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Radical Young's Diagrams Core

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الخلاصة:

هذا البحث هو أساساً في موضوع نظرية التمثيل Representation theory وتحديداً في جبر شور من النوع q (q-Schur algebra) وجبر هيكا (Hecke algebra) حيث تلعب أعداد β لأي تجزئة μ من العدد الصحيح الموجب r دوراً مهماً في كل من هذان النوعان من الجبور. في هذا العمل وسعنا نتائج Fayers، الذي قام بإضافة عمود (runner) جديد لإعداد β، وقمنا بإضافة عدة أعمدة وأطلقنا عليها بالشجرة هذا من جانب، من جانب أخر سنقدم اختزال الأعمدة وصولاً إلى حالة أطلقنا عليها اسم الجذر (radical) مقدمين الحل رياضياً وبرمجياً.

Abstract:

This is research is basically deals with the representation theory that is specifically on Hecke algebra and q-Schur algebra, where β-numbers of a partition μ has sufficient effect in both types of algebra.

The objective of this work is to expand the results of Fayers that is by adding new runners to β-numbers, to represent a "tree", and by other hand, we decide to reduce the runners reaching another new definition "radical" by using both mathematically and computer programming ways.

Keywords: Hecke algebra, q-Schur algebra, β-numbers and e- Core

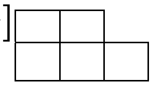
1. Introduction:

Let F be a field, q an invertible element of F, r a non-negative integer and G_r a symmetric group. We define e > 1 to be minimal such that $1+q+.....+q^{e-1} = 0$, with $e = \infty$ if no such integer exists, then we shall assume that e is finite.

A composition μ of r is a sequence $(\mu_1,\mu_2,....,\mu_n)$ of non-negative integers such that $|\mu| = \sum_{i=1}^{n} \mu_i = r$. A composition μ is a partition if

$\mu_i \geq \mu_{i+1}$ for all i≥1. The diagram of Young of a composition μ is the subset :

$$[\mu] = \{(x,y) \mid 1 \leq y \leq \mu_x \text{ and } x \geq 1\} \text{ of } N \times N,$$

it is useful to represent the diagram of μ as an array of boxes in the plane, for example, if μ=(2,3) then $[\mu]$  .

We denote the μ-composition of r as $\mu \models$ r and denote the μ-partition of r as $\mu \vdash$ r. The best introduction to the representation theory of Iwahori- Hecke algebra and q-Schur algebra can be found in Mathas's book [9], as the following: Let $H_r = H_{F,q}(r)$ be the Iwahori-Hecke algebra of G_r and let $S_q(n,r)$ be the corresponding q-Schur algebra. H_r is the associative F-algebra with basis $\{T_w \mid w \in G_r\}$ and multiplication determined by:

$$T_{S_i} T_W = \begin{cases} T_{S_i W} & \text{if} \quad i^w < (i+1)^w \\ qT_{S_i W} + (q-1)T_W & \text{otherwise} \end{cases},$$

where $w \in G_r$ and s_i= (i,i+1), for i= 1,2,3,…..,r-1. The q-Schur algebra is the endomorphism algebra

$$S_q(n,r) = \mathrm{End}_{H_r}\left(\bigoplus_{\mu \vdash r} x_\mu H_r\right),$$

where $x_\mu = \sum_{w \in G_r} T_w$ and a Young subgroup

$$G_\mu = G_{\mu_1} \times G_{\mu_2} \times \dots\dots \times G_{\mu_n} \quad \text{of } G_r \;.$$

Dipper and James in [1] defined the Specht module S^μ; for each partition μ of r there is a right H_r-module S^μ. A partition μ is *e*-regular if it does not have *e* non-zero equal parts. If μ is *e*- regular then S^μ has an irreducible cosocle D^μ. Also, a Weyl module W^μ is defined as, for any partition μ of r, there is a right S_q(n,r)-module W^μ. The cosocle L^μ of W^μ is irreducible.

