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The Order of Elements in T_n and P_n Semigroups

رتبة العناصر في شبه الزمرة T_n وشبه الزمرة P_n

Neeran Tahir Al-Khafaji sajda Kadhum Mohammed
Dep.of Math. , College Of Education for Girls, Al-Kufa University
Yiezi Kadham Mahdi
Dep.of Math. , College Of Education, Babylon University

Abstract:

In this paper we obtain a formula for the order of elements in T_n (*The symmetric Semigroup)* and P_n(*The Partial symmetric Semigroup*) semigroups .

الخلاصة //

في هذا البحث أوجدنا صيغة لحساب رتبة العناصر في شبه الزمرة التناظرية T_n وفي شبه الزمرة التناظرية الجزئية P_n .

1. Introduction:

Let $X_n=\{1,2,\ldots,n\}$ then a partial transformation $\alpha:\mathrm{Dom}\,\alpha\subseteq X_n \to \mathrm{Im}\alpha\subseteq X_n$ is said to be *a full or total transformation* if $\mathrm{Dom}\,\alpha = X_n$, otherwise it is called *strictly partial* .The three fundamental semigroups of transformations under the usual composite of mapping that have been extensively studied are : T_n *the full transformation semigroup* (or *the symmetric Semigroup*); I_n ,*the semigroup of partial one-one mappings* (or t*he symmetric inverse semigroup*); and P_n *the semigroup of partial transformations*(or the *Partial symmetric Semigroup*), [1],[2] .

In a semigroup S, the order of an element in S is the order of the cyclic subsemigroup of S generated by this element ,[3]. Let S be the T_n or P_n semigroup so we will know the order of these semigroup if we get a repeated element or the 0 element, [4] where

$$0=\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ - & - & \ldots & - \end{pmatrix}\in P_n \text{ ,that is mean } \alpha:\mathrm{Dom}\,\alpha=\Phi\subseteq X_n \to \mathrm{Im}\alpha\subseteq X_n .$$

In this paper we want to find the order of each element in T_n and P_n .

Definition 1.1: A semigroup S is generated by a subset G if every element of S can be written as a product of some elements of G. A semigroup generated by a subset consisting of a single element is *a cyclic semigroup* .If A is a non- empty subset of a semigroup S, then the set

$\{a_1 a_2 \ldots a_n : a_i \in A \text{ and n is arbitrary}\}$,

is *the subsemigroup of S generated by* A ,[3] .

Definition 1.2: *The order of a semigroup* S is the number of its element if S is finite, otherwise S is of infinite order .*The order of an element* s of a semigroup S is the order of the cyclic subsemigroup of S generated by s ,[3] .

Exampe 1.1:p_2 is the set of all mapping $\alpha_1, \ldots , \alpha_9$ where

if Dom $\alpha = X_2$,

$\alpha_1(1) = \alpha_1(2) =1$

$\alpha_2(1) = \alpha_2(2) =2$

$\alpha_3(1) =1,\alpha_3(2) =2$ is the identity element (e)

$\alpha_4(1) =2,\alpha_4(2) =1$

if Dom $\alpha \subset X_2$ and contains one elements

$\alpha_5(1) =1,\alpha_5(2) = -$,where $\alpha_5(2)=-$ means that $2\notin$ Dom α_5 .

$\alpha_6(1) =2,\alpha_6(2) = -$

$\alpha_7(1) = -,\alpha_7(2) =1$

$\alpha_8(1) = -,\alpha_8(2) =2$

if Dom α= $\Phi \subset X_2$
$\alpha_9(1) = -, \alpha_9(2) = -$ is the zero element .
We must note that { $\alpha_1, \dots, \alpha_4$}=$T_2$. Because we can deal with these mapping easily we will write the mapping above as following:

$$\alpha_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \alpha_5 = \begin{pmatrix} 1 & 2 \\ 1 & - \end{pmatrix},$$
$$\alpha_6 = \begin{pmatrix} 1 & 2 \\ 2 & - \end{pmatrix}, \alpha_7 = \begin{pmatrix} 1 & 2 \\ - & 1 \end{pmatrix}, \alpha_8 = \begin{pmatrix} 1 & 2 \\ - & 2 \end{pmatrix}, \alpha_9 = \begin{pmatrix} 1 & 2 \\ - & - \end{pmatrix}.$$

2. The Main Result:

The subset T_n of P_n consisting of all mappings from X_n into X_n form a subsemigroup **.** Now inside T_n and P_n we find one or more than one element of the form

$$\alpha = \begin{pmatrix} x_1 x_2 & \dots & x_{k-1} & x_k \\ x_2\ x_3 & \dots & x_k & x_k \end{pmatrix} \text{ , where } k > 1 \text{ and } \forall\, x_i \in \text{Dom}\,\alpha \text{ , } i = k+1, \dots, n \text{ are fixed .}$$

We shall call α the element of the first kind I with length k and we say x_k is fixed element in α where $\alpha(x_{k-1})= x_k$ and $\alpha(x_k)= x_k$ .We must note that every element of the form

