Right Closed Multiplication sets in Prime Near-Ring (α, β)-Derivation

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Abstract

In this paper , we extended the concept given in [1] which they used a semi group ideal to get the result has been given in [2] . We introduce a new body which is call it a right closed multiplicative set with zero . This structure give use similar results , and any semi group ideal satisfy the conditions of right closed multiplicative set . We prove that for any prime near-ring N and a multiplicative set I , if I , is abelian , then N is abelian . These results depends on many papers for example [3] , [4] , [5] , [6] , [7] , [8] .

المجموعات المغلقة بالضرب من اليمين في الحلقة المقتربة الأولية مع الاشتقاق (α, β)

الخلاصة

في هذا البحث , عملنا على تعميم الفكرة في البحث المنشور في [1] , حيث استفادوا من تعريف المثاليات الأولية للحصول على النتائج التي ظهرت في [2] بينما نحن استحدثنا تعريف ذو شروط اقل و يقوم بنفس المهمة في النتائج و برهنا تلك النتائج . إن التعريف الذي استدلينا علية و أسميناه المجموعة المغلقة على عملية الضرب من اليمين و تحتوي على الصفر . واهتدينا إلى النتيجة الرئيسية التي تنص على أن الحلقة المقترب الأولية N تكون أبدالية مع الجمع أذا احتوت على مجموعه مغلقة من اليمين تحتوي على الصفر و هي أبدالية مع الجمع . هذه النتائج تعتمد على بعض البحوث منها[3] , [4] , [5] , [6] , [7] , [8] .

1.Introduction :-

In this section we introduce a necessary conditions and definitions to get our results .

Definition(1.1)[1]:-An additive mapping $D: N \to N$ is said to be derivation on N if $D(xy) = xD(y) + D(x)y$ for all $x, y \in N$.

Notation(1.2)[1]:-In this paper N will be denoted a left near-ring with multiplicative center $Z(N)$, the symbol $[x, y]$ will denote the commutator $xy - yx$, the symbol (x, y) will denote the additive commutator $x + y - y - x$, $[x, y]_{\alpha, \beta}$ will denote the $(\alpha, \beta) -$commutator $\beta(x)y - y\alpha(x)$ and a near-ring N is called a zero symmetric if $0x = 0$, for all $x \in N$.

Definition(1.3)[1]:- An additive mapping $D: N \to N$ is called a $(\alpha, \beta) -$derivation if there exists "as automorphisms" $\alpha, \beta : N \to N$ such that $D(xy) = \beta(x)D(y) + D(x)\alpha(y)$ for all $x, y \in N$.

Definition(1.4)[1]:- The $(\alpha, \beta) -$derivation D will be called $(\alpha, \beta) -$commuting if $[x, D(x)]_{\alpha, \beta} = 0$ for all $x \in N$.

Definition(1.5)[1]:-A near-ring N is said to be prime if $aNb = 0$ implies that $a = 0$ or $b = 0$. Further an element $x \in N$ for which $D(x) = 0$ is called a constant.

Definition (1.6) :- A subset I of a near-ring N is called a right closed multiplication set contain zero , if $NI \subseteq I$. We will use right closed multiplication set contain zero (RCM) for this set .

2.Main result :-

In this section , we give some results which depend on section one .

<u>Lemma(2.1)</u> :- An additive endomorphism D on a near-ring N is (α,β) −derivation if and only if $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$ for all $x,y \in N$.

<u>Proof</u> :- Let D be a (α,β) −derivation on a near-ring N . Since $x(y+y) = xy + xy$ we obtain ,

$$D\big(x(y+y)\big) = \beta(x)D(y+y) + D(x)\alpha(y+y)$$

$$= \beta(x)D(y) + \beta(x)D(y) + D(x)\alpha(y) + D(x)\alpha(y)....(2.1)$$

for all $x,y \in N$, on the other hand , we have ;

$$D(xy + xy) = D(xy) + D(xy)$$

$$= \beta(x)D(y) + D(x)\alpha(x) + \beta(x)D(y) + D(x)\alpha(y)....(2.2$$

) for all $x,y \in N$, combining (2.1) and (2.2) , we find $\beta(x)D(y) + D(x)\alpha(y) = D(x)\alpha(y) + \beta(x)D(y)$ for all $x,y \in N$. Thus , we have $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$.

