On Some Relationships Between Spectral Sequences And The New Exact Sequences

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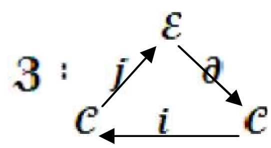
الخلاصة

هذا البحث يصف علاقة ربط بين مفهوم المتتالية الطيفية والمتتالية الجديدة المضبوطة . وجدت أن المتتالية الجديدة المضبوطة هي المشتقة الثانية للثنائي المضبوط .
ليكن

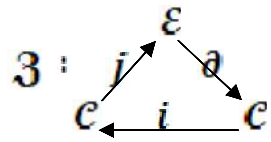


ثنائي مضبوط . نشتق من $\mathfrak{Z}$ الثنائي المضبوط (الثاني) $\mathfrak{Z}^1 = \mathbf{NES}$ الذي هو المتتالية الجديدة المضبوطة ونشتق من $\mathfrak{Z}^1$ الثنائي المضبوط (الثالث) $\mathfrak{Z}^2$ ونستمر بهذا العمل . عملية الاشتقاق هذهِ تولد متتالية غير منتهية من الثنائيات المضبوطة .
حصلنا على بعض النتائج ، منها :
يشكل صف كل الثنائيات المضبوطة والتشاكلات بين هذهِ الثنائيات فصيلة $\mathfrak{T} = (\mathfrak{Z}, \mathcal{F})$ ، يوجد مقرن من فصيلة المجمعات cw الى $\mathfrak{T}$.
أذا كان كلاً من $\mathbf{K}, \mathbf{L}$ مجمعاً متصلاً وكانا متكافئان هوموتوبياً فـــأن $\mathfrak{Z}^{\mathrm{r}}(\mathbf{K}) \cong \mathfrak{Z}^{\mathrm{r}}(\mathbf{L}) \quad \forall \mathrm{r}, \mathrm{r} = 0, 1, 2, \cdots$.
أذا كان K مجمعا متصلا من نوع $(n-1)$ فـــأن
$\mathcal{E}^r_{p,q} = 0 \quad \forall r \ \& \forall p, p = 1,2,\cdots,n-1$ **, and** $\mathcal{E}^r_{n,q} \cong \mathcal{C}^r_{n,q} \quad \forall r$ **.**

Abstract

In this paper I introduce and study relationships between spectral sequences and the new exact sequences . I will show that the new exact sequence is the second derived exact couple .

Let



be an exact couple , we derive $\mathfrak{Z}$ to get the second exact couple $\mathfrak{Z}^1 = NES$ which is the new exact sequence . the process of derivation can be iterated indefinitely, yielding an infinite sequence of exact couples .

I establish some results ,for examples ;

The class of all exact couples and morphisms between these couples formed a category $\mathfrak{T} = (\mathfrak{Z}, \mathcal{F})$. There is a functor from the category of $cw - complexes$ into $\mathfrak{T}$.

If $K \equiv L$ $(are\ cw - complexes),\ then$
$\mathfrak{Z}^r(K) \cong \mathfrak{Z}^r(L) \quad \forall r, r = 0, 1, 2, \cdots$.

If K be any $(n-1) - connected\ complex$, then $\mathcal{E}^r_{p,q} = 0 \quad \forall r\ \&\ \forall p, p = 1, 2, \cdots, n-1$, and $\mathcal{E}^r_{n,q} \cong \mathcal{C}^r_{n,q} \quad \forall r$.

Introduction

In this work I introduce and study relationships between spectral sequences and the new exact sequences (in short NES) introduced in [D] . I mean by the term " complex " in the sequel a " connected cw-complex ", see [H] .

This work contains two sections ; in first section , I construct a spectral sequence from exact couples and I construct a categories whose objects are exact couples and morphisms between these couples .

In second section ,I establish some results about our work , some of these results are purely algebraic and others depend on the topology of space , for examples ;

If K be any $(n-1) - connected\ complex$, then $\mathcal{E}^r_{p,q} = 0 \quad \forall r\ \&\ \forall p, p = 1, 2, \cdots, n-1$, and $\mathcal{E}^r_{n,q} \cong \mathcal{C}^r_{n,q} \quad \forall r$.