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_u & x_{u+1} & \dots x_h \\ x_2 & x_3 & \dots & x_1 & x_{u+2} & \dots & x_i \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & \dots & x_u & x_{u+1} & \dots x_h \\ x_2 & x_3 & \dots & x_1 & x_{u+1} & \dots & x_h \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \dots & x_u & x_{u+1} & \dots & x_h \\ x_1 & x_2 & \dots & x_u & x_{u+2} & \dots & \alpha^{-1}(x_i) \end{pmatrix}$$
$$= \lambda\gamma$$

, where i=1, … u and $\alpha^{-1}(x_i)$ is the pre-image of the x_i such that it is belong to the cyclic λ we shall called this element is the element of the first kind II with length h,, then α can be written as a product of cyclic and element of the first kind .also we find one or more than one element of the form

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_j & x_j \\ x_2 & x_3 & \dots & x_j & - \end{pmatrix}, \text{ where either } x_i \in \text{Dom}\alpha \text{ is fixed or } x_i \in X_n / \text{Dom}\alpha\ \ \forall\, i = j+1, \dots, n.$$

We shall call s α the element of the second kind with length j .

Note: we will use L.C.M.(least common multiple for lengths of its cycles) for the order of m cycles , h for the length of the longest element of the first kind I , k for the length of the longest element of the first kind II and f for the length of the longest element of the second kind , and we will not care for the element of the form $\alpha(x_i)=x_i$ in α where x_i is not the fixed element that is because it doesn't affect on the order of α .We say h=0 if there is no element of the first kind I in α also k=0 if there is no element of the first kind II in α and f=0 if there is no element of the second kind in α.

In this paper we interested in finding the order of each element in T_n and P_n .

Theorem 2.1: let $\alpha \in T_n$ such that α has m cycles and L elements of the first kind I or of the first kind II then the order of

$$o(\alpha)=\begin{cases} h-1 & \text{if } \quad L.C.M.=0 \\ L.C.M.+h-2 & \text{if } \quad 2\le L.C.M. \text{ and } h-1\ge k \text{ or } k=0 \text{ and } 2\le L.C.M. \\ L.C.M.+k-1 & \text{if } \quad 2\le L.C.M. \text{ and } h-1<k \text{ or } h=0 \text{ and } 2\le L.C.M. \\ L.C.M. & h=0 \text{ and } L.C.M.\ge 1 \end{cases}.$$

Proof: We have several cases:
Case 1: m= 0 , $L\ge 1$ so L.C.M.=0.

First, let L=1 so α $=w_1$ with length h so $\alpha=\begin{pmatrix} x_1x_2 & \dots & x_{h-1} & x_h \\ x_2\ x_3 & \dots & x_h & x_h \end{pmatrix}$ .By using the operation of T_n then $\alpha^{h-1}(x_i)=x_h \forall x_i\in \mathrm{Dom}\,\alpha$ where x_k is the fixed element in α therefore the order of α=o(α)= h-1, now, let L > 1 so α $=w_1w_2\dots w_L$, where $w_1, w_2, \dots, w_L$ are elements of the first kind I with lengths $s_1, s_2, \dots s_L$ respectively , $s_1\le s_2\le \dots \le s_L=h$.

$\alpha^{s_1-1}(x_i)=$ the fixed element in $w_1\ \forall x_i\in \mathrm{Dom}\, w_1$ and

$\alpha^{s_2-1}(x_i)=$ the fixed element in $w_2 \forall x_i\in \mathrm{Dom}\, w_2$, since $s_1\le s_2$ then

$\alpha^{s_2-1}(x_i)=$ the fixed element in $w_2\forall x_i\in \mathrm{Dom}\, w_2$ and the fixed element in $w_1\forall x_i\in \mathrm{Dom}\, w_1$,

and so on, therefore α^{s_L-1} make the image of each element belong to the Dom w_j equal to the image of the fixed element in w_j, where j=1, … L , thus

order of α= o(α) = the length of the longest element in it (s_L=h) -1 .
so we can say the order of the element of the first kind I =h-1.

case 2: $m\ge 1$, L=0 so h=0 and L.C.M.≥ 1 .
so $\alpha\in S_n\subset T_n$ and o(α) =L.C.M. of lengths for its cycles ,[5] , where S_n is a symmetric group. it is clear that o(the identity element)=o(e)=1

Case 3:A. $m\ge 1$, L=1 and L.C.M. ≥ 2
first if L is the number of elements of the first kind II so
α $=\lambda_1\lambda_2\dots\lambda_m w_1$, where $\lambda_1, \lambda_2, \dots, \lambda_m$ are cycles and w_1 is the element of the first kind II with length k, where $k\ne 0$ because if it is so, we have just proved in case 2 , so by using the operation of T_n we have $\alpha^k(x_i)$ will belong to one of $\lambda_1, \lambda_2, \dots, \lambda_m$ which related to it ,now by case 2,
$\alpha^{L.C.M.}(x_i)=x_i\forall x_i\in \mathrm{Dom}\,\lambda_1,\dots,\lambda_m$ and $\alpha^{1+L.C.M.}(x_i)=\alpha^1(x_i)\ \forall x_i\in \mathrm{Dom}\,\lambda_1,\dots,\lambda_m$ but not in $<\alpha>$ since $\alpha^{1+L.C.M.}(x_i)\ne\alpha^1(x_i)\ \forall x_i\in \mathrm{Dom}\, w_1$ unless k=1, $\alpha^{2+L.C.M.}(x_i)=\alpha^2(x_i)$ $\forall x_i\in \mathrm{Dom}\,\lambda_1,\dots,\lambda_m$ but not in $<\alpha>$ since $\alpha^{2+L.C.M.}(x_i)\ne\alpha^2(x_i)\ \forall x_i\in \mathrm{Dom}\, w_1$ unless k=2 ,and so on until $\alpha^{k+L.C.M.}(x_i)=\alpha^k(x_i)\ \forall x_i\in \mathrm{Dom}\,\lambda_1,\dots,\lambda_m$ and $\forall x_i\in \mathrm{Dom}\, w_1$ so $\alpha^{k+L.C.M.}$ will be repeated element in $<\alpha>$ so

o(α)= L.C.M. +length of w_1-1= L.C.M+k-1.