Given partitions μ and λ of r, with *e*-regular, let $[S^\mu : D^\lambda]$ be the multiplicity of D^λ as a composition factor of S^μ. Similarly, let $[W^\mu : L^\lambda]$ be the multiplicity of L^λ as a composition factor of W^μ. With μ is *e*-regular, $([S^\mu : D^\lambda])_{\mu,\lambda \vdash r}$ is the decomposition matrix of H_r and $([W^\mu : L^\lambda])_{\mu,\lambda \vdash r}$ is the decomposition matrix of S_q(n,r), see [8].

2. β-numbers and e-Core :

Choose an integer b greater than the number of parts of a partition μ, and define

$$\beta_j = \mu_j + b - j, \quad \text{for} \quad j=1,2,\dots,b.$$

The set $\{\beta_1,\dots,\beta_b\}$ is said to be a set of beta-number for μ. For example, if $\mu=(5,3^2,2,1)$, then the number of parts of μ is 5. Let b=7, then β-numbers are (11,8,7,5,3,1,0).

We consider an abacus with e vertical runners, labeled 0,1,….,e-1 from left to right. And label the partition on runner j as j, j+e, j+2e, …. from the top downwards. We call the bead position me, me+1,…, me+e-1 row m of the abacus configuration for μ with b beads is the abacus configuration obtained by placing a bead at position β_j for j=1,2,….,b.

0	1	2	…	e-1
e	e+1	e+2	…	2e-1
2e	2e+1	2e+2	…	3e-1 .
.	.	.		.
.	.	.		.
.	.	.		.

From the above example,
if e =2,

0	1		•	•	
2	3		-	•	
4	5	→	-	•	,
6	7		-	•	
8	9		•	-	
10	11		-	•	

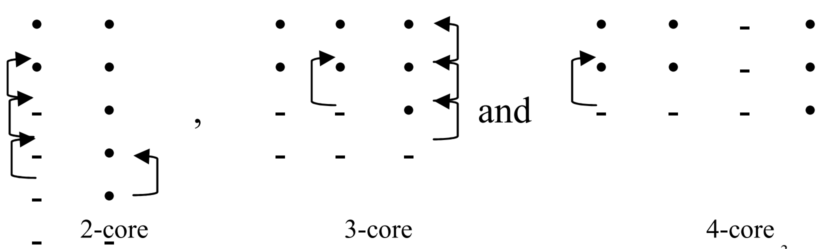
e =3,

0	1	2		•	•	-	
3	4	5		•	-	•	
6	7	8	→	-	•	•	,
9	10	11		-	-	•	

and if e =4, then

0	1	2	3		•	•	-	•	
4	5	6	7	→	-	•	-	•	.
8	9	10	11		•	-	-	•	

Given an abacus configuration for μ we can create a new abacus configuration by moving all beads as high as possible on each runner. The partition; denoted by ρ, corresponding to this new abacus configuration is called the e-core of μ ,



2-core ρ =(3,2,1)   3-core ρ =(2)   4-core ρ =(5,2,1^3)

<u>Rule (2.1):</u> We can find an easy rule for finding any partition of any e-core and as follows:

"first we count the spaces in all runners before the last bead, which is equal to ρ_1. Then we subtract the spaces from ρ_1 for the last bead with the one before the last, the result would be denoted as ρ_2. This procedure will be repeated on ρ_2, that is to subtract the spaces from ρ_2 for the bead before the last with the one before it, and denoted by ρ_3, and so on ….".

<u>Theorem (2.2)</u> : [7]

Each partition has a uniquely e-core.

If ρ is the e-core of μ then e-weight of μ is: $W = \frac{|\mu| - |\rho|}{e}$, for the above example,

$\mu = (5,3^2,2,1) \quad \Rightarrow \quad |\mu| = 14, \quad |\rho|_{e=2} = 6, \quad |\rho|_{e=3} = 2$ and $|\rho|_{e=4} = 10.$

Then $W_{e=2} = \frac{14-6}{2} = 4, \quad W_{e=3} = \frac{14-2}{3} = 4$ and $W_{e=4} = \frac{14-10}{4} = 1.$

For more application with e-weight, see [2], [5] and [6].

