Goeken-Johnson Sixth-Order Runge-Kutta Method

Mohammed M. Ismail
Department of Computer Sciences
College of Computer and Mathematical Sciences
University of Mosul

الملخص

تم في هذا البحث اشتقاق طريقة رن ج–كوتا الجديدة من الرتبة السادسة اعتماداً على طريقة رنج–كوتا من الرتبة الخامسة لل باحثين David Goeken و Olin Johnson، ما يميز هذه الطريقة أنها تتطلب خمسة تقييمات لـ f فقط، في حين أن الطريقة الاعتيادية تتطلب ستة أو سبعة تقييمات لـ f، بعد ذلك نتم مقارنة هذه الطريقة بطريقة رنج–كوتا الجديدة من الرتبة الخامسة.

ABSTRACT

In this paper we drive a new sixth-order Runge-Kutta method, depending on the new fifth order Runge-Kutta method of David Goeken and Olin Johnson, the property of this method is that it needs five function evaluations only where the standard method needs six or seven function evaluations, then this method is compared with the new fifth order Runge-Kutta method.

1- Introduction:-

Given y' = f(y), standard Runge-Kutta methods perform multiple evaluations of f(y) in each integration sub-interval as required for a given accuracy. Evaluations of $y'' = f_y(x, y)f(x, y)$ or higher derivatives are not considered due to the assumption that the calculations involved in these functions exceed those of f. However, y" can be approximated to sufficient accuracy from past and current evaluations of f to achieve a higher order of accuracy than is available through current functional evaluations alone.

In July of 1998 at the ANODE (Auckland Numerical Ordinary Differential Equations) Workshop, the two scientists David Goeken and Olin Johnson introduced a new class of Runge-Kutta methods based on this observation [4],[11]. They presented a third order method which requires

only two evaluations of f and a fourth-order method which requires three and fifth order method which requires four. This paper reviews the fifth-order methods and gives the general solution to the equations generated by the sixth-order methods of this new class. Interestingly, these sixth-order methods require only five functional evaluations per step whereas standard Runge-Kutta methods require six or seven.

2- Third-order method:-

Goeken and Johnson consider initial value problems expressed in autonomous form. Starting with the non-autonomous form, they assume that f(x, y) is a continuous function with domain D in IR^{n+1} where $x \in R$, $y \in IR^n$ and $(x, y) \in D$ .They assume that:- [10], [12]

$$\left\| f(x, y_1) - f(x, y_2) \right\|_2 \le L \left\| y_1 - y_2 \right\|_2$$

for all $(x, y_1), (x, y_2) \in D$(where L is a Lipschitz constant); thus the problem

$$y' = f(x, y)$$
$$y(x_0) = y_0 \text{ with } (x_0, y_0) \in D$$

has a unique solution.

In autonomous form, y and f have n + 1 components with $y_{n+1} = f_{n+1}(x, y) = x$ and $f_{n+1}(y) = 1$. The initial value problem is then written:

$$y' = f(y)$$
$$y(x_0) = y_0 \text{ where } (y_0)_{n+1} = f_{n+1}(x_0, y_0) = x_0$$

Most efforts to increase the order of the Runge-Kutta methods have been accomplished by increasing the number of Taylor's series terms used and thus the number of functional evaluations required [10],[5],[12] and [9]. The use of higher order derivative terms has been proposed for stiff problems [16] and [7]. The method adds higher order derivative terms to the Runge-Kutta k_i terms (i > 1) to achieve a higher order of accuracy, For more details see [3] [6]. For example, the new third-order method, GJ3, for autonomous systems, is:- [10]

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2$$

and $k_1 = hf(y_n)$. However, they introduce additional terms by assigning:- [4], [10],[12].

$$k_2 = hf(y_n + a_{21} k_1 + ha_{22} f_y (y_n) k_1)$$

Using Taylor's series expansion techniques, the above is uniquely satisfied to $O(h^3)$ as follows:

$$k_1 = hf(y_n)$$

$$k_2 = hf(y_n + \frac{2}{3} k_1 + \frac{2}{9} hf_y (y_n) k_1)$$

$$y_{n+1} = y_n + \frac{1}{4} k_1 + \frac{3}{4} k_2$$

3- Fourth-order method:-

Similarly, the fourth-order method, GJ4, for autonomous systems, is:- [10],[12]