<u>Section 1</u> (Construction)

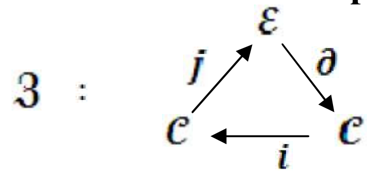
Let K be a cw-complex . For each integer p and q , let $\mathcal{E}_{p,q}\ be\ \pi_{p+q}(K^p, K^{p-1})\ if\ q \geq -p$ and zero otherwise , and

$\mathcal{C}_{p,q}$ *be* $\pi_{p+q}(K^p)$ *if* $q \geq -p$ **and zero otherwise .**

Consider the following sequence which is known to be exact , see [D] ,

$$\cdots \to \pi_{p+q}(K^{p-1}) \xrightarrow{i_{p,q}} \pi_{p+q}(K^p) \xrightarrow{j_{p,q}} \pi_{p+q}(K^p, K^{p-1}) \xrightarrow{\partial_{p,q}} \pi_{p+q-1}(K^{p-1}) \to \cdots$$

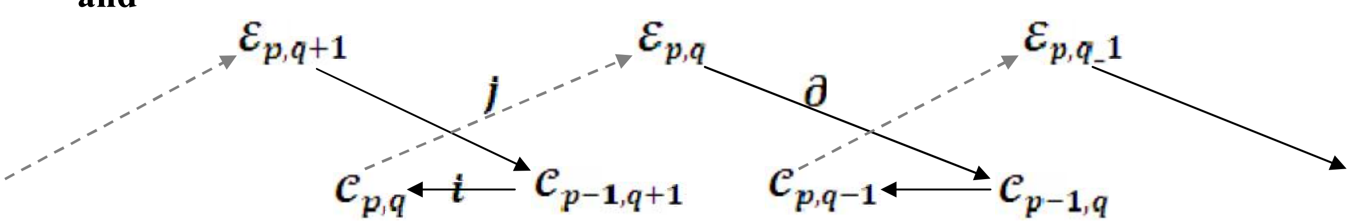
which forms a first exact couple ,



where

∂ *whose degree* $(-1,0)$
i *whose degree* $(1,-1)$
j *whose degree* $(0,0)$

and



From this exact couple, we construct a second exact couple, which is itself the new exact sequence (NES), (see [D]), by taking

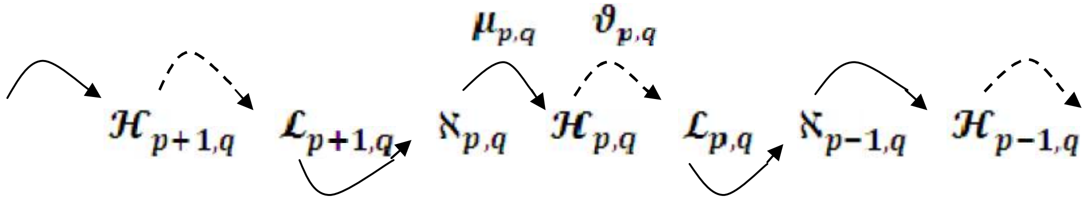
$$\mathcal{E}^1_{p,q} = {}^{ker.d_{p,q}}\!\big/_{im.d_{p-1,q}} \quad (= \mathcal{H}_{p,q})$$

$$\mathcal{C}^1_{p,q} = {}^{\mathcal{C}_{p,q}}\!\big/_{im.\partial_{p+1,q}} \quad (= \aleph_{p,q})$$

$$\mathcal{C}^1_{p,q} = ker.j_{p,q} \quad (= \mathcal{L}_{p,q})$$

where $d_{p,q} = j_{p-1,q} \circ \partial_{p,q}$.

and

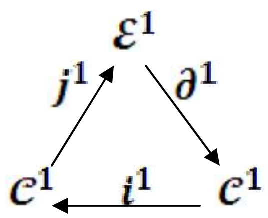


$\theta_{p,q}$

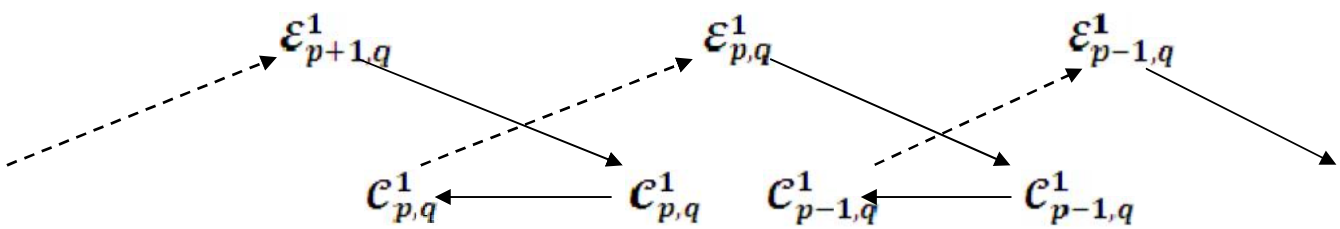
It is easy to see that the NES is an exact sequence, (for details see [D])

Hence we have a second exact couple ;

$\mathfrak{Z}^1$:



where



Now, we will construct a third exact couple .

Let $d^1_{p,q} = j^1 \circ \partial^1 : \mathcal{E}^1_{p,q} \to \mathcal{E}^1_{p-1,q}$.