The definition of e-weight is equivalent exactly to e-quotient; see [7]:

"We write μ_b^a for the number of unoccupied positions above the bth lowest bead on runner a", then $\mu(a) = (\mu_1^a, \mu_2^a, ...)$ is a partition, and we refer to the sequence $(\mu(0),......, \mu(e-1))$ as the e-quotient of μ.

Then we have ((3), (1)), ((0), (1), (1^3)) and ((1), (0), (0), (0)) if we use e =2, e =3 and e =4 respectively.

According to the beads, we can find many different cases having the same weight for μ but also having the same core. These cases can be shown from the last example and achieve the aim:

- • • - • • • - • • - - $(3^2,2^3,1^2)$,	- • • • - • - • • • - - $(3^3,2,1^3)$,	- • • • • • - - - • - • $(5,4,1^5)$,	• • • - - • - - • • • - $(4^3,2)$,
• - • - • • - • • • - - $(3^3,2^2,1)$,	• • • - - • - • - • - • $(5,4,3,2)$,	• • • - • - - - • • - • $(5,4^2,1)$,	• • • - - • • - - - • • $(5^2,2^2)$,
• - • - • • • • - - - • $(5,2^4,1)$,	• - • - • • • - • - • - $(4,3,2^3,1)$,	• - • • - • - • • - • - $(4,3^2,2,1^2)$,	• - • • • • - - - - • • $(5^2,1^4)$,

• • • • - - - - • - • • $(5^2,4)$	and	• • - • - • - • • - - • $(5,3^2,2,1)$.

Theorem (2.3) "Nakayama conjecture": [9]

The two modules of Weyl W^{μ} and W^{λ} belong to the same block if and only if μ and λ have the same weight and the same core, similarly, for two modules of Specht S^{μ} and S^{λ}.

Rule (2.4): The maximum weight (max_w) can be calculated and found for any core which is equal to the sum of all products the number of beads by the number of spaces of the same runner.

As a result, the max_w for the case $\mu=(5,3^2,2,1)$ and e =3 is (2×2)+(2×2)+(3×1)=11, where max_w =4 when e =4.

3. Trees and Radicals :

According to Fayers in [4], who was able to make an easy way to insert one runner to the β-numbers by putting number of beads under consideration that the last bead location in this runner does not exceed the location of β_1 but left with a space, otherwise, if this bead exceeds β_1 without making a space, this case will be calculated by using Fayers research of this insertion as follows:

"Given a partition μ and a non-negative K, we constrict a new partition μ^{+K} as follows. Take $b \geq \mu_1'$ and constrict the abacus display for μ with b beads. Write b+K = ce + d with 0 ≤ d ≤ e-1, and add a runner to the abacus display immediately to the left of runner d; now put c beads on this new runner in the top c position. The partition whose abacus display is obtained, is μ^{+K}".

By the pervious example, when $\mu=(5,3^2,2,1)$, if we picked e =2 then

0	1
•	•
-	•
-	•
-	•
•	-
-	•

.

Therefore, we'll have the following cases for μ^{+K}:

$$\mu^{+0} = \begin{array}{ccc} 0 & & 1 \\ \bullet & \bullet & \bullet \\ - & \bullet & \bullet \\ - & \bullet & \bullet \\ - & - & \bullet \\ \bullet & - & - \\ - & - & \bullet \end{array} = (8,4^2,2^2,1^2) \quad , \qquad \mu^{+1} = \begin{array}{ccc} & 0 & 1 \\ \bullet & \bullet & \bullet \\ \bullet & - & \bullet \\ \bullet & - & \bullet \\ \bullet & - & \bullet \\ - & \bullet & - \\ - & - & \bullet \end{array} = (7,4,3,2^2,1^2) \quad ,$$

