Certain Properties of N- Proper Functions

خصائص معينة للدوال الفعلية-N

المدرس هيام حسن كاظم أ.د. هادي جابر مصطفى
قسم الرياضيات- كلية الرياضيات وعلوم الحاسبات - جامعة الكوفة

Abstract

In this work, we introduce and study N-proper function by using the concept of N-closed function .A function f: $(X,\tau)\rightarrow(Y,\sigma)$ from topological space (X,τ) into a topological space (Y,σ) is said to be a N-proper function if f is continuous , N- closed ,and for every $y\in Y$, $f^{-1}(y)$ is compact . Several properties of N-proper functions are proved.

المستخلص //

في هذا البحث ، قدمنا ودرسنا الدالة الفعلية-N باستخدام مفهوم الدالة المغلقة-N . الدالة $f: (X,\tau)\rightarrow(Y,\sigma)$ من الفضاء التبولوجي (X,τ) إلى الفضاء التبولوجي (Y,σ) ،تسمى دالة فعلية-N أذا كانت f مستمرة ، مغلقة-N، ولكل $y\in Y$ يكون $f^{-1}(y)$ متراص. وبرهنا عدة خصائص للدالة الفعلية-N .

1.Introduction:

In [1] , Bourbaki study in details the concept of proper mapping . In [4] , Hadi Jaber Mustaffa study proper functions and semi- proper functions. In this work, we introduce and study N- proper functions using the concept of N- closed functions.

Now, let us state the following definitions from [1,2] . Let (X ,τ) be a topological space , let $A\subseteq X$ and let $p\in X$, we say that :

i) p is a limit point (L.P.) of A if and only if, given G open in X and $p\in G$ then $(G-\{p\})\cap A\neq\phi$, $A'=$ the set of all limit points of A ,and A is closed if and only if, $A'\subseteq A$.

ii) p is a C-point(C.P.) of A if and only if, given G open in X and $p\in G$ then $G\cap A$ is a countable subset of A (we emphases that $G\cap A$ is infinite).

$(A')_c=$ the set of all C- points of A ,and we say that A is C- closed if and only if, $(A')_c\subseteq A$.

iii) p is a N-point(N.P.) of A if and only if, given G open in X and $p\in G$ then $G\cap A$ is a uncountable subset of A.

$(A')_N=$ the set of all N- points of A ,and we say that A is N- closed if and only if, $(A')_N\subseteq A$.

Some time we use X to denote the topological space (X ,τ) and we will use $|A|$ to indicate the cardinality of A , w means cardinality of countable sets and the symbol ۞ to indicate the end of the proof , .

1.1 Remark:

i) Every N-point is C-point and every C-point is a limit point . N.P→C.P→L.P

So, every closed set is a C- closed set and every C- closed set is a N- closed set .

N- closed←C- closed← closed

ii) The complement of N- closed (C- closed) is called N-open(C-open).

2.Basic definitions and examples:

In this section , we introduce and recall the basic definitions needed in this work.

First, we state the following definition:

2.1 Definition[1,2]:

Let f: $X\to Y$ be a function from a topological space X into a topological space Y, we say that:

i) f is closed if and only if, the image of every closed set in X is closed in Y.

ii) f is continuous if and only if, the inverse image of every closed set in Y is closed in X.

Next, we introduce the following definition:

2.2 Definition:

Let f: $X\to Y$ be a function from a topological space X into a topological space Y, we say that:

i) f is C- closed if and only if, the image of every closed set in X is C- closed in Y.

ii) f is N- closed if and only if, the image of every closed set in X is N- closed in Y.

2.3 Definition[1, 4]**:**

Let f: $X\to Y$ be a function from a topological space X into a topological space Y, we say that f is proper if and only if, f is continuous , closed ,and for every $y\in Y$, $f^{-1}(y)$ is compact.