It is clear that $d^1 \circ d^1 = 0$ **, implies that** $\mathcal{E}^1$ **is a chain complex .**

Denote

$$\mathcal{E}^2 = {ker.d^1}/{im.d^1}$$

$$\mathcal{C}^2 = ker.j^1 = im.i^1$$

$$\mathcal{C}^2 = {\mathcal{C}^1}/{ker.i^1}$$

Define

$$i^2 : \mathcal{C}^2 \longrightarrow \mathcal{C}^2 \quad by \quad i^2(x) = [x], \qquad \forall x \in \mathcal{C}^2 ,$$

$$j^2 : \mathcal{C}^2 \longrightarrow \mathcal{E}^2 \quad by \quad j^2([y]) = [j^1(y)], \ \forall [y] \in \mathcal{C}^2 ,$$

and $\partial^2 : \mathcal{E}^2 \longrightarrow \mathcal{C}^2 \quad by \quad \partial^2([z]) = \partial^1(z), \ \forall [z] \in \mathcal{E}^2$.

<u>Remark 1.1</u>

The homomorphisms i^2 , j^2 and ∂^2 are well defined .

To prove j^2 is well defined ;

Let $[y] \in \mathcal{C}^2 = \mathcal{C}^1/_{ker.i^1}$, $y \in \mathcal{C}^1 \Rightarrow j^1(y) \in \mathcal{E}^1$

$$\Rightarrow \quad d^1\left(j^1(y)\right) = j^1 \circ \partial^1\left(j^1(y)\right) = 0$$

$$\Rightarrow \quad j^1(y) \in ker.d^1 \quad \Rightarrow \quad [j^1(y)] \in \mathcal{E}^2$$

Now , suppose $[y_1] = [y_2]$

$$\Rightarrow \quad y_1 - y_2 \in ker.i^1 = im.\partial^1$$

$$\Rightarrow \exists\, x \in \mathcal{E}^1 \quad such\,that \quad \partial^1(x) = y_1 - y_2$$

$$\Rightarrow \quad j^1(y_1 - y_2) = j^1\partial^1(x) \in im.d^1 \quad \Rightarrow \quad [j^1(y_1)] = [j^1(y_2)] .$$

To prove ∂^2 is well defined ;

Let $[z] \in \mathcal{E}^2 = ker.d^1/_{im.d^1}$, $z \in ker.d^1$

$$\Rightarrow \quad 0 = d^1(z) = j^1 \circ \partial^1(z) \quad \Rightarrow \quad \partial^1(z) \in ker.j^1 = im.i^1 = \mathcal{C}^2 .$$

Now , suppose $[z_1] = [z_2]$

$$\Rightarrow \quad z_1 - z_2 \in im.d^1 , \Rightarrow \exists\, y \in \mathcal{E}^1 \quad such\,that \quad d^1(y) = z_1 - z_2$$

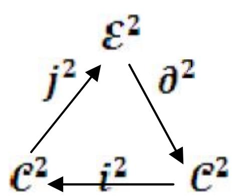
$$\Rightarrow \quad \partial^1(z_1 - z_2) = \partial^1\left(d^1(y)\right) = \partial^1\left(j^1 \circ \partial^1(y)\right) = 0$$

$$\Rightarrow \quad \partial^1(z_1) = \partial^1(z_2) .$$

And it is clear that i^2 is well defined ■

<u>Theorem 1.2</u>

The diagram



is an exact couple .

<u>proof</u>

First , to prove ; $im.j^2 = ker.\partial^2$.

Let $a \in \mathcal{C}^2$, $a = [x]$, $where$ $x \in \mathcal{C}^1$

$$\partial^2\left(j^2(a)\right) = \partial^2([j^1(x)]) = \partial^1\left(j^1(x)\right) = 0 , \Rightarrow im.j^2 \subseteq ker.\partial^2 .$$

Let $b \in \mathcal{E}^2$, $b = [z]$, $where$ $z \in ker.d^1$

assume that $\partial^2(b)=0 \Rightarrow \partial^1(z)=0 \Rightarrow z\in ker.\partial^1 = im.j^1$
$\Rightarrow z=j^1(y)$ $for\,some$ $y\in\mathcal{C}^1 \Rightarrow [y]\in\mathcal{C}^2$
so we let $j^2([y])=[j^1(y)]=[z]=b \Rightarrow ker.\partial^2\subseteq im.j^2$.