Second if L is the number of element of the first kind I and w_1 is the element of the first kind I with length $h\ne 0$ so $\alpha^{h-1}(x_i)=x_h\forall x_i\in \mathrm{Dom}\, w_1$ where x_h is the fixed element in w_1.
$\alpha^{1+L.C.M.}(x_i)=\alpha^1(x_i)\ \forall x_i\in \mathrm{Dom}\,\lambda_1,\dots,\lambda_m$ but not in $<\alpha>$ since by case1,
$\alpha^{1+L.C.M.}(x_i)\ne\alpha^1(x_i)\ \forall x_i\in \mathrm{Dom}\, w_1$ unless h-1=1, $\alpha^{2+L.C.M.}(x_i)=\alpha^2(x_i)\ \forall x_i\in \mathrm{Dom}\,\lambda_1,\dots,\lambda_m$ but not in $<\alpha>$ since $\alpha^{2+L.C.M.}(x_i)\ne\alpha^2(x_i)\ \forall x_i\in \mathrm{Dom}\, w_1$ unless h-1=2
and so on until $\alpha^{h+L.C.M.-1}(x_i)=\alpha^{h-1}(x_i)\forall x_i\in \mathrm{Dom}\,\lambda_1,\dots,\lambda_m$ and $\forall x_i\in \mathrm{Dom}\, w_1$ so $\alpha^{h-1+L.C.M}$ will be repeated element in $<\alpha>$ so

o(α)= L.C.M. +length of w_1-2= L.C.M.+h-2.

B. $m \geq 1$, L=2 and L.C.M. ≥ 2

$\alpha = \lambda_1 \lambda_2 \ldots \lambda_m w_1 w_2$, so that w_1 is the element of the first kind I and w_2 the element of the first kind II with lengths h, k respectively if h-1>k where h-1 is the order of w_1 and k is the length of w_2 , now since w_2 is the element of the first kind II

therefore by case 3: A , $\alpha^{k+L.C.M.}(x_i) = \alpha^{k}(x_i)$ $\forall x_i \in Dom \lambda_1, \ldots, \lambda_m$ and $\forall x_i \in Dom\, w_2$

and since w_1 is the element of first kind I so by case 3: A , $\alpha^{h+L.C.M.-1}(x_i) = \alpha^{h-1}(x_i)$

$\forall x_i \in Dom \lambda_1, \ldots, \lambda_m$ and $\forall x_i \in Dom\, w_1$ so,

Since h-1>k

$\alpha^{k+L.C.M.+1}(x_i) = \alpha^{k+1}(x_i)$ $\forall x_i \in Dom \lambda_1, \ldots, \lambda_m$ and $\forall x_i \in Dom\, w_2$ but not in $< \alpha >$ since by case1, $\alpha^{k+L.C.M.+1}(x_i) \neq \alpha^{k+1}(x_i)$ $\forall x_i \in Dom\, w_1$ unless k+1=h-1,and so on,

$\alpha^{h+L.C.M.-1}(x_i) = \alpha^{h-1}(x_i)$ $\forall x_i \in Dom \lambda_1, \ldots, \lambda_m$ and $\forall x_i \in Dom\, w_1, w_2$,so

o(α)= L.C.M. +length of the longest element(w_1)-2= L.C.M+h-2.

Now ,if k>h -1

$\alpha^{h+L.C.M.-1}(x_i) = \alpha^{h-1}(x_i)$ $\forall x_i \in Dom \lambda_1, \ldots, \lambda_m$ and $\forall x_i \in Dom\, w_1$ but not in $< \alpha >$ since

$\alpha^{h+L.C.M.-1}(x_i) \neq \alpha^{h-1}(x_i)$ $\forall x_i \in Dom\, w_2$ unless k=h-1,

$\alpha^{h+L.C.M.}(x_i) = \alpha^{h}(x_i)$ $\forall x_i \in Dom \lambda_1, \ldots, \lambda_m$ and $\forall x_i \in Dom\, w_1$ but not in $< \alpha >$ since

$\alpha^{h+L.C.M.}(x_i) \neq \alpha^{h}(x_i)$ $\forall x_i \in Dom\, w_2$ unless h=k we will continue in this way until

$\alpha^{k+L.C.M.}(x_i) = \alpha^{k}(x_i)$ $\forall x_i \in Dom \lambda_1, \ldots, \lambda_m$ and $\forall x_i \in Dom\, w_1, w_2$,, so

o(α)= L.C.M. +length of the longest element(w_2)-1= L.C.M+k-1.