$$\mu^{+2} = \begin{array}{ccc} 0 & & 1 \\ \bullet & \bullet & \bullet \\ - & \bullet & \bullet \\ - & \bullet & \bullet \\ - & \bullet & \bullet \\ \bullet & - & - \\ - & - & \bullet \end{array} = (7,3^3,2^2,1^2) \quad , \qquad \mu^{+3} = \begin{array}{ccc} & 0 & 1 \\ \bullet & \bullet & \bullet \\ \bullet & - & \bullet \\ \bullet & - & \bullet \\ \bullet & - & \bullet \\ \bullet & \bullet & - \\ - & - & \bullet \end{array} = (6,3^3,2^2,1^2) \quad ,$$

$$\mu^{+4} = \begin{array}{ccc} 0 & & 1 \\ \bullet & \bullet & \bullet \\ - & \bullet & \bullet \\ - & \bullet & \bullet \\ - & \bullet & \bullet \\ \bullet & \bullet & - \\ - & - & \bullet \end{array} = (6,3^4,2^2,1^2) \quad , \qquad \mu^{+5} = \begin{array}{ccc} & 0 & 1 \\ \bullet & \bullet & \bullet \\ \bullet & - & \bullet \\ \bullet & - & \bullet \\ \bullet & - & \bullet \\ \bullet & \bullet & - \\ \bullet & - & \bullet \end{array} = (5,4,3^3,2^2,1^2) \quad ,$$

$$\mu^{+6} = \begin{array}{ccc} 0 & & 1 \\ \bullet & \bullet & \bullet \\ - & \bullet & \bullet \\ - & \bullet & \bullet \\ - & \bullet & \bullet \\ \bullet & \bullet & - \\ - & \bullet & \bullet \end{array} = (5^2,3^4,2^2,1^2) \quad , \qquad \mu^{+7} = \begin{array}{ccc} & 0 & 1 \\ \bullet & \bullet & \bullet \\ \bullet & - & \bullet \\ \bullet & - & \bullet \\ \bullet & - & \bullet \\ \bullet & \bullet & - \\ \bullet & - & \bullet \\ \bullet & - & - \end{array} = (5^2,4,3^2,2^2,1^2) \quad .$$

In this study we are able to insert many runners to an β-numbers and get $\mu^{+K,+K',+K'',\ldots\ldots}$ which will from a " Tree " or " group of trees ".

Rule (3.1):

1) For choosing +K, we follow the previous steps offered by Fayers in [4].
2) For choosing +K', we'll depend on the value of

$$b' = b + c$$
$$= b + \frac{b + K - d}{e}$$

under condition that d is fit to the value of b+K-d is divisible by e. Obviously, e' = e+1, and the solution is continued using the same technique of Fayers.

3) Repeat the same way in c for the rest of steps by :

$$\overbrace{b}^{\text{"... m-times ..."}} = \overbrace{b}^{\text{"...(m-1)-times. .."}} + \overbrace{c}^{\text{"...(m-1)-times ..."}}$$

$$\text{and} \quad \overbrace{e}^{\text{"... m-times ..."}} = \overbrace{e}^{\text{"...(m-1)-times. .."}} + 1 .$$

We denote these cases by trees.

For example, the following tree: $\mu^{+1,+1,+2}$ is the extension to μ^{+1} where $\mu = (5,3^2,2,1)$:

•	•	•	•	•	
•	•	•	-	•	
•	•	•	-	•	$= (11,6,3,2^4,1^4)$.
•	•	•	-	•	
-	-	-	•	-	
-	-	-	-	•	