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2 + b_3 k_3$$

and

$$k_1 = hf(y_n)$$
$$k_2 = hf(y_n + a_{21}k_1 + ha_{22}f_y(y_n)k_1)$$
$$k_3 = hf(y_n + a_{31}k_1 + a_{32}k_2 + ha_{33}f_y(y_n)k_1 + ha_{34}f_y(y_n)k_2)$$

The Taylor's series expansion of these higher order methods is tedious and error prone. Goeken and Johnson used modern symbolic mathematical packages to expand and then to solve the resulting systems of nonlinear equations that were generated. In this work, they used the symbolic mathematical packages [14],[10], [11], and [7].

The general solution to the system of equations (with $a_{34} = 0$) has been found with example solutions are shown in:

$$y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{1}{6}k_2 + \frac{2}{3}k_3$$

and

$$k_1 = hf(y_n)$$

$$k_2 = hf(y_n + k_1 + \frac{1}{2}hf_y(y_n)k_1)$$

$$k_3 = hf(y_n + \frac{3}{8}k_1 + \frac{1}{8}k_2)$$

4- Fifth order method:-

In July of 1998, Goeken and Johnson presented [11] this numerical integration technique at a meeting attended by John Butcher. Using his tree-based approach [5], Butcher suggested a fifth-order method. Since the meeting, his technique has been verified using Taylor's series expansion techniques to determine the general solution for the fifth-order methods. The fifth-order method, GJ5, for autonomous systems, is:- [10],[12]

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4$$

and

$$k_1 = hf(y_n)$$
$$k_2 = hf(y_n + a_{21}k_1 + ha_{22}f_y(y_n)k_1)$$
$$k_3 = hf(y_n + a_{31}k_1 + a_{32}k_2 + ha_{33}f_y(y_n)k_1)$$
$$k_4 = hf(y_n + a_{41}k_1 + a_{42}k_2 + a_{43}k_3 + ha_{44}f_y(y_n)k_1)$$

The solution presented by Butcher and verified using the above system of equations is:

$$k_1 = hf(y_n)$$

$$k_2 = hf(y_n + \frac{1}{3}k_1 + \frac{1}{18}hf_y(y_n)k_1)$$

$$k_3 = hf(y_n - \frac{152}{125}k_1 + \frac{252}{125}k_2 - \frac{44}{125}hf_y(y_n)k_1)$$

$$k_4 = hf(y_n + \frac{19}{2}k_1 - \frac{72}{7}k_2 + \frac{25}{14}k_3 + \frac{5}{2}hf_y(y_n)k_1)$$

and

$$y_{n+1} = y_n + \frac{5}{48}k_1 + \frac{27}{56}k_2 + \frac{125}{336}k_3 + \frac{1}{24}k_4$$

5- Sixth order method:-

We drive now the Sixth order, GJ6, for autonomous systems, lets:

$$y_{n+1} = y_n + b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + b_5 k_5$$

and

$$k_1 = hf(y_n)$$
$$k_2 = hf(y_n + a_{21}k_1 + ha_{22}f_y(y_n)k_1)$$
$$k_3 = hf(y_n + a_{31}k_1 + a_{32}k_2 + ha_{33}f_y(y_n)k_1)$$
$$k_4 = hf(y_n + a_{41}k_1 + a_{42}k_2 + a_{43}k_3 + ha_{44}f_y(y_n)k_1)$$
$$k_5 = hf(y_n + a_{51}k_1 + a_{52}k_2 + a_{53}k_3 + a_{54}k_4 + ha_{55}f_y(y_n)k_1)$$

