G(t)),M(t)+q(b,H(t)),t)dt+$$
$$\Phi(W(t)+p(a,G(T))) \qquad (20)$$

subject to the solutions of:

$$\frac{\partial F[a,b]}{\partial a} = 0, \text{for arbitrary selection vector } b, \text{or}$$

$$\frac{\partial F}{\partial a_i} = 0, i = 1,2,\ldots,n_2, n_1 + 1,\ldots,n_2,\ldots,n_n,$$

$$\text{where } b = (b_1, b_2,\ldots,b_{m_1},\ldots,b_{m_m}),$$

from problem (20) , one have to solve the set of non-linear algebraic equations for each given vector b in the objective function to obtain the desired ordered pair (a, b) that make the solution optimum, i.e., satisfy the $\frac{\partial F[a,b]}{\partial a} = 0$ and optimize the cost function J(a, b) . This procedure cal also be covered with equality and inequality constraints that imposed the state $x(t) \in R^n$ and the control input $u(t) \in R^m$.

4.Illustration examples

Problem (4.1):

consider the following nonlinear optimal control problem with equality and inequality constraints [6]:

$$\min_{u(.)} 0.5\int_0^1 u^2(t)dt, \tag{21}$$

subject:

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{22}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 5/6 \end{bmatrix}, \tag{23}$$

and

$$|u(t)| \le 1, \text{ for } 0 \le t \le 1, \tag{24}$$

then, we have

$$L_1x(t)=f(t), \tag{25}$$

where L_1 is a linear operator:

$$L_1x(t) = \begin{bmatrix} \frac{dx_1}{dt} - x_2 \\ \frac{dx_2}{dt} \end{bmatrix}, \quad f(t) = \begin{bmatrix} 0 \\ u(t) \end{bmatrix},$$

since, the linear operator L_1 is not symmetric relative to the classical bilinear form , then a non-classical variational approach can be defined as:

Step(1): define arbitrary symmetric bilinear form (u, v) as:

$$(u, v) = \int_0^1 u(t)v(t)dt\,.$$

Step(2): define new one as follows:

$<x_1, x_2>=(x_1, L_1x_2)$, $x_1 \in U_2$ and $x_2 \in D(L_1)$.

Step(3): construct the functional as:

$F[x]=1/2<L_1x, x>-<f(t), x>,$ (26)

$=1/2(L_1x,L_1x)-(f(t), L_1x),$

from step (3), we have:

$$F[x] = \tfrac{1}{2}\int_0^1 \left[\begin{bmatrix} \frac{dx_1(t)}{dt} - x_2(t) \\ \frac{dx_2(t)}{dt} \end{bmatrix}^T \begin{bmatrix} \frac{dx_1(t)}{dt} - x_2(t) \\ \frac{dx_2(t)}{dt} \end{bmatrix} - \begin{bmatrix} 0 \\ u(t) \end{bmatrix} \begin{bmatrix} \frac{dx_1(t)}{dt} - x_2(t) \\ \frac{dx_2(t)}{dt} \end{bmatrix} \right] dt$$

$$= \tfrac{1}{2}\int_0^1 \left[\frac{dx_1(t)}{dt} - x_2(t) \right]^2 + \left[\frac{dx_2(t)}{dt} \right]^2 - \left[u(t)\left(\frac{dx_2(t)}{dt} \right) \right] dt\,. \quad (27)$$

Step(4): select as suitable number of Ritz basis as follows:

Let $n_1=3$,

$x_1(t)=x_{10}+t(x_{11}-x_{10})+a_1t+a_2t^2+a_3t^3$

$$= w(t) + \sum_{i=1}^{n_1} a_i G_i(t) \quad (28)$$

$x_1(0)=x_{10}$, where $x_{10}=0$,

$x_1(1)=0.333=x_{11}$,

then from equation (28), one can get the following algebraic relation:

$x_1(1)=x_{11}+a_1+a_2+a_3=x_{11}$,

then:

$a_1+a_2+a_3=0$

$a_1=-a_2-a_3.$ (29)

From equations (28) and (29), we have that :

$x_1(t)=0.333t+(t^2-t)a_2+(t^3-t)a_3.$

Rename the variable to get :

$x_1(t)=0.333t+(t^2-t)a_1+(t^3-t)a_2$ (30)

also,

$x_2(t)=0.5t+a_4t+a_5t^2$ (31)

$x_2(1)=0.5,$

then:

$0.5+a_4+a_5=0.5,$

and $a_4+a_5=0$, which implies that:

$a_4=-a_5$. (32)

From (31) and (32) , we have that:

$x_2(t)=0.5t+(t^2-t)a_5.$

Rename a_5 to a_3

$x_2(t)=0.5t+(t^2-t)a_3.$ (33)

Then

$x'_1(t)=0.333+(2t-1)a_1+(3t^2-1)a_2$ (34)

$x'_2(t)=0.5+(2t-1).$ (35)

Also, let

$$u(t)=b_0+b_1t+b_2t^2=\sum_{i=0}^{m_1} b_i H_i(t). \quad (36)$$

Then, back substitute these equations into the functional, we have:

$$F=\frac{1}{2}\int_0^1\Big[\big[(0.333-0.5t)+(2t-1)a_1+(3t^2-1)a_2+(t-t^2)a_3\big]^2+$$
$$\big[0.5+(2t-1)a_3\big]^2-\big[0.5+(2t-1)a_3\big]\big[b_0+b_1t+b_2t^2\big]\Big]dt.$$

Then:

$$F=\frac{1}{2}\int_0^1\big[h_0(t)+h_1(t)a_1+h_2(t)a_2+h_3(t)a_3\big]^2+\big[G_0(t)+G_1(t)a_3\big]^2-$$
$$\big[G_0(t)+G_1(t)a_3\big]\big[b_0+b_1t+b_2t^2\big]dt \quad (37)$$

Then, the comparison results between the numerical solution by non classical variational approach and the given analytical solution are shown in the following table (4.1). The good agreement between the numerical and exact solutions is obtained.

Table (4.1) show the numerical result which are compared with given analytical solution.

Time	Approx.x_1	Exact x_1	Approx.x_2	Exact x_2	Approx.u	Exact.u
0	0	0	0	0	0.999	1

0.1	0.04837	0.04833	0.09495	0.095	0.8991	0.9
0.2	0.01867	0.018667	0.1799	0.18	0.7992	0.8
0.3	0.0405	0.0405	0.271566	0.2555	0.6993	0.7
0.4	0.06933	0.06933	0.31988	0.32	0.5994	0.6
0.5	0.10416	0.10416	0.37488	0.375	0.4995	0.5
0.6	0.14399	0.144	0.41998	0.42	0.3996	0.4
0.7	0.18782	0.18783	0.45489	0.455	0.2997	0.3
0.8	0.23465	0.23467	0.47992	0.48	0.1998	0.2
0.9	0.283492	0.283505	0.49495	0.495	0.0999	0.1
1	0.333	0.3333	0.5	0.5	0	0

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