2.4 Definition:

Let f: $X\to Y$ be a function from a topological space X into a topological space Y, we say that:

i) f is C- proper if and only if, f is continuous , C- closed ,and for every $y\in Y$, $f^{-1}(y)$ is compact.

ii) f is N- proper if and only if, f is continuous , N- closed ,and for every $y\in Y$, $f^{-1}(y)$ is compact.

2.5 Remarks and examples:

i) Every proper function is C- proper and every C- proper is N- proper.

Proper $\to$C- proper$\to$ N- proper

ii)Let X={a,b,c} with indiscrete topology τ_I define on X ,and let A={a}.We have A′= {b,c,d}, so we observe that b is L.P but not N.P ,and $A'\not\subseteq A$, but $(A')_N = \phi\subseteq A$, that is A is N- closed but not closed.

iii)Let X={1,2,3}and Y={a,b,c,d},with discrete topology τ_d define on X and indiscrete topology σ_i define on Y. Let f: $X\to Y$ be a function defined by :

$$f(x)= a,\forall x\in X.$$

Let A={a} be a subset of Y . In this example, we have $A'= \{b,c,d\}\not\subseteq A$ so A is not a closed set in Y , but A is an N- closed set in Y. Now , f is N- proper but not a proper function.

3.Main Results:

In this section , we prove that several properties of N-proper functions.

3.1 Theorem:

If (X ,τ) be a compact topological space and $A\subseteq X$ be N- closed , then A is also compact.

Proof: Let $\mathcal{L} = \{\omega_\alpha \mid \alpha\in\Omega\}$ be an open cover of A. Since A is N- closed , then A^c is N-open , so , for each $x \in A^c$ there is an open set V_x such that $x \in V_x$ and $|V_x \cap A|\leq w$ ($|V_x\cap A|$=cardinality of $V_x \cap A$ and w = cardinality of countable sets) .Let $\mathcal{L}_1=\{\nu_x \mid x \in A^c\}$ .Now $\mathcal{L}\cup\mathcal{L}_1$ is an open cover of X, but X is compact. So , $\mathcal{L} \cup\mathcal{L}_1$ has a finite sub cover say $\{\omega_{\alpha_1},\ldots,\omega_{\alpha_n}\}\cup\{\nu_{x_1},\ldots,\nu_{x_m}\}$, but $\bigcup_{i=1}^{m}\{\nu_{x_i}\cap A\}$ is finite, so for each $x_t\in \bigcup_{i=1}^{m}\{\nu_{x_i}\cap A\}$ choose $\omega_{x_t}\in\mathcal{L}$ such that $x_t\in \omega_{x_t}$. Then, $\{\omega_{\alpha_1},\ldots,\omega_{\alpha_n}\}\cup\{\omega_{x_t} \mid x_t\in \bigcup_{i=1}^{m}\{\nu_{x_i}\cap A\}\}$ is a finite sub cover of A. So, A is compact. ۞

From remark (1.1) part (i), we can get the following corollary:

3.2 Corollary: Every closed subset of a compact space is compact.

3.3 Theorem: Let f: $X\to Y$ be a continuous function from a space X onto a space Y , then f is N-closed if and only if, for each $y\in Y$ and any open set W in X such that $f^{-1}(y)\subseteq W$, there exists an N-open set V_y in Y such that $y\in V_y$ and $f^{-1}(V_y)\subseteq W$.

Proof:⇒) Suppose that f: X→Y is N- closed and $y\in Y$, let W be any open set in X such that $f^{-1}(y)\subseteq W$, then X-W is closed in X . So, f (X-W) is N- closed in X. Let V_y = Y-f(X-W), then V_y is N-open in Y and $y\in V_y$ such that $f^{-1}(V_y)$=X- f^{-1}(f(X-W)) ⊆W.