The sixth order Goeken-Johnson can be able to generate the Taylor's series expansion of the above, and we get the following systems of equations:- [10]

$$b_1 + b_2 + b_3 + b_4 + b_5 = 1$$

$$b_2 a_{21} + b_3(a_{31} + a_{32}) + b_4(a_{41} + a_{42} + a_{43}) + b_5(a_{51} + a_{52} + a_{53} + a_{54}) = \frac{1}{2}$$

$$b_2 a_{21}{}^2 + b_3(a_{31} + a_{32})^2 + b_4(a_{41} + a_{42} + a_{43})^2 + b_5(a_{51} + a_{52} + a_{53} + a_{54})^2 = \frac{1}{3}$$

$$b_2 a_{21}{}^3 + b_3(a_{31} + a_{32})^3 + b_4(a_{41} + a_{42} + a_{43})^3 + b_5(a_{51} + a_{52} + a_{53} + a_{54})^3 = \frac{1}{4}$$

$$b_2 a_{21}{}^4 + b_3(a_{31} + a_{32})^4 + b_4(a_{41} + a_{42} + a_{43})^4 + b_5(a_{51} + a_{52} + a_{53} + a_{54})^4 = \frac{1}{5}$$

$$b_2 a_{21}{}^5 + b_3(a_{31} + a_{32})^5 + b_4(a_{41} + a_{42} + a_{43})^5 + b_5(a_{51} + a_{52} + a_{53} + a_{54})^5 = \frac{1}{6}$$

$$b_3 a_{22}a_{32} + b_4(a_{21}a_{32}a_{43} + a_{22}a_{42} + a_{33}a_{43}) + b_5(a_{21}a_{32}a_{43}a_{54} + a_{22}a_{52} + a_{33}a_{53} + a_{44}a_{54}) = \frac{1}{24}$$

$$b_4 a_{21}a_{32}a_{43} + b_5 a_{21}a_{32}a_{43}a_{54} = \frac{1}{120}$$

$$b_5 a_{21}a_{32}a_{43}a_{54} = \frac{1}{720}$$

$$a_{21} + a_{22} = \frac{1}{2}$$

$$a_{31} + a_{32} + a_{33} = \frac{1}{2}$$

$$a_{41} + a_{42} + a_{43} + a_{44} = 1$$

$$a_{51} + a_{52} + a_{53} + a_{54} + a_{55} = 1$$

Now we can form the sixth order formula (with $a_{21} = \frac{1}{4}$) we get:- [10]

$$a_{22} = \frac{1}{4}, a_{31} = \frac{-1}{4}, a_{32} = \frac{1}{2}, a_{33} = \frac{1}{4}, a_{41} = \frac{1}{4}, a_{42} = \frac{1}{4}, a_{43} = \frac{1}{4}, a_{44} = \frac{1}{4},$$

$$a_{51} = \frac{-1}{4}, a_{52} = \frac{1}{2}, a_{53} = \frac{1}{4}, a_{54} = \frac{1}{4}, a_{55} = \frac{1}{4}$$

$$b_5 a_{21} a_{32} a_{43} a_{54} = \frac{1}{720} \Rightarrow b_5(\frac{1}{4})(\frac{1}{2})(\frac{1}{4})(\frac{1}{4}) = \frac{1}{720} \Rightarrow \frac{1}{128} b_5 = \frac{1}{720} \Rightarrow b_5 = \frac{8}{45}$$

$$b_4 a_{21} a_{32} a_{43} + b_5 a_{21} a_{32} a_{43} a_{54} = \frac{1}{120} \Rightarrow \frac{1}{32} b_4 + \frac{8}{45}(\frac{1}{4})(\frac{1}{2})(\frac{1}{4})(\frac{1}{4}) = \frac{1}{120}$$

$$\frac{1}{32} b_4 + \frac{1}{720} = \frac{1}{120} \Rightarrow b_4 + \frac{4}{90} = \frac{4}{15} \Rightarrow b_4 = \frac{4}{18}$$

$$b_3 a_{22} a_{32} + b_4(a_{21} a_{32} a_{43} + a_{22} a_{42} + a_{33} a_{43}) + b_5(a_{21} a_{32} a_{43} a_{54} + a_{22} a_{52} + a_{33} a_{53} + a_{44} a_{54}) = \frac{1}{24}$$