The following program is for finding this tree :

```
function
matrix=check_acception_matrix(may_be_matrix,accepted_x_position,acc
epted_y_position)
matrix=1;
mm=max(find(may_be_matrix(end,:)==1));
if mm>accepted_y_position
   matrix=0;
end

function insertion1=ins_column(a,v,k);
% /a/ is the matrix that we  must insert the first elements of  vector/v/ in it
before the
% position /k/ here the value of /k/may be {0,1,2,3,...,size(a,2)-1}
[m n]=size(a);
mm=length(v);
if mm~=m
   ' The dimension of the vector  is not correct to put it in the matrix'
%#ok<NOPRT>
return;
```

```
end
if (k>=n)||(k<0)||(round(k)~=k)
   ' There is an error in the value of /k/ it is very large or minus'
%#ok<NOPRT>
elseif k==0
   b=[v a];
elseif k==1
   b=[a(:,1) v a(:,2:end)];
else
   b=[a(:,1:k) v a(:,k+1:end)];
end
insertion1=b;

clear
clc
 %First Point  Mu is the input series
 %Type the value of /Mu/ here
Mu=[5 3 3 2 1];
mu=Mu;
if any(mu~=fix(mu)) | any(mu<=0)
    disp('Please check Mu ( the original series)')
    disp('Mu must be a composition of positive integers')
   break
end

 %Second Point  check for Partition
sort_mu=fliplr(sort(mu));
if any(mu~=sort_mu)
    disp('Please check Mu ( the original series)')
    disp('Mu must be of descending order')
    break
end


 %Third Point /b/
 %Type the value of /b/ here
b=7;
if b<length(mu)
    disp('Please check /b/ ')
    disp('/b/  must be >= the number elements of /Mu / ')
    break
end
```

```
 %Forth Point Beta_Numbers /b_num/
residue1=zeros(1,b-length(mu));
 b_num=[mu residue1]+b-[1:b];
%++++++++++++++++++++++++Begin                                    to
Delete++++++++++++++++++++++++++++
 %for example please, to delete the following b_num
%b_num=[0 2 5];
%b_num=[11    8    7    5    3    1]
%++++++++++++++++++++++++End                                      to
Delete++++++++++++++++++++++++++++

%Fifth Point  /e/
 %Type the value of /e/  here it must be >1
e=2;
if (e<=1)|(e~=fix(e))
   disp('/e/  must be a positive integer number  >=2 ')
    break
end

 %Step 5  to make the table called here /E_Pure /
 %depending on the /b_num/ and the /e/.
 %to put the numbers in the series /b_num  into the matrix /E_Pure/
%   [ 0 1 2 … (e-1);
 %   e  e+1 e+2 ... 2e-1;
%        …              ]
bmax=max(b_num);
 %D is the length of the new matrix
d=fix((bmax)/e)+1;
% /Big_Value/ used to demonstrate the free positions from /m1/
Big_Value=1000000;
C1=ones(d,e)*Big_Value;
 % %function
 % %b_num is the series (1*n)
 % %C1 is the corresponding Matrix for it
```

```
E_Pure=C1(:);
b_num=b_num+1;
for i=1:length(E_Pure)
    if any(b_num==i)
        m11=find(b_num==i);
        E_Pure(i)=b_num(m11(1));
    end
end
E_Pure=(reshape(E_Pure,[size(C1')])-1),

 %Step 6   to evaluate /E-Core  /
 %Now we must make the balls go up to fill each empty space
m2=E_Pure;
for i=1:size(m2,2)
    ci=m2(:,i);
    c0=find(ci~=(Big_Value-1));
    c1=length(c0);
    if c1==0
        continue
    else
    for j=1:length(c0)
        m2(j,i)=ci(c0(j));
    end
    m2(c1+1:end,i)=Big_Value-1;
    end;
end
E_Core=m2
%disp('E_Core is : ')
%disp(E_Core)

%Step 7  Expansion  Process
%disp('                                                        ')