Conversely , let $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$.............(2.3) for all $x,y \in N$, then ;

$$D\big(x(y+y)\big) = D(x)\alpha(y+y) + \beta(x)D(y+y)$$

$$= D(x)\alpha(y) + D(x)\alpha(y) + \beta(x)D(y) + \beta(x)D(y)...(2.4) \quad \text{for} \quad \text{all}$$

$x, y \in N$. Also ;

$$D(xy + xy) = D(xy) + D(xy)$$

$$= D(x)\alpha(y) + \beta(x)D(y) + D(x)\alpha(y) + \beta(x)D(y).....(2.5) \quad \text{for} \quad \text{all}$$

$x, y \in N$ combining (2.4) and (2.5), we obtain .

$$D(xy) = D(x)\alpha(y) + \beta(x)D(y) = \beta(x)D(y) + D(x)\alpha(y). \quad \text{Thus}$$

$$D(xy) = \beta(x)D(y) + D(x)\alpha(y) \text{ for all } x, y \in N. \blacksquare$$

Lemma(2.2) :- Let N be a prime near-ring and I be a non zero RCM . If (I ,+) is a abelian then (N,+) is abelian .

Proof :-Let $x, y \in N$ and $a \in I$, then $xa, ya \in I$. So $xa + ya = ya + xa$. Then we get $(x + y - y - x)a = 0$ for all $a \in I$ and $x, y \in N$. This means that $(x + y - y - x)I = (x + y - y - x)NI = 0$ because I is a non zero RCM set . Since N is a prime near-ring we have $x + y - y - x = 0$ for all $x, y \in N$. Thus (N,+) is abelian . $\blacksquare$

Lemma(2.3) :- Let N be a prime near-ring and I be a RCM set of N .

(i)- If z is a non zero element in $Z(N)$, then z is not zero divisor.

(ii)- If there exist a non zero element z of $Z(N)$ such that $z + z \in Z(N)$, then $(I, +)$ is abelian .

Proof :- (i) – If $z \in Z(N) - \{0\}$, and $zx = 0$ for some $x \in I$.

Left multiplying this equation by b , where $b \in N$, we get $bzx = 0$.

Since N is multiplicative with center $Z(N)$, we get $zbx = 0$, for all

$b \in N, x \in I$. Hence , $zNx = 0$. Since N is a prime near-ring and

z is a non zero element , we get $x = 0$. ∎

(ii) - Let $z \in Z(N) - \{0\}$, such that $z + z \in Z(N)$,

and let $x, y \in I$, then $(x+y)(z+z) = (z+z)(x+y)$.

Hence $xz + xz + yz + yz = zx + zy + zx + zy$ since $z \in Z(N)$,

we get $zx + zy = zy + zx$. Thus , $z(x + y - x - y) = 0$ then be (i)

$z \neq 0$ we get $(x, y) = 0$ hence $(I, +)$ is abelian . ∎

Lemma (2.4) :- Let D be a non zero (α, β) −derivation on a

prime near-ring N and I be a non zero RCM set of N. such that

$\alpha(I) = I$ and $\beta(I) = I$, let $x \in I$ then :

(i)- If $\beta(x)D(I) = 0$ then $x = 0$. (ii)- If $xD(I) = 0$ then

$x = 0$.

Proof :- (i)- For $a, b \in I$, we have $\beta(x)D(ab) = 0$,

so $\beta(x)\big(\beta(a)D(b) + D(a)\alpha(b)\big) = 0$, to get

$\beta(x)\beta(a)D(b) + \beta(a)D(a)\alpha(b) = 0$, the second summand in

this equation equal zero by the hypothesis , so get

$\beta(x)\beta(a)D(b) = 0$, for $a, b, x \in I$. $\beta(I) = I$, we get $\beta(x)ID(b) = 0$,

since I is a RCM set of N , we get $\beta(x)NID(b) = 0$. Since N is a

prime near-ring , I is a non zero RCM set of N , D is a non zero

(α, β) −derivation of N , we get $\beta(x) = 0$, for all $x \in I$. Since

$\beta(I) = I$, thus get $x = 0$. ∎

(ii)- For all $a, b \in I$, we get $xD(ab) = 0$.