$$\frac{1}{8} b_3 + \frac{4}{18}(\frac{1}{32} + \frac{1}{16} + \frac{1}{16}) + \frac{8}{45}(\frac{1}{128} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16}) = \frac{1}{24}$$

$$\frac{1}{16} b_3 + \frac{5}{144} + \frac{33}{720} = \frac{1}{24} \Rightarrow b_3 + \frac{5}{9} + \frac{33}{45} = \frac{4}{6} \Rightarrow b_3 + \frac{58}{45} = \frac{2}{3} \Rightarrow b_3 = \frac{-28}{45}$$

$$b_2 a_{21} + b_3(a_{31} + a_{32}) + b_4(a_{41} + a_{42} + a_{43}) + b_5(a_{51} + a_{52} + a_{53} + a_{54}) = \frac{1}{2}$$

$$\frac{1}{4} b_2 - \frac{28}{45}(\frac{-1}{4} + \frac{1}{2}) + \frac{4}{18}(\frac{1}{4} + \frac{1}{4} + \frac{1}{4}) + \frac{8}{45}(\frac{-1}{4} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}) = \frac{1}{2}$$

$$\frac{1}{4} b_2 - \frac{7}{45} + \frac{1}{6} + \frac{2}{15} = \frac{1}{2} \Rightarrow \frac{1}{2} b_2 - \frac{14}{45} + \frac{2}{6} + \frac{4}{15} = 1 \Rightarrow \frac{1}{2} b_2 + \frac{13}{45} = 1 \Rightarrow b_2 = \frac{16}{45}$$

$$b_1 + b_2 + b_3 + b_4 + b_5 = 1 \Rightarrow b_1 + \frac{16}{45} - \frac{28}{45} + \frac{4}{18} + \frac{8}{45} = 1 \Rightarrow b_1 + \frac{12}{90} = 1 \Rightarrow b_1 = \frac{78}{90}$$

By using the above systems of equations we get the following systems of general solution:

$$y_{n+1} = y_n + \frac{78}{90} k_1 + \frac{16}{45} k_2 - \frac{28}{45} k_3 + \frac{4}{18} k_4 + \frac{8}{45} k_5$$

and

$$k_1 = hf(y_n)$$

$$k_2 = hf(y_n + \frac{1}{4} k_1 + \frac{1}{4} hf_y(y_n)k_1)$$

$$k_3 = hf(y_n - \frac{1}{4} k_1 + \frac{1}{2} k_2 + \frac{1}{4} hf_y(y_n)k_1)$$

$$k_4 = hf(y_n + \frac{1}{4} k_1 + \frac{1}{4} k_2 + \frac{1}{4} k_3 + \frac{1}{4} hf_y(y_n)k_1)$$

$$k_5 = hf(y_n - \frac{1}{4} k_1 + \frac{1}{2} k_2 + \frac{1}{4} k_3 + \frac{1}{4} k_4 + \frac{1}{4} hf_y(y_n)k_1)$$

As we know that the standard sixth order method has 6 or 7 k_i's function evaluations but in this method the k_i's function evaluations are 5 only.

6- Stability Region of Sixth order method:-

The stability properties of numerical methods of Ordinary Differential equations determined by study the behavior of these methods, by using test problem:- [1],[2],[13]

$$y' = \lambda y, \; y(x_0) = y_0$$

Where λ is real, the real solution of the above problem when $x = x_n$ is:

$$y(x_{n+1}) = e^{\lambda h} y(x_n)$$

This implies the use of the general formula of Runge-Kutta method:- [1],[2],[13]

$$y_{n+1} = y_n + h\phi(y_n, h) \quad ....(1)$$

Where:

$$\phi(y_n, h) = \sum_{r=1}^{R} b_r k_r$$

$$k_r = f(y_n + h\sum_{s=1}^{R} a_{rs} k_s) \;, r = 1,2,.......,R$$

$$b_r = \sum_{s=1}^{R} a_{rs} \;, r = 1,2,............R$$

By using test problem we get the difference equation:- [1],[2],[13]

