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Topological Quasi Projective Modules

المقاسات الشبه اسقاطية التبولوجية

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Abstract

In this search obtain the following results on a topological quasi projective modules and on a topological quasi projective cover :

1. A topological direct sum of two topological quasi projective modules each one has a topological projective cover is a topological quasi projective module has a topological projective cover which is a direct sum of topological projective cover of it is summand.
2. A topological finite direct sum of a topological quasi projective modules is topological quasi projective if each one has a topological projective cover.
3. A topological module has a topological projective cover if it has a topological quasi projective cover which is unique up to a topological isomorphism over the identity on topological module.

المستخلص //

في هذا البحث حصلنا على النتائج الاتية في المقاسات شبه الاسقاطية التوبولوجية وكذلك في الاغطية شبه الاسقاطية التوبولوجية:

١. مركبة جمع مباشرتوبولوجي لمقاسين شبه اسقاطيين توبولوجيين وكل واحد منهما يمتلك غطاءً اسقاطياً توبولوجيا مقاس شبه اسقاطي توبولوجي يمتلك غطاءً اسقاطياً توبولوجياً عبارة عن مركبة جمع باشرتوبولوجي لغطائين اسقاطيين توبولوجيين لمركبتي الجمع المباشر التوبولوجي.

٢. الجمع المباشر المنتهي التوبولوجي للمقاسات شبه الاسقاطية التوبولوجية يكون مقاس شبه اسقاطي توبولوجي اذا كان كل واحد منها يمتلك غطاءً اسقاطياً توبولوجيا .

٣.اذا امتلك المقاس التوبولوجي غطاءً اسقاطياً توبولوجيا فانه يمتلك غطاءً شبه اسقاطياً توبولوجيا ويكون وحيد عندما يكون التشاكل التوبولوجي متقابل بالنسبة للعنصر المحايد للمقاسات التوبولوجية .

Introduction:

Kaplanski is the first scientist who introduced the definition of topological module and topological sub module in 1955,after that, a topological module studied by many scientists like Alberto Tolono and Nelson. In this research a necessary and sufficient conditions for a topological module to be topological direct projective has been established ,the kind of topology don't determinant and the results in this research algebraically satisfied . we will satisfying the results topologically.

This research divided into two sections. In section one includes some necessary definitions while section two includes propositions , references [1],[2]and[3]are used to construct some definitions in this research .

***Section " ONE" Some Definitions*:**

In this section introduce some necessary definitions on a topological quasi rojective modules and on topological quasi projective covers.

Definition (1–1): Let *R* be a topological ring. A non empty set *E* is said to be a topological module on *R* if:
1. E is a module on *R*;
2. E is a topological group;

3. A mappings $\pi : E \times E \to E$ and $\mu : E \to E$ are continuous, where μ defined as $\mu(x) \cong x^{-1}$, and $E \times E$ is the product of two topological spaces.

Definition (1–2) : Let *E* be a topological module on a topological ring *R*. A subset M of E is said to be a topological sub module of *E* if:
1. M is a sub module of *E*;
2. M is a topological subgroup of *E*.

Definition (1–3): A topological module *E* is said to be finitely generated topological module if it is generated by a finite set. If it is generated by one element it is called cyclic topological module.

Definition (1–4): Let E, E^* be two topological modules on a topological ring *R*. A mapping $f : E \to E^*$ is called a topological module morphism if :
1. f is a homomorphism;
2. f is continuous.

Definition (1–5): A topological module epimorphism is an onto topological module morphism.

Definition (1–6): A topological module monomorphism is a one to one topological module morphism.

Definition (1–7): Let E, E^* be two topological modules and $f : E \to E^*$ a topological morphism. A topological kernel of *f* is the set $Kerf \cong \{t \in E : f(t) \cong e'\}$, where e' the identity element of E^*.

Definition (1–8) [5]: Let $\{E_\lambda\}_{\lambda \in \Omega}$ be a family of topological modules on topological ring *R*, the set $E = \prod_{\lambda \in \Omega} E_\lambda$ with projection morphism is a topological direct product module on *R*, where a continuous morphism $f : R \times E \to E$ defined as $f(\alpha(X_\lambda))_{\lambda \in \Omega} = \alpha(X_\lambda)_{\lambda \in \Omega}$.