Thus $x\beta(a)D(b) + xD(a)\alpha(b) = 0$, the second summand in this equation equal zero by the hypothesis and I is a RCM set of N , we get $x\beta(a)D(b) = 0$, for all $x, a, b \in I$. But $\beta(I) = I$, then $xID(b) = 0$, this means that $xNID(I) = 0$. By the same way in (i) we get $x = 0$. ∎

Lemma(2.5) :- Let D be $(\alpha, \beta)-$derivation on a near-ring N and I be a RCM set of N such that $\alpha(I) = I$ and $\beta(I) = I$. Suppose $u \in I$ is a not a left zero divisor . If $[u, D(u)] = 0$, then (x, u) is a constant for every $x \in I$.

Proof :- From $u(u + x) = u^2 + ux$, apply D for both sided to have $D\big(u(u + x)\big) = D(u^2 + ux)$. Expanding this equation , to have $D\big(u(u + x)\big) = \beta(u)D(u + x) + D(u)\alpha(u + x)$

$$= \beta(u)D(u) + \beta(u)D(u) + D(u)\alpha(u) + D(u)\alpha(x) \ . \text{ and}$$

$D(u^2 + ux) = D(u^2) + D(ux)$

$$= \beta(u)D(u) + D(u)\alpha(u) + \beta(u)D(u) + D(u)\alpha(u) \ , \text{ for}$$

all $u, x \in I$. since $D\big(u(u + x)\big) = D(u^2 + ux)$ which reduces to $\beta(u)D(x) + D(u)\alpha(u) = D(u)\alpha(u) + \beta(u)D(x)$, for all $u, x \in I$.

By using the hypothesis $[u, D(u)] = 0$, this equation is expressible as

$\beta(u)\Big((D(x) + D(u) - D(x) - D(u))\Big) = 0 = \beta(u)D((x, u))$. so

$\beta(I) = I$, $uD((x, u)) = 0$. From u is not a left zero divisor , we get $D((x, u)) = 0$ .Thus , (x, u) is a constant for every $x \in I$. ∎

Proposition(2.6) :- Let N be a near-ring and I is a RCM set of N have no non zero divisors of zero. If N admits a non zero (α, β) −derivation on D which is commuting on I, then $(N, +)$ is abelian.

Proof :- Let c be any additive commutator in I . Then , by lemma (2.5) , yields that c is a constant . Now for any $x \in I$, cx is also an additive commutator in I . Hence , also a constant . Thus , $0 = D(cx) = \beta(c)D(x) + D(c)\alpha(x)$. Second summand in this equation equal zero, we get $\beta(c)D(x) = 0$, for all $x \in I$ and an additive commutator c in I. By lemma (2.4) (i) , we get $c = 0$ for all additive commutator c in I .Hence , $(I, +)$ is abelian . By lemma (2.2) , we get $(N, +)$ is abelian . ∎

Lemma (2.7) :- Let N be a prime near-ring , I be a non zero RCM set of N and D be a non zero (α, β) −derivation on N , $\beta(I) = I$. If $D((x, y)) = 0$, for all $x, y \in I$, then $(I, +)$ is abelian .

Proof :- Suppose that $D\big((x, y)\big) = 0$, for all $x, y \in I$. Taking xu instead of x and yu instead of y , where $u \in I$ we get $0 = D((xu, yu)) = D((x, y)u) = \beta((x, y))D(u) + D((x, y))\alpha(u)$, for all $x, y, u \in I$. By the hypothesis have $\beta((x, y))D(u) = 0$, for all $x, y, u \in I$. Hence , $\beta((x, y))D(I) = 0$. Using lemma (2.4) (i) , to get $(x, y) = 0$, for all $x, y \in I$. Thus , $(I, +)$ is abelian . ∎

lemma (2.8) :- Let N be a prime near-ring and I be a non zero RCM set of N . If I is a commutative then N is a commutative near-ring .