Where $h \neq 0$ and $k \neq 0$ because if they are so, we have just proved in case3: A.

In general: let $\alpha \in T_n$ such that $m \geq 1$, $L \geq 1$ and L.C.M. ≥ 2

$\alpha = \lambda_1 \lambda_2 \ldots \lambda_m w_1 \ldots w_L$, where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are cycles and $w_1, \ldots, w_L$ are the elements of the first kind I or the first II with lengths $s_1, \ldots, s_L$ respectively , $s_1 \leq s_2 \leq \ldots \leq s_L$. By case 2, the order of $\lambda_1 \lambda_2 \ldots \lambda_m$ is L.C.M. , we have three cases:

A: if $w_1, \ldots, w_L$ are the elements of the first kind I so by case 1, $\alpha^{s_L - 1}(x_i)$ =fixed element in w_j $\forall x_i \in w_j$ such that j=1, … , L, by case 2, o($\lambda_1 \lambda_2 \ldots \lambda_m$) = L.C.M. therefore

$\alpha^{1+L.C.M.}(x_i) = \alpha^{1}(x_i)$ $\forall x_i \in Dom \lambda_1, \ldots, \lambda_m$ but not in $< \alpha >$ since $\alpha^{1+L.C.M.}(x_i) \neq \alpha^{1}(x_i)$ $\forall x_i \in Dom\, w_1, \ldots, w_L$ unless s_L-1=1,so it is new element in $< \alpha >$and

$\alpha^{2+L.C.M.}(x_i) = \alpha^{2}(x_i)$ $\forall x_i \in Dom \lambda_1, \ldots, \lambda_m$ but not in $< \alpha >$ since

$\alpha^{2+L.C.M.}(x_i) \neq \alpha^{2}(x_i)$ $\forall x_i \in Dom\, w_1$,.. ,w_L unless s_L-1=2 , and so on until

$\alpha^{s_L+L.C.M.-1}(x_i) = \alpha^{s_L-1}(x_i)$ $\forall x_i \in Dom \lambda_1, \ldots, \lambda_m$ and $\forall x_i \in Dom\, w_1, \ldots, w_L$ so

o(α)= L.C.M. + the length of the longest element(s_L)-2 = L.C.M. +h-2,

Where $s_L = h \neq 0$ because if it is so, we have just proved in case2.

B : if $w_1, \ldots, w_L$ are the elements of the first kind II ,if L=1 then by case3 ,o(α)= L.C.M. +length of w_1-1= L.C.M+s_1-1.Now if L>1 then

$\alpha^{s_1}(x_i)$ belong to the related cyclic $\forall x_i \in Dom\, w_1$ and

$\alpha^{s_2}(x_i)$ belong to the related cyclic $\forall x_i \in Dom\, w_2$, since $s_1 \leq s_2$ then

$\alpha^{s_2}(x_i)$ belong to the related cyclic $\forall x_i \in Dom\, w_2$ and $\forall x_i \in Dom\, w_1$,

and so on, therefore $\alpha^{s_L}(x_i)$ belong to the related cyclices $\forall x_i \in Dom\, w_1, \ldots, w_L$

and o($\lambda_1 \lambda_2 \ldots \lambda_m$) = L.C.M. , $\alpha^{1+L.C.M.}(x_i) = \alpha^{1}(x_i)$ $\forall x_i \in Dom \lambda_1, \ldots, \lambda_m$ but not in $< \alpha >$ since

$\alpha^{1+L.C.M.}(x_i) \neq \alpha^1(x_i)$ $\forall x_i \in \text{Dom}\, w_1, \ldots, w_L$ unless s_L=1 , and so on until
$\alpha^{s_L+L.C.M.}(x_i) = \alpha^{s_L}(x_i)$ $\forall x_i \in \text{Dom}\, \lambda_1, \ldots, \lambda_m$ and $\forall x_i \in \text{Dom}\, w_1, \ldots, w_L$ so

o(α)= L.C.M. + the length of the longest element(s_L)-1= L.C.M. +k-1,

Where s_L= k $\neq 0$ because if it is so, we have just proved in case2.

C: if some of $w_1, \ldots, w_L$ are the elements of the first kind II and the other are the elements of the first kind I. let h= the length of the longest element of the first kind I and k= the length of the longest element of the first kind II ,by the general case: A, we have $\alpha^{h+L.C.M.-1}(x_i) = \alpha^{h-1}(x_i)$ $\forall x_i \in \text{Dom}\, \lambda_1, \ldots, \lambda_m$ and for each x_i belong to the domain of the elements of the first kind I also by the general case : B. we have $\alpha^{k+L.C.M.}(x_i) = \alpha^k(x_i)$ $\forall x_i \in \text{Dom}\, \lambda_1, \ldots, \lambda_m$ and for each x_i belong to the domain of the elements the first kind II, now
if h-1>k then
$\alpha^{k+L.C.M.+1}(x_i) = \alpha^{k+1}(x_i)$ $\forall x_i \in \text{Dom}\, \lambda_1, \ldots, \lambda_m$ and for each x_i belong to the domain of the elements the first kind II but not in $< \alpha >$ since $\alpha^{k+L.C.M.+1}(x_i) \neq \alpha^{k+1}(x_i)$ for each x_i belong to the domain of the elements of the first kind I unless k+1=h-1 ,and so on until $\alpha^{h+L.C.M.-1}(x_i) = \alpha^{h-1}(x_i)$ $\forall x_i \in \text{Dom}\, \lambda_1, \ldots, \lambda_m$ and for each x_i belong to the domain of the elements of the first kind I and for each x_i belong to the domain of the elements of the first kind II so

o(α)= L.C.M. +length of the longest element(h)-2= L.C.M+h-2.