Definition (1–9) [5]: A topological module *P* is said to be a topological projective module if for each topological module morphism $f : P \to B$ and for each topological module epimorphism $g : A \to B$ there is a topological module morphism $f^* : P \to A$ for which the following diagram commutes

$$\begin{tikzcd} & P \arrow[ld, dashed, "f^*"'] \arrow[d, "f"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

Definition (1–10) [5]: A topological module epimorphism $f^*: P \to M$ called a topological projective cover of M if P is a topological module and if f^* is small topological module epimorphism.

Definition (1–11): Let R be a topological ring, and $0 \to L \to A_1 \to B \xrightarrow{f} A_2 \to 0$ a short exact sequence of topological *R*-module morphism, then the sequence called a topological splits sequence if $fof^{-1} \cong I_{A_2}$.

Definition(1–12) : $Q(P(M)) \xrightarrow{\pi'} M \longrightarrow 0$ will be called a topological quasi projective cover of M provided ;

1. $Q(P(M))$ is a topological quasi projective ;

2. If $X \oplus Ker\pi' \cong Q(P(M))$ then $X \cong Q(P(M))$;

3. $0 \neq T \subseteq Ker\pi'$ then $Q(P(M))/T$ is a topological quasi projective.

Definition (1–13): A Projection morphism is a topological morphism from a topological module onto each one of it is summands.

Definition (1–14): $End(M)$ is the set of all one to one topological morphism from M onto M .

Definition (1–15): A topological module *M* is said to be a topological quasi projective if for each topological module morphism $\alpha: N \to M$ and for each topological module monomorphism $f: M \to M$ there is a topological module morphism $f^*: M \to N$ for which the following diagram commutes

$$\begin{tikzcd} & & M \arrow[dll, dashed, "f^*"'] \arrow[d, "f"] \\ f \quad N \arrow[rr, "\alpha"'] & & M \end{tikzcd}$$

Zorn’s Lemma [6] : Let *A* be a set that is strictly partially ordered. If every simply ordered subset of *A* has an upper bound in *A*, then *A* has a maximal element.

Section” TWO” Some Propositions:

In this section introduce some necessary propositions and the relation between a topological projective cover and a topological quasi projective cover on a topological quasi projective module.

Proposition (2-١) : If M is a topological quasi projective module and has a topological projective cover $P(M) \xrightarrow{\pi} M \longrightarrow 0$ and $P(M) \cong P(M_1) \oplus P(M_2)$ then $M \cong M_1 \oplus M_2$ and $P(M_1) \xrightarrow{\pi_1} M_1 \to 0$ is a topological projective cover of M_1 where $\pi_1 \cong \pi|p(M_1)$. also $P(M_2) \xrightarrow{\pi_2} M_2 \to 0$ is a topological projective cover of M_2 where $\pi_2 \cong \pi|p(M_2)$.

Proof : Since $Ker\pi$ is a topological submodule of $P(M)$,also $Ker\pi_1$ is a topological submodule of $P(M_1)$, $Ker\pi_2$ is a topological submodule of $P(M_2)$, we have $Ker\pi \cong (Ker\pi_1) \oplus (Ker\pi_2)$ .Thus we can induce a decomposition in M_1 by letting $M_1 \cong P(M_1) \oplus (Ker\pi_1)$ it follows that $0 \longrightarrow (Ker\pi_1) \longrightarrow P(M_1) \xrightarrow{\pi_1} M_1 \longrightarrow 0$ is exact and $M_2 \cong P(M_2) \oplus (Ker\pi_2)$ it follows that $0 \longrightarrow (Ker\pi_2) \longrightarrow P(M_2) \xrightarrow{\pi_2} M_2 \longrightarrow 0$ is also exact, then $M_1 \oplus M_2 \cong P(M_1) \oplus (Ker\pi_1) \oplus P(M_2) \oplus (Ker\pi_2) \cong M$ so that $M \cong M_1 \oplus M_2$. Now if $X \subseteq P(M_1)$ Such that $X \oplus Ker\pi_1 \cong P(M_1)$ then $X \oplus P(M_2) \oplus Ker\pi \cong P(M)$ So $X \oplus P(M_2) \cong P(M)$. Therefore $X \cong P(M_1)$ and it is follows that $P(M_1) \xrightarrow{\pi_1} M_1 \longrightarrow 0$ is a projective cover of M_1, by same way $P(M_2) \xrightarrow{\pi_2} M_2 \to 0$ is a topological projective cover of M_2 where $\pi_2 \cong \pi | p(M_2)$.