%disp('                                                        ')
%disp(' Here we will make EXPANSION to our Matrix /E_Pure/ ')
%function [Expansions]=expansion(E_Pure,Big_Value,e)

 %First we must check for the right positions technique
```

```
 %that is  /accepted_x_position/ and /accepted_y_position/
pqpq=1;
ddd=size(E_Pure);
 d1=ddd(1);d2=ddd(2);
great_x_value=1;
great_y_value=1;
for i=1:d1
    for j=1:d2
        if E_Pure(i,j)~=(Big_Value-1)
          great_x_value=i;
          great_y_value=j;
        end
    end
end
if great_y_value==d2
   accepted_x_position=great_x_value+1;
   accepted_y_position=1;
else
   accepted_x_position=great_x_value;
   accepted_y_position=great_y_value+1;
end
our_balls=length(find(E_Pure~=(Big_Value-1)));
b=our_balls;
 %Now we must evaluate the equation : b+k=c*e+d
 %here we call the value of all possible solutions by /may_be_roots/
may_be_roots =[ ];
for k=0:accepted_x_position*e+e-1-b
    for d=0:e-1
         for c=0:accepted_x_position
 if (b+k)==(c*e+d)
     may_be_roots=[may_be_roots;[b k c e d]];
 end
         end
    end
end
disp('++++++++++++++++++++++++++++++++++')
disp(' We can insert /c/ points at the left of the column /d / ')
disp('      All Solutions for our E_Pure are : ')
disp('    b  +  k  =  c  *  e  +  d ')
```

```
disp('- - -    - - -    - - -    - - -    - - -     ')
disp(may_be_roots)
disp('++++++++++++++++++++++++++++++++++')
 %insert_column as the rows of /may_be_roots/ for the matrix /E_Pure/
 %function z=Check_Insertion(E_Pure,Big_Value,may_be_roots)
ddd=size(E_Pure);
 d1=ddd(1);d2=ddd(2 );
great_x_value=1;
great_y_value=1;
for i=1:d1
    for j=1:d2
        if E_Pure(i,j)~=(Big_Value-1)
          great_x_value=i;
          great_y_value=j;
        end
    end
end
if great_y_value==d2
   accepted_x_position=great_x_value+1;
   accepted_y_position=1;
else
   accepted_x_position=great_x_value;
   accepted_y_position=great_y_value+1;
end
Origion_E_Pure=E_Pure;
for i=1:d1
    for j=1:d2
        if E_Pure(i,j)~=(Big_Value-1)
 %            here /1/ means there is an origion ball in this position
            Origion_E_Pure(i,j)=1;
        else
 %            here /0/ means that there is nothing ball in this position
          Origion_E_Pure(i,j)=0;
        end
    end
end
if great_y_value==d2
    w=zeros(size(Origion_E_Pure,2));
    w=w(1, : );
 Origion_E_Pure=[Origion_E_Pure;w];
```

```
end
Origion_E_Pure %#ok<NOPTS>
solutions(pqpq).E_Pure=Origion_E_Pure;
solutions(pqpq).equation=zeros(size(may_be_roots(1 , :);

 %Now we will simulate the case of all solutions
 %at first we must evaluate the vector/c4/ that will insert it in the
 %matrix Origion_E_Pure to make all solutions may be possible.

    v4=zeros(size(Origion_E_Pure,1));
    v4=v4(:,1;(
tt=size(may_be_roots,1);
 for pkpk=1:tt
c4=v4;
for popo=1:may_be_roots(pkpk,3)
    c4(popo)=1;
end
may_be_matrix=ins_column(Origion_E_Pure,c4,may_be_roots(pkpk,5));
s(pkpk).m=may_be_matrix;
new_matrix=check_acception_matrix(may_be_matrix,accepted_x_positi
on,accepted_y_position);
if new_matrix==0
    continue
else
    pqpq=pqpq+1;
    solutions(pqpq).E_Pure=may_be_matrix;
    solutions(pqpq).equation=may_be_roots(pkpk,:)
end
end
clc
solutions(1:end).E_Pure
```