Proof :- For all $a, b \in I$, $a.b = b.a$ we get $a.b - b.a = 0$, since , $a.b - b.a = [a, b] = 0$ so $[a, b] = 0$. Taking xa instead of a and yb instead of b , where $x, y \in N$, to get $0 = [xa, yb] = xayb - ybxa = xyab - yxab = [x, y]ab$, for all $a, b \in I$ and $x, y \in N$. Thus , $[x, y]ab = [x, y]I^2 = 0$. Since

$NI \subseteq I$, we get $[x, y]NI^2 = 0$, for all x , $y \in N$. Since N is a prime near-ring and I is a non zero , we get $[x, y] = 0$, for all x , $y \in N$. Hence , N is a commutative near-ring . ∎

Lemma(2.9) :- Let N be a prime near-ring admits , a non zero (α, β) −derivation D and I be a RCM set of N . If $D(I) \subseteq Z(N)$ then $(I, +)$ is abelian .

Proof :- Since $D(I) \subseteq Z(N)$ and D is a non zero (α, β) −derivation . There exists a non zero element x in I, such that $z = D(x) \in Z(N) - \{0\}$ so $z + z = D(x) + D(x) = D(x + x) \in Z(N)$. Hence $(I, +)$ is abelian by lemma$((2.3)$ (ii)) . ∎

Corollary (2.10) :- Let N be a prime near-ring admits a non zero (α, β) −derivation and I be a RCM set of N . If $D(I) \subseteq Z(N)$ then $(N, +)$ is abelian .

Proof :-Using lemma (2.9) , to have $(I, +)$ abelian , then using lemma (2.2) , we get $(N, +)$ is abelian . ∎

Proposition (2.11) :- Let N be a prime near-ring admitting a non zero (α, β) −derivation D such that $\beta D = D\beta$, I be a RCM set of N such that $\beta(I) = I$ and β is homomoriphism on N. If $[D(I), D(I)] = 0$, then $(N, +)$ is abelian .

Proof :- By the hypothesis $[D(I), D(I)] = 0$, for all x , y , $t \in I$ we have $D(t + t)D(x + y) = D(x + y)D(t + t)$. Hence , $D(t)D(x) + D(t)D(y) - D(x)D(t) - D(y)D(t) = 0$. By application the hypothesis in this equation , we get ,

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$$\big(D(x) + D(y) - D(x) - D(y)\big)D(t) \;=\; (D(x+y-x-y))D(t) = 0$$

then $\quad D((x,y))D(t) = 0 \quad$, for all $\;x,y,t \in I\;$. Since β is homomorphism on N, we get $\beta(D((x,y)))\beta(D(t)) = 0$. By using $\beta D = D\beta$, obtain $\beta(D((x,y)))D(\beta(t)) = 0$, for all $x,y,t \in I$. Using lemma $((2.4)(i))$. Obtain $D((x,y)) = 0$, for all $x,y \in I$. Then using lemma (2.7), we get $(I,+)$ is abelian. So using lemma (2.2), to get $(N,+)$ is abelian. $\blacksquare$

References :-

1.   A. H. Majeed , Hiba, A. A , 2009 , Semi group Ideal in Prime Near-Ring with (α,β) −Derivations , Baghdad University, 368-372 .

2.   Mohammad. A , Asma. A and Shakir. A , 2004 , (α,β) −Derivations on prime near –ring , Tomus , 40 , 281-286.

3. Bell, H. E. 1997, On derivations in near-ring II , Kluwer Academic Publishers Netherlands , 191-197.

4. Bell, H. E. and Daif , M. N. , 1995 ,On derivation and commutatively in prime rings , Acta .Math. Hungar . 66 , No . 4 , 337-343.

5. Posner, E.C.1957, Derivations in Prime ring . Proc. Amer . Math. Soc. 8 , 1093-1100.

6. Pliz . G , 1983 , Near-ring 2nd Ed , North-Holland , Amsterdam .

7. Argac , N. 2004 , On near-ring with two sided α-derivation , Turk. J. Math. , 28, 195-204.

8. Wang , X . K. , 1994 , Derivations in prime near-ring . Proceedings of American Mathematical Society, 121 , no. 2 , 361-366.