$$y_{n+1} = E(\lambda h)y_n \;, n = 0,1,2,.........$$

Where $E(\lambda h)$ is the stability function and converges to $e^{\lambda h}$ and $E(\lambda h) < 1$, by using equation (1) in the difference equation we get:- [1],[2],[13]

$$y(x_{n+1}) - y_{n+1} = O(h^{p+1})$$

Now by using test problem we get:- [1],[2]

$$E(\bar{h}) = r = 1 + \bar{h} + \frac{\bar{h}^2}{2!} + \frac{\bar{h}^3}{3!} + ... + \frac{\bar{h}^p}{p!} + O(\bar{h}^{p+1})$$

Where $\bar{h} = \lambda h$, by using Taylor series we get:- [1],[2],[13]
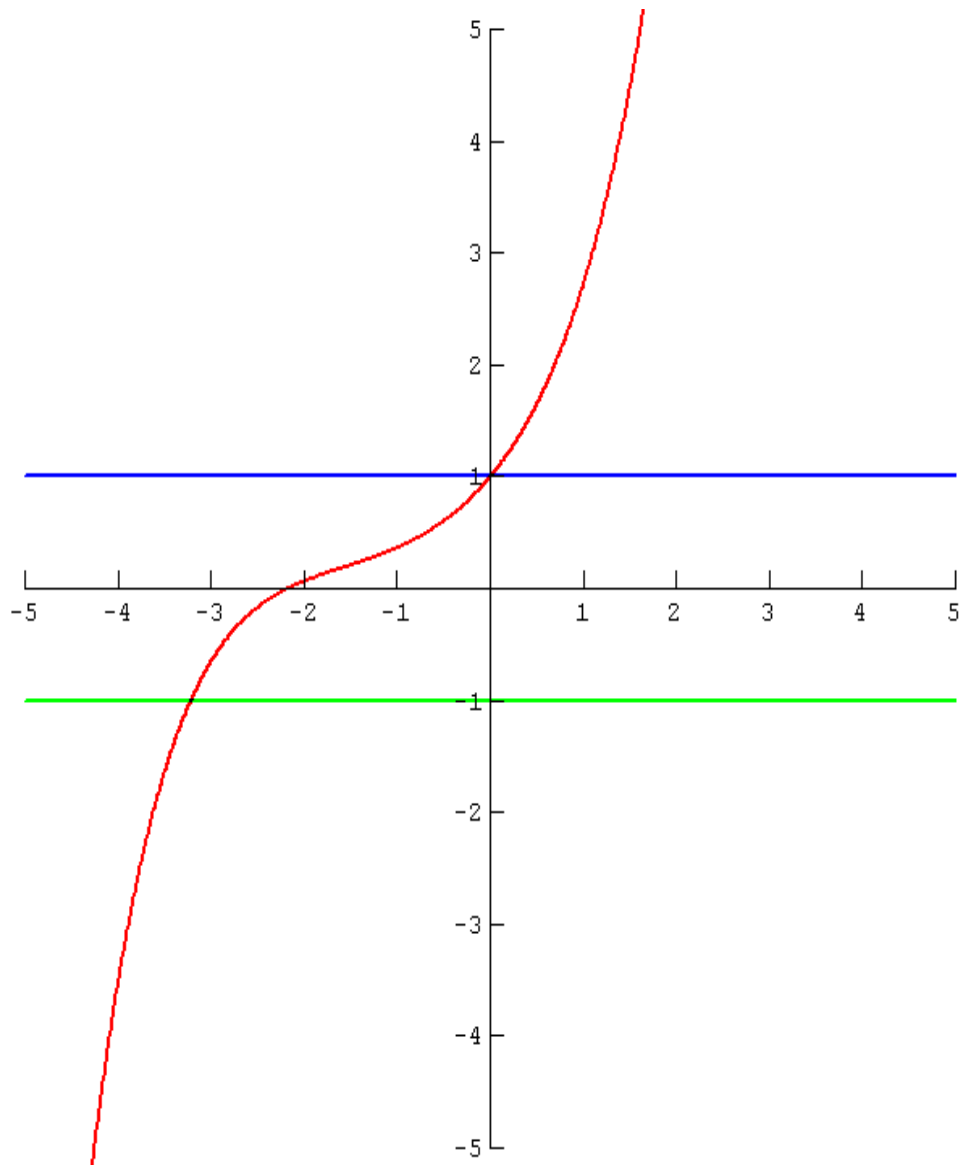
$$r = \exp(\bar{h}) + O(\bar{h}^{p+1})$$