Now ,if k>h -1 by the same manner we have $\alpha^{k+L.C.M.}(x_i) = \alpha^k(x_i)$ $\forall x_i \in \text{Dom}\, \lambda_1, \ldots, \lambda_m$ and for each x_i belong to the domain of the elements of the first kind I and for each x_i belong to the domain of the elements of the first kind II so

o(α)= L.C.M. +length of the longest element(k)-1= L.C.M+k-1.

We must note that $h \neq 0$ or $k \neq 0$ because if it is so, we have just proved case3 or case2 .

Example2.1:case1:m=0, L>=1

Suppose that L=1, L.C.M.=0

Let α= $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix} \in T_3$, h=3

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix},\ \langle\alpha\rangle = \left\{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}\right\}$$ so O(α)=2.

Suppose that L >1 , L.C.M.=0

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 4 & 6 & 8 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 6 & 8 & 7 & 8 \end{pmatrix} = w_1 w_2$,where α(7)=7,

$\alpha \in T_8$ it is clear that length of w_1 = 4 and length of w_2 = 3 where

so $\alpha^{4-1}(x_i) = 4$ $\forall x_i \in \text{Dom}\, w_1$ where α(4)=4 is a fixed element in w_1

$\alpha^{3-1}(x_i) = 8$ $\forall x_i \in \text{Dom}\, w_2$ where α(8)=8 is a fixed element in w_2,

$\alpha^3(x_i) = 4$ $\forall x_i \in \text{Dom}\, w_1$

$\alpha^3(x_i) = 8$ $\forall x_i \in \text{Dom}\, w_2$

$$\langle\alpha\rangle = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 4 & 6 & 8 & 7 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 4 & 4 & 8 & 8 & 7 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 4 & 4 & 4 & 8 & 8 & 7 & 8 \end{pmatrix}\right\}$$

So o(α) = the length of the longest element in it (4) -1 =3

Case 2: m=2 , L=0

Let α= $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 5 & 3 & 6 \end{pmatrix}$ = (12)(345) $\in T_6$, $\alpha \in S_6$ where S_6 is the symmetric group so by [5] ,
o(α) = L.C.M.=6
to explain the general case A. we take this example: let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 1 & 4 & 5 & 3 & 7 & 8 & 9 & 10 & 10 & 12 & 12 \end{pmatrix} \in T_{12}$$, ,where m=2 and L=2

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 1 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 4 & 5 & 3 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 3 & 4 & 5 & 7 & 8 & 9 & 10 & 10 & 11 & 12 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 12 & 12 \end{pmatrix} = \lambda_1\lambda_2 w_1 w_2$$

it is clear that L.C.M.=6 and h=S_L=5
o(α)=9,where $\alpha^9 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 1 & 3 & 4 & 5 & 10 & 10 & 10 & 10 & 10 & 12 & 12 \end{pmatrix}$ we must note that $\alpha^{10} = \alpha^4$,i.e., $\alpha^{10}$is a repeated element in < α >.
Now let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 5 & 3 & 7 & 8 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 4 & 5 & 3 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 7 & 8 & 2 \end{pmatrix} = \lambda_1\lambda_2 w_1$$

such that $\alpha^{-1}(2)=1$ and $1 \in \lambda_1$, it is clear that L.C.M.=6 and k=S_L= the length of the element of the first kind II =3, so o(α)= L.C.M.+k-1=8 since $\alpha^9 = \alpha^3$.

Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 \\ 2 & 3 & 4 & 1 & 6 & 5 & 8 & 9 & 10 & 3 & 12 & 13 & 14 & 15 & 16 & 16 & 18 & 19 & 6 \end{pmatrix} \in T_{19}$

L.C.M.=4 , h= 6 , k= 4 since $\alpha^9 = \alpha^5$ so o(α)=8.

Corollary2.1 : let $\alpha \in P_n$ and $\alpha = (\alpha_1 \ldots \alpha_m \lambda_1 \lambda_2 \ldots \lambda_L)(\theta_1 \ldots \theta_g) = w_1 w_2$,where $\alpha_1, \ldots, \alpha_m$ are cycles , $\lambda_1, \lambda_2, \ldots, \lambda_L$ are elements of the first kind I or of the first kind II and $\theta_1, \ldots, \theta_g$ are elements of the second kind then

$$o(\alpha) = \begin{cases} f & \text{if } (h = k = 0 \text{ or } f \ge h-1, k = 0) \text{ and L.C.M.} = 0 \\ o(w_1) & \text{if } h-1 \ge f, \text{L.C.M} = 0 \text{ or } f = 0 \text{ or } (h-1 > k, h-1 > f, \text{ or } h-1 < k, f < k, \\ & \text{or , or } k < f, f < h-1 \text{ or } k \ge f \text{ or } h = 0 \text{ or } h-1 \ge f, k = 0) \text{ and L.C.M.} \ge 2 \\ \text{L.C.M.} + f - 1 & \text{if}(f > h-1, h-1 > k, \text{or } k > h-1, f > h-1, f > k, \text{or } h = k = 0 \text{or } f \ge k, h = 0 \\ & \text{or } f \ge h-1, k = 0) \text{ and L.C.M.} \ge 2 \end{cases}$$