⇐)Let A be any closed subset of X and y∈Y-f(A), so $f^{-1}(y)\subseteq$ X-A=W. Then, there exists an N-open set V_y such that $y\in V_y$ and $f^{-1}(V_y)\subseteq$W=X-A.Now, $V_y\subseteq$Y-f(A). Hence, Y-f(A) is N-open, then f(A) is N- closed which means that f: X→Y is N- closed.✪

3.4 Remark: If we add to the above theorem the condition that : for each y∈Y , $f^{-1}(y)$ is compact , then f: X→Y will be N-proper from the definition of N-proper function .

3.5 Example: Let X=($\Re,\tau_u$) where ℜ is the set of real numbers and τ_u is the usual topology on ℜ . Consider A=[0,1]. Now, A is compact , also A is a G_δ-set , but A is not N-open (recall that A is called a G_δ-set if A is the intersection of a countable number of open sets).

3.6 Theorem: Let f: X→Y be N-proper function from a space X onto a compact space Y , then X is also compact.

Proof: Let $\mathcal{L} = \{\omega_\alpha | \alpha\in\Omega\}$ be an open cover of X, since $f^{-1}(y)$ is compact , $f^{-1}(y)\subseteq \bigcup_{i=1}^{m} w_{\alpha_i}$ .Let V_y= Y-f(X- $\bigcup_{i=1}^{n} w_{\alpha_i}$) . Now, f is N- closed , so V_y is N- open for each y∈Y , so there exists an open set V_y^* of y such that $|V_y^*\cap V_y^c|\leq w$. Now, V_y^* =[$V_y\cap V_y^*$] ∪ [$V_y^*\cap V_y^c$] , also $\mathcal{L}_1$={ V_y^* | y∈Y} is an open cover of Y and Y is compact , so $\mathcal{L}_1$ has a finite sub cover , therefore X is the union of finite many members of { $f^{-1}(V_y^*)$| y∈Y} . Since each $f^{-1}(y)$ is contained many members of $\mathcal{L}$, so X is compact . ✪

Similarly, we can prove:

3.7 Theorem: Let f: X→Y be N-proper function of a space X onto a Lindelöf space Y , then X Lindelöf .

Before , we state the next theorem , we recall the followings:

3.8 Definition[3]: A function f: X→Y is called a compact function if the inverse of each compact set in Y is compact in X.

3.9 Definition[2]: A space X is called a P- space if the intersection of countable number of open sets is open (that is , each G_δ-set is open)[2].

Similarly X is called a P^*- space if each G_δ-set is N-open .

3.10 Theorem: Let f: X→Y be a continuous function from a space X onto a space Y , where Y is compact Hausdorff P^*- space , then f is N- proper if and only if, f is a compact function.

Proof:⇒) Follows from theorem 3.6 .

⇐) Let f: X→Y be a continuous compact function . It suffices to show that f is N-closed let F be a closed subset of X , assume that f(F) is not N-closed , so there exists a point $y_o\in$Y- f(F) such that for every open set V of y_o , |V∩ f(F) |>w since Y is locally compact , there is an open set G of y_o such that $\overline{G}$ = cl(G) is compact. Now, f(F)∩$\overline{G}$ is not compact if it is compact , then it will be N-closed , so there exists an open set M of y_o such that | M∩ f(F) |≤ w which is impossible . Now, $\overline{G}$ is compact , so $f^{-1}(\overline{G})$ is compact and F ∩ $f^{-1}(\overline{G})$ is a compact subset of X, therefore f(F ∩ $f^{-1}(\overline{G})$) = f(F)∩$\overline{G}$ is compact , which is a contradiction . Hence , f(F) is N-closed which means that f is N-closed . ✪

References:

[1] N. Bourbaki, General Topology, Vol. 1 , Addison-Wesley Puplishing Company, Reading-Massachussetts , 1966.

[2] R. Engelking, Out Line of General Topology, North Holland Puplishing Company , Amsterdam , 1968 .

[3] Hadi J. Mustaffa and D. Ibraheem , Compact Functions , M.Sc thesis, College of Education , AL- Mustansiriyah University ,2001.

[4] Hadi J. Mustaffa , Proper Functions , Journal of Babylon University , 2002.