Where r is polynomial of R stage in $\bar{h}$, if the method of stage R then it is of rank p, therefore, $R \geq p$. Then we get:- [1],[2],[13]

$$r = 1 + \bar{h} + \frac{\bar{h}^2}{2!} + \frac{\bar{h}^3}{3!} + ... + \frac{\bar{h}^p}{p!}$$

Then from Goeken-Johnson sixth order Runge-Kutta method r is:

$$r = 1 + \bar{h} + \frac{\bar{h}^2}{2!} + \frac{\bar{h}^3}{3!} + \frac{\bar{h}^4}{4!} + \frac{\bar{h}^5}{5!}$$

and the stability region of Goeken-Johnson sixth order Runge-Kutta method is (-3.2,0), we got the region by plotting r in Figure(1).

Figure(1): Stability region of Goeken-Johnson sixth order Runge-Kutta method

7- Comparison:-

In this section we compare with the fifth-order method, GJ5, and the sixth-order method, GJ6, and we get the solutions of the two methods have the same results, and the GJ6 is useful for solving any problems of Ordinary Differential equations, In this comparison we use the example $y' = -y, y(0) = 1$, See table(1) and Figure(2) (The Figure is plotted in Matlab).

x	Theoretical solution	Numerical solution of GJ5	Numerical solution of GJ6	Errors (GJ5-GJ6)
0	1.0	1.0	1.0	0
0.002	0.998001998667333	0.998001998600667	0.998000799822222	1.198778444400084e-006
0.004	0.996007989343991	0.996007989210925	0.996005596445795	2.392765129810570e-006
0.006	0.994017964053935	0.994017963854734	0.994014381880313	3.581974421296152e-006
0.008	0.992031914837061	0.992031914571990	0.992027148151344	4.766420645840874e-006
0.01	0.990049833749168	0.990049833418492	0.990043887300400	5.946118092237107e-006
0.012	0.988071712861931	0.988071712465912	0.988064591384901	7.121081011085551e-006
0.014	0.986097544262862	0.986097543801764	0.986089252478148	8.291323615128299e-006
0.016	0.984127320055285	0.984127319529368	0.984117862669289	9.456860079026797e-006
0.018	0.982161032358301	0.982161031767827	0.982150414063287	1.061770453991695e-005
0.02	0.980198673306755	0.980198672651984	0.980186898780887	1.177387109696504e-005
0.022	0.978240235051210	0.978240234332400	0.978227308958589	1.292537381181180e-005
0.024	0.976285709757909	0.976285708975320	0.976271636748612	1.407222670857244e-005
0.026	0.974335089608749	0.974335088762638	0.974319874318864	1.521444377405867e-005
0.028	0.972388366801247	0.972388365891871	0.972372013852914	1.635203895744564e-005
0.03	0.970445533548508	0.970445532576124	0.970428047549953	1.748502617093806e-005
0.032	0.968506582079198	0.968506581044060	0.968487967624770	1.861341928943716e-005
0.034	0.966571504637507	0.966571503539870	0.966551766307719	1.973723215098477e-005
0.036	0.964640293483123	0.964640292323242	0.964619435844685	2.085647855643025e-005
0.038	0.962712940891200	0.962712939669327	0.962690968497057	2.197117226987455e-005
0.04	0.960789439152323	0.960789437868711	0.960766356541692	2.308132701867027e-005
0.042	0.958869780572485	0.958869779227385	0.958845592270891	2.418695649331060e-005
0.044	0.956953957473047	0.956953956066710	0.956928667992362	2.528807434798441e-005
0.046	0.955041962190715	0.955041960723391	0.955015576029191	2.638469420013223e-005
0.048	0.953133787077505	0.953133785549444	0.953106308719813	2.747682963089027e-005
0.05	0.951229424500714	0.951229422912164	0.951200858417979	2.856449418497942e-005