Proof: We have several cases:
case1:A. m=0 , L= 0 , g=1 therefore h=k=0 ,L.C.M.=0
Let

$\alpha = \begin{pmatrix} x_1 & x_2 & \ldots & x_{f-1} & x_f \\ x_2 & x_3 & \ldots & x_f & - \end{pmatrix}$, where either $x_i \in Dom\alpha$ is fixed or $x_i \in X_n / Dom\alpha \ \forall i = f+1, \ldots, n$ .By using the operation of P_n we have $\alpha^f(x_i) = - \ \forall x_i \in Dom\alpha$ so o(α) =the length of α =f.
B. m=0 ,L=0 , g > 1.

Let $\alpha = w_2 = \theta_1 \dots \theta_g$, where $\theta_1, \dots, \theta_g$ are elements of the second kind with lengths $s_1, \dots, s_g$ respectively , $s_1 \le s_2 \le \dots \le s_g = f$, so by case1:A,

$\alpha^{s_1}(x_i) = - \ \forall x_i \in \mathrm{Dom}\,\theta_1$ otherwise $\alpha^{s_1}(x_i) \ne -$ then

$\alpha^{s_2}(x_i) = - \ \forall x_i \in \mathrm{Dom}\theta_2$ otherwise $\alpha^{s_2}(x_i) \ne -$

Since $s_1 < s_2$ so $\alpha^{s_2}(x_i) = - \ \forall x_i \in \mathrm{Dom}\,\theta_1, \theta_2$, and so on ,

$\alpha^{s_g}(x_i) = - \ \forall x_i \in \mathrm{Dom}\,\theta_1, \dots, \theta_g$,where $s_1 \le \dots \le s_g = f$,therefore

o(α) = the length of the longest element in α=f .

case 2: $m \ge 1, L=0, g \ge 1$ so L.C.M. ≥ 2 and h=k=0

so $\alpha = \alpha_1 \dots \alpha_m \theta_1 \dots \theta_g$

by case1: B, $\alpha^{s_g}(x_i) = - \ \forall x_i \in \mathrm{Dom}\,\theta_1, \dots, \theta_g$ so $\alpha^{s_g+1}(x_i) = \alpha^{s_g} = - \ \forall x_i \in \mathrm{Dom}\,\theta_1, \dots, \theta_g$ and since $\alpha^{1+L.C.M.}(x_i) = \alpha^1(x_i) \ \forall x_i \in \mathrm{Dom}\,\alpha_1, \dots, \alpha_m$ but not in < α > since $\alpha^{1+L.C.M.}(x_i) \ne \alpha^1(x_i)$ $\forall x_i \in \mathrm{Dom}\,\theta_1, \dots, \theta_g$ unless $s_g=1$ so $\alpha^{1+L.C.M}$ is new element in < α > unless $s_g=1$,and so on until $\alpha^{s_g+L.C.M.}(x_i) = \alpha^{s_g}(x_i) \ \forall x_i \in \mathrm{Dom}\,\alpha$,therefore

o(α) =L.C.M. +the length of the longest element in α -1= L.C.M. + s_g -1= L.C.M. + f -1.

C: if m≥1 , L≥ 1,g=0 so f=0 ,

$\alpha = \alpha_1 \dots \alpha_m \lambda_1 \lambda_2 \dots \lambda_L$ therefore $\alpha \in T_n$ where we discussed this in theorem above .

Case 3 : m=0 ,L>1 , g > 1 and k=0 ,

A. L.C.M.=0 and f ≥h-1

$\alpha = \lambda_1 \lambda_2 \dots \lambda_L \theta_1 \dots \theta_g$, suppose that $\lambda_1, \lambda_2, \dots, \lambda_L$ are elements of the first kind I ,since f ≥h-1 then by case1:B, $\alpha^f(x_i) = - \ \forall x_i \in \mathrm{Dom}\,\theta_1, \dots, \theta_g$, by theorem above $\alpha^{h-1}(x_i) =$ the fixed element in λj where j= 1, …,L, so $\alpha^h(x_i) =$ the fixed element in λj where j= 1, …,L and by case 1, $\alpha^h(x_i) \ne - \ \forall x_i \in \mathrm{Dom}\,\theta_1, \dots, \theta_g$ unless f=h so α^h is new element in < α > unless f=h , $\alpha^{h+1}(x_i) =$ the fixed element in λj where j= 1, …,L but $\alpha^{h+1}(x_i) \ne - \ \forall x_i \in \mathrm{Dom}\,\theta_1, \dots, \theta_g$ unless f=h+1, and so on, we have $\alpha^f(x_i) =$ the fixed element in λj where j= 1, …,L and $\alpha^f(x_i) = - \ \forall x_i \in \mathrm{Dom}\,\theta_1, \dots, \theta_g$ therefore $\alpha^{f+1}(x_i) = \alpha^f(x_i) \ \forall x_i \in \mathrm{Dom}\,\alpha$ so

o(α)=the length of the longest element in α =f.