This research attempt to find "radical (s)" from a given tree. That is to go back to this tree's base.

Rule (3.2):

To find the radical (s), we'll follow:

1) Count the beads from the given tree which we attempt to find its radicals, and let it denote by b.
2) Sorting the runners in the form below:

   runner 1 by -1
   runner 2 by 0

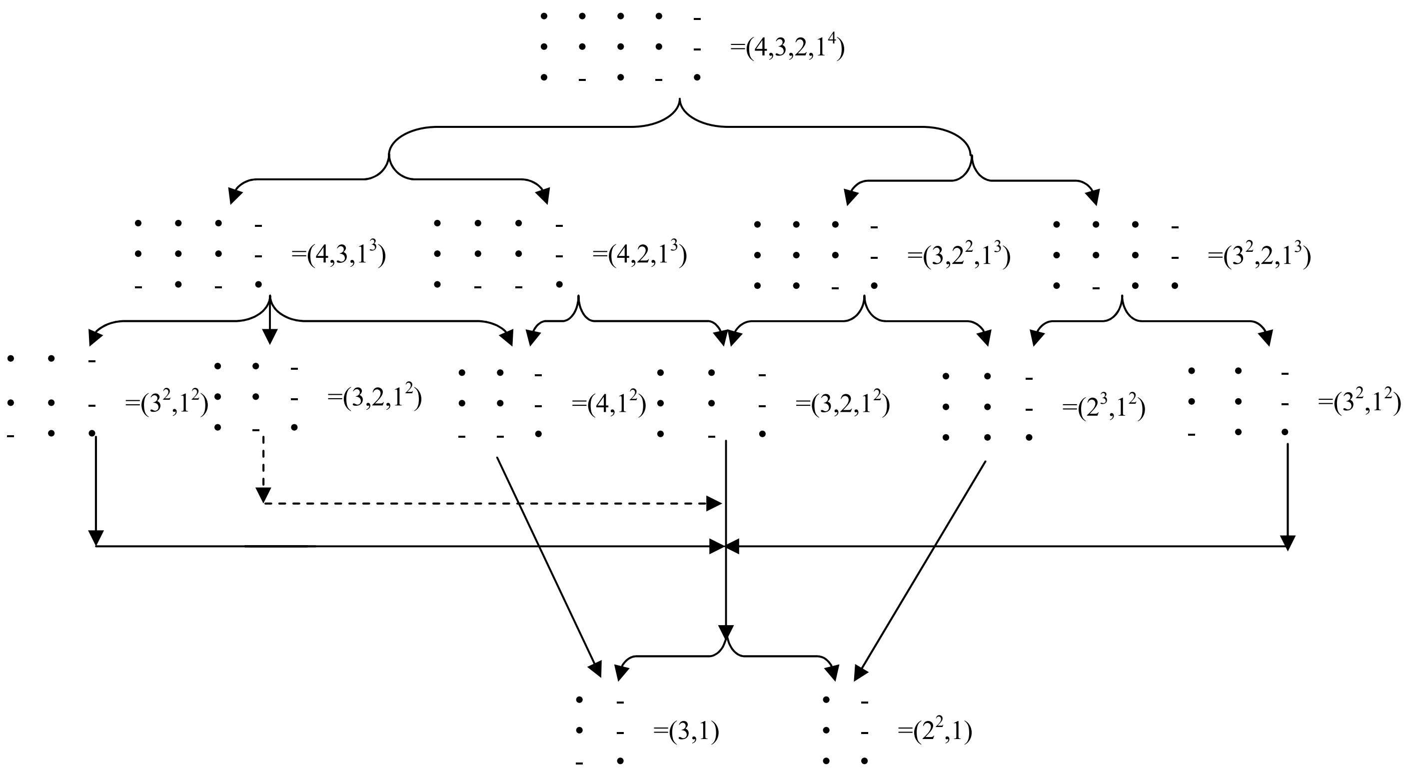
runner 3 by 1

⋮

runner e by e-2

3) Let the number of runner is e' = e-1.
4) Obviously, the last runner and the runners which hold β_1, they will not be discarded.
5) We note the number of runners which are full up with beads without any spaces by t_0, and $b' = b-t_0$.
6) Applying Fayers's rule: $b'+u = t_0e'+d'$, $0 \le d' \le e'-1$, if a value found that carry out the above equation by deleting from the beads and then delete the runner, our reduction is succeed. Otherwise, this is "sterile" or "useless" and will be neglected. Then, we seek another runner having the same beads feature t_1 and apply the same previous step: $b'+K = t_1e'+d'$.
7) repeat step (6) and neglect all sterile cases, and continue with the useful one. Then apply all steps from (1) to (6) on it.
8) When there is no case can keep on with it, this means all cases are steriles. This case considers the radical for the given tree.

For example :



Then the radicals of this example are (3,1) and $(2^2,1)$.

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