Table(1): Comparison between GJ5 and GJ6
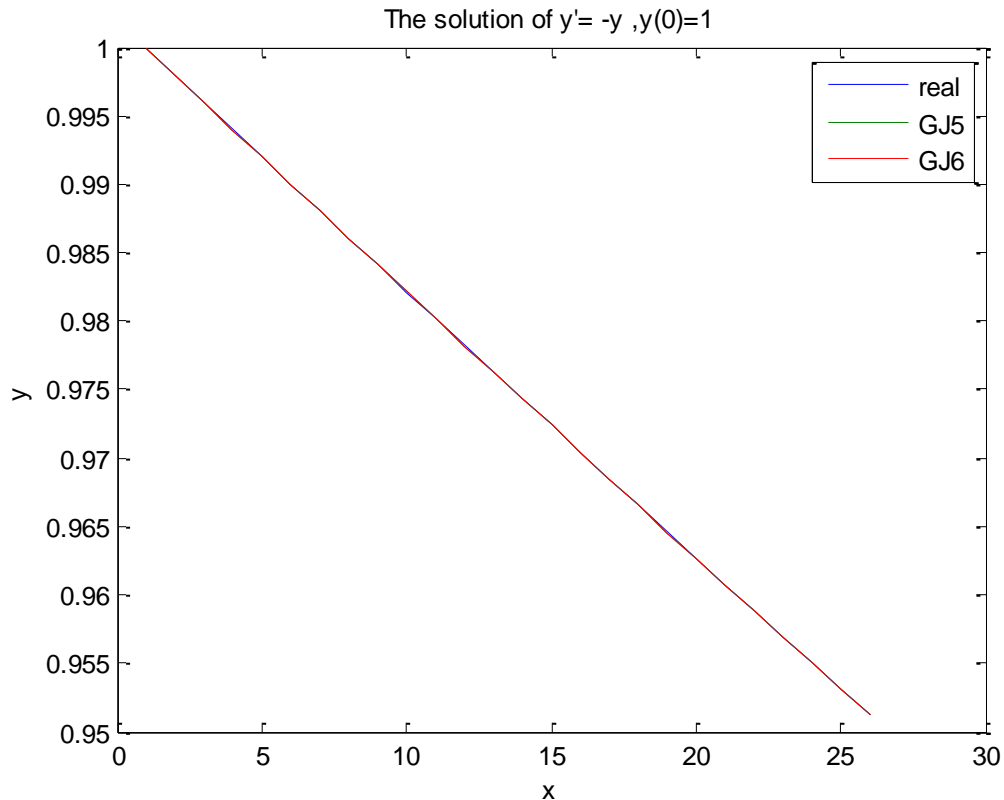
Figure(2): the Solution of $\mathbf{y'} = -\mathbf{y}, \mathbf{y(0)} = \mathbf{1}$

8- Conclusions:-

New third-, fourth-, fifth-, and sixth-order numerical integration techniques inspired by the Runge-Kutta method have been presented. The new methods exploit the use of higher order derivatives, specifically f_y. In particular, a technique utilizing an approximation to y'' has been presented resulting in a multistep Runge-Kutta method. The proposed methods are more efficient than the standard Runge-Kutta methods for cases:- [4],[10],[12]

1. f_y or y'' is cheaper to evaluate than f .
2.   The use of historical values of f is cheaper than evaluating f ,because the number of functional evaluations of standard Runge-Kutta methods are from 4 to 6 or 7 while the number of functional evaluations of new Runge-Kutta methods are from 2 to 5 only .

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