B. L.C.M.=0 and f <h-1

since h-1>f then by the same argument $\alpha^{f+1}(x_i) = - \ \forall x_i \in \mathrm{Dom}\,\theta_1, \dots, \theta_g$ so by the theorem above $\alpha^{f+1}(x_i) \ne$ the fixed element in λj where j= 1, …,L unless f+1=h -1,and so on we have $\alpha^{h-1}(x_i) =$ the fixed element in λj where j= 1, …,L and $\alpha^{h-1}(x_i) = - \ \forall x_i \in \mathrm{Dom}\theta_1, \dots, \theta_g$ so

o(α)=the length of the longest element in α-1 = h-1.

In general: $m \ge 1, L \ge 1, g \ge 1$ such that L.C.M. ≥ 2,

Let $\alpha = (\alpha_1 \dots \alpha_m \lambda_1 \lambda_2 \dots \lambda_L)(\theta_1 \dots \theta_g) = w_1 w_2$

Case1: h-1>k then by theorem above $\alpha^{h+L.C.M.-1}(x_i) = \alpha^{h-1}(x_i) \ \forall x_i \in \mathrm{Dom}\,\alpha_1, \dots, \alpha_m, \lambda_1, \dots, \lambda_L$ and by case 2, $\alpha^{L.C.M.+f}(x_i) = \alpha^f(x_i) = - \ \forall x_i \in \mathrm{Dom}\,\theta_1, \dots, \theta_g$ and

$\alpha^{L.C.M.+f}(x_i) = \alpha^f(x_i) \ \forall x_i \in \mathrm{Dom}\,\alpha_1, \dots, \alpha_m$ there are two cases:

A. if h-1<f , h-1>k

$\alpha^{h+L.C.M.}(x_i)=\alpha^h(x_i)\ \forall x_i \in Dom\,\alpha_1,\ldots,\alpha_m,\lambda_1,\ldots,\lambda_L$ but by case 2 above , $\alpha^{h+L.C.M.}(x_i)\neq -$ $\forall x_i \in Dom\,\theta_1,\ldots,\theta_g$ unless f=h therefore will be have new element in< α> unless f=h ,
$\alpha^{h+L.C.M.+1}(x_i)=\alpha^{h+1}(x_i)\ \forall x_i \in Dom\,\alpha_1,\ldots,\alpha_m,\lambda_1,\ldots,\lambda_L$ but not in < α>since
$\alpha^{L.C.M.+h+1}(x_i)=\neq -\ \forall x_i \in Dom\,\theta_1,\ldots,\theta_g$ unless h+1=f
and so on , $\alpha^{L.C.M.+f}(x_i)=\alpha^f(x_i)\forall x_i \in Dom\,\alpha$ so $\alpha^{f+L.C.M}$ is repeated element in < α> so

o(α)=L.C.M.+f-1,special case if k=0

B. if h-1>f since h-1>k
by case 2 above , $\alpha^{L.C.M.+f}(x_i)=\alpha^f(x_i)=-\ \forall x_i \in Dom\,\theta_1,\ldots,\theta_g$ and
$\alpha^{L.C.M.+f}(x_i)=\alpha^f(x_i)\ \forall x_i \in Dom\,\alpha_1,\ldots,\alpha_m$ but
$\alpha^{L.C.M.+f}(x_i)\neq\alpha^f(x_i)\ \forall x_i \in Dom\,\alpha_1,\ldots,\alpha_m,\lambda_1,\ldots,\lambda_L$ unless h-1=f therefore will be have new element in< α> unless f=h-1, $\alpha^{L.C.M.+f+1}(x_i)=\alpha^{f+1}(x_i)\ \forall x_i \in Dom\,\alpha_1,\ldots,\alpha_m,\theta_1,\ldots,\theta_g$ but not in < α> since $\alpha^{L.C.M.+f+1}(x_i)\neq\alpha^{f+1}(x_i)\ \forall x_i \in Dom\,\lambda_1,\ldots,\lambda_L$ unless f+1=h-1, and so on until
$\alpha^{L.C.M.+h-1}(x_i)=\alpha^{h-1}(x_i)\ \forall x_i \in Dom\alpha$ so $\alpha^{L.C.M.+h-1}$ repeated element in < α> so

o(α)=L.C.M.+h-2=o(w_1) where h-1>k in theorem above, special case if k=0

Case2: h-1<k then by theorem above $\alpha^{k+L.C.M.}(x_i)=\alpha^k(x_i)\ \forall x_i \in Dom\,\alpha_1,\ldots,\alpha_m,\lambda_1,\ldots,\lambda_L$
A. if f>h-1 there is two cases first f<k so $\alpha^{L.C.M.+f}(x_i)=\alpha^f(x_i)\forall x_i \in Dom\,\theta_1,\ldots,\theta_g,\alpha_1,\ldots,\alpha_m$ and for each element of the first kind I but $\alpha^{f+L.C.M.}(x_i)\neq\alpha^f(x_i)$ for each element of the first kind II, unless f=k therefore $\alpha^{f+L.C.M.}$ will be new element in <α> unless f=k,
$\alpha^{L.C.M.+f+1}(x_i)=\alpha^{f+1}(x_i)\forall x_i \in Dom\,\theta_1,\ldots,\theta_g,\alpha_1,\ldots,\alpha_m$ and for each element of the first kind I but not in < α> since $\alpha^{f+L.C.M.+1}(x_i)\neq\alpha^{f+1}(x_i)$ for each element of the first kind II unless f+1=k, and so on until $\alpha^{k+L.C.M.}(x_i)=\alpha^k(x_i)\ \forall x_i \in Dom\,\alpha_1,\ldots,\alpha_m,\lambda_1,\ldots,\lambda_L$ and $\forall x_i \in Dom\,\theta_1,\ldots,\theta_g$ so $\alpha^{L.C.M.+k}$ repeated element in < α> therefore