Proposition (2-2) : If a topological quasi projective modules M_i has a projective covers $\overset{n}{\underset{i=1}{\oplus}} P(M_i)$ then $\overset{n}{\underset{i=1}{\oplus}} M_i$ also quasi projective module.

Proof : Let $0 \longrightarrow Ker\pi \longrightarrow P(M) \xrightarrow{\pi} M \longrightarrow 0$ be a projective cover of M , since a projective cover of $\overset{n}{\underset{i=1}{\oplus}} M_i$ is $\overset{n}{\underset{i=1}{\oplus}} P(M_i)$, now it is sufficient to show a topological kernel of

$\overset{n}{\underset{i=1}{\oplus}} P(M_i) \xrightarrow{\overset{n}{\underset{i=1}{\oplus}}\pi_i} \overset{n}{\underset{i=1}{\oplus}} M_i \longrightarrow 0$ is a topological submodule of $\overset{n}{\underset{i=1}{\oplus}} P(M_i)$ .A topological kernel of the above morphism is $\overset{n}{\underset{i=1}{\oplus}} Ker(\pi_i)$ .since it is clear that any morphism $f_{ij} : P_i(M) \longrightarrow P_j(M)$ from the i^{th} copy of $P(M)$ to the j^{th}) must carry $Ker(\pi_i)$ in to $Ker(\pi_j)$ (because $Ker\pi$ is a $(P(M))$ topological module), it follows that $\overset{n}{\underset{i=1}{\oplus}} Ker\pi_i$ is a topological sub module of $\overset{n}{\underset{i=1}{\oplus}} P(M_i)$.

We have the following existence theorem for quasi projective cover ;

Theorem (2-3) : If a topological module M has a topological projective cover $P(M) \xrightarrow{\pi} M \longrightarrow 0$ then it has a topological quasi projective cover $Q(P(M)) \xrightarrow{\pi'} M \longrightarrow 0$ which is unique up to a topological isomorphism over the identity on M .

Proof : Let X be the unique topological maximal submodule of $P(M)$ contained in $Ker\pi$. Existence is assumed by Zorn's lemma and uniqueness follows from the fact that a topological direct sum of two topological submodules of $P(M)$ contained in $Ker\pi$ is again contained in $Ker\pi$. Now let $Q(P(M)) \cong P(M)|X$ mapping onto M by the induced mapping π'. By definition(1–1٢) $Q(P(M))$ is topological quasi projective cover. If $Y' \oplus Ker\pi' \cong Q(P(M))$ then $Y \oplus Ker\pi \cong P(M)$ where Y is the pre-image in $P(M)$ of Y' . It follows that $Y \cong P(M)$ and $Y' = Q(P(M))$ condition (3) for a topological quasi projective cover is satisfied by the maximality of X . Now we test uniqueness . Suppose that M has another topological quasi projective cover $Z \xrightarrow{\delta} M \longrightarrow 0$ satisfying conditions (1),(2) and (3) for a topological quasi projective cover. Since $P(M)$ is a topological projective cover we have the following commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\delta"] & M \arrow[r, dashed] & 0 \\ P(M) \arrow[u, "\mu"] \arrow[r, "\pi"'] & M \arrow[u, dashed, "i_M"'] \arrow[r] & 0 \end{tikzcd}$$

Since $\operatorname{Im}\mu \oplus Ker\delta = Z$ it follows from condition (2) that μ is an epimorphism and $Ker\mu \subseteq Ker\pi$ .Therefore the mapping $P(M) \xrightarrow{\delta} Z \longrightarrow 0$ is a topological projective cover of Z and by proposition (٢-٢) $Ker\mu$ is a topological submodule of $P(M)$. By the choice of X above we see $X \supseteq Ker\mu$ where X properly contained $Ker\mu$ we would have that $0 \neq \mu(X) \subseteq Keri_{\mu}$ with $Z|\mu(X) \cong P(M)|X \cong Q(P(M))$ which is a topological quasi projective . But that would contradict condition (3)for $Z \longrightarrow M \longrightarrow 0$ to be a topological quasi projective cover therefore $X = Ker\mu$ and μ induces an topological isomorphism μ' so that the following diagram is commutative

$$\begin{tikzcd} Z \arrow[r, "\delta"] & M \arrow[r] & 0 \\ Q(P(M)) \arrow[u, "\mu'"] \arrow[r, "\pi"'] & M \arrow[u, dashed, "i_M"'] \arrow[r] & 0 \end{tikzcd}$$

This completes the proof of this proposition .

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