o(α)=L.C.M.+k-1=o(w_1) where h-1<k in theorem above, special case if h=0

Second if f>k then by the similar way above we have $\alpha^{f+L.C.M.}(x_i)=\alpha^f(x_i)$ for each element of the first kind II and $\forall\,\alpha_1,\ldots,\alpha_m$ since f>h-1 $\alpha^{f+L.C.M}(x_i)=\alpha^f(x_i)$ for each element of the first kind II and $\forall\,\alpha_1,\ldots,\alpha_m$ so
$\alpha^{f+L.C.M.}(x_i)=\alpha^f(x_i)\ \forall x_i \in Dom\,\alpha_1,\ldots,\alpha_m,\lambda_1,\ldots,\lambda_L$ and $\forall x_i \in Dom\,\theta_1,\ldots,\theta_g$ so

o(α)=L.C.M.+f-1, special case if h=0 .

B. f<h-1and h-1<k so f<k,
by the same above way and since f<k so $\alpha^{k+L.C.M.}(x_i)=\alpha^k(x_i),\ \forall x_i \in \alpha_1,\ldots,\alpha_m$ and $\forall x_i \in Dom\,\theta_1,\ldots,\theta_g$,since k>h-1 so by theorem above
$\alpha^{k+L.C.M.}(x_i)=\alpha^k(x_i)\ \forall x_i \in Dom\,\alpha_1,\ldots,\alpha_m,\lambda_1,\ldots,\lambda_L$ and since f<k so
$\alpha^{k+L.C.M.}(x_i)=\alpha^k(x_i)\ \forall x_i \in Dom\,\alpha_1,\ldots,\alpha_m,\lambda_1,\ldots,\lambda_L,\theta_1,\ldots,\theta_g$ that is mean this is repeated element in < α> so

o(α)=L.C.M.+k-1,special case if h=0 .

we must note that f $\neq$ 0 because if it is so, we have just proved in case1:C, also L $\neq$ 0 since if it is then h=k=0 so, we proved it in case2 .

Example 2.2:let $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & - \end{pmatrix} \in p_4$, L.C.M.=0 and f=4 so o(α)=4,

Where $<\alpha>=\left\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & - \end{pmatrix},\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & - & - \end{pmatrix},\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & - & - & - \end{pmatrix},\begin{pmatrix} 1 & 2 & 3 & 4 \\ - & - & - & - \end{pmatrix}\right\}$.

l et α=$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 1 & 4 & 5 & 6 & 7 & 7 & 9 & 10 & - \end{pmatrix} \in P_{10}$,

α=$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 1 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 4 & 5 & 6 & 7 & 7 & 8 & 9 & 10 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 10 & - \end{pmatrix} = w_1 w_2$,

L.C.M.=2 , h=5 ,f=2,k=0,note that h-1=4>f and

$$\langle\alpha\rangle = \left\{\begin{matrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 1 & 4 & 5 & 6 & 7 & 7 & 9 & 10 & - \end{pmatrix},\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 5 & 6 & 7 & 7 & 7 & 10 & - & - \end{pmatrix},\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 1 & 6 & 7 & 7 & 7 & 7 & - & - & - \end{pmatrix} \\ ,\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 7 & 7 & 7 & 7 & 7 & - & - & - \end{pmatrix},\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 1 & 7 & 7 & 7 & 7 & 7 & - & - & - \end{pmatrix}\end{matrix}\right\}$$

So o(α)=5.

Now let α=$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 1 & 6 & 6 & 8 & 9 & - \end{pmatrix} \in P_9$,

α=$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 1 & 5 & 6 & 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 6 & 6 & 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & - \end{pmatrix} = w_1 w_2$

h=2 ,f=3and L.C.M.=4 ,so by corollary above o(α)= L.C.M.+f-1=6 since $\alpha^7 = \alpha^3$

3.conclusions:

In this paper we have discussed two important semigroups in the semigroup theory ,namely T_n and P_n . The element in a symmetric semigroup (T_n) is a mapping from $\mathrm{Dom}\,\alpha = X_n \to X_n$ and the binary operation defined on it is the usual composite of mapping .The order of an element in any semigroup practical T_n semigroup is the order of a cyclic subsemigroup generated by this element and since the elements of T_n can be classified into elements of the first kind I or of the first kind II, we observation that every element in T_n can be written as a product of one or more than one of cycles or elements of the first kind I or of the first kind II as we explain above we use these fact to find the order for each element in T_n by compute the number of elements in the subsemigroup generated by this element ,by similar way and by observation that every element in P_n can be written as a product of one or more than one of cycles or elements of the first kind I or of the first kind II or element of the second kind we find the order for each element in P_n semigroup .

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