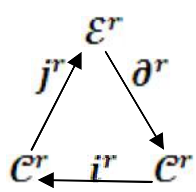
Second , to prove ; $im.i^2 = ker.j^2$.
Let $a\in\mathcal{C}^2$, $a=[x]$, $where$ $x\in\mathcal{C}^1$
assume $j^2(a)=0 \Rightarrow [j^1(x)]=0$ in $\mathcal{E}^2 = {ker.d^1}/{im.d^1}$
$\Rightarrow j^1(x)\in im.d^1$ $so\,that$ $d^1(y)=j^1(x)$ $for\,some$ $y\in\mathcal{E}^1$
thus $j^1\partial^1(y)=j^1(x) \Rightarrow \left(x-\partial^1(y)\right)\in ker.j^1 = im.i^1=\mathcal{C}^2$
so we let $i^2\left(x-\partial^1(y)\right)=[x]=a \Rightarrow ker.j^2\subseteq im.i^2$.
Let $x\in\mathcal{C}^2 = im.i^1 = ker.j^1$
$\Rightarrow j^2\left(i^2(x)\right)=j^2([x])=[j^1(x)]=0 \Rightarrow im.i^2\subseteq ker.j^2$.

Final , to prove ; $im.\partial^2 = ker.i^2$.
Let $b\in\mathcal{E}^2$, $b=[z]$, $where$ $z\in ker.d^1$
$\Rightarrow i^2\left(\partial^2(b)\right)=i^2\left(\partial^2([z])\right)=i^2\left(\partial^1(z)\right)=0 \Rightarrow im.\partial^2\subseteq ker.i^2$.
Let $x\in\mathcal{C}^2 = im.i^1 = ker.j^1$
assume $i^2(x)=[x]=0$ in $\mathcal{C}^2 = {\mathcal{C}^1}/{ker.i^1}$
$\Rightarrow x\in ker.i^1 = im.\partial^1 \Rightarrow \partial^1(y)=x$, $for\,some$ $y\in\mathcal{E}^1$
so $d^1(y)=j^1\partial^1(y)=j^1(x)=0 \Rightarrow y\in ker.d^1$
Let $[y]\in\mathcal{E}^2 \Rightarrow \partial^2([y])=\partial^1(y)=x \Rightarrow x\in im.\partial^2$
$\Rightarrow ker.i^2\subseteq im.\partial^2$ ■

Remark 1.3

The exact couple $\mathfrak{Z}^1$ is called the derived couple of the original exact couple $\mathfrak{Z}$. the process of derivation can be iterated indefinitely, yielding an infinite sequence of exact couples ;

$\mathfrak{Z}^r$:

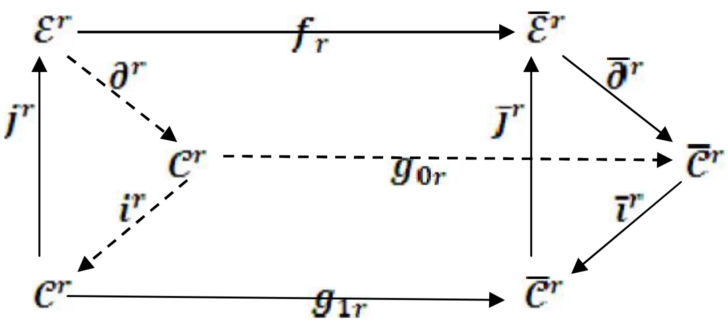


$r=0,1,2,\cdots$

such that $\mathfrak{Z}^0 = \mathfrak{Z}$, $\mathfrak{Z}^1 = NES$ *and* $\mathfrak{Z}^{r+1}$ is the derived couple of $\mathfrak{Z}^r$. The endomorphism $d^r = j^r \circ \partial^r$ has the property that $d^r \circ d^r = 0$, so that $\mathcal{E}^r$ is a chain complex under d^r, whose homology group is $\mathcal{E}^{r+1}$. In this way we obtain a spectral sequence .

<u>Definition 1.4</u>

Define a morphism between two exact couple $\mathfrak{Z}^r , \overline{\mathfrak{Z}}^r$ $\mathcal{F}_r : \mathfrak{Z}^r \to \overline{\mathfrak{Z}}^r$, we mean a family of homomorphisms (f_r, g_r) showing in the following diagram



such that

$$\overline{\partial}^r \circ f_r = g_{0r} \circ \partial^r$$
$$\overline{i}^r \circ g_{0r} = g_{1r} \circ i^r$$
$$\overline{j}^r \circ g_{1r} = f_r \circ j^r$$

So that $\mathcal{F}_r$ is a chain morphism and induces a morphism $\mathcal{F}_r^1 = \mathcal{F}_{r+1}$, and f_r^1 , g_r^1 defined a maps between the derived couples , such that ;

$f_{r+1} : \mathcal{E}^{r+1} \to \overline{\mathcal{E}}^{r+1}$ *defined by* $f_{r+1}([z]) = [f_r(z)]$, $\forall [z] \in \mathcal{E}^{r+1}$.
$g_{0r+1} : \mathcal{C}^{r+1} \to \overline{\mathcal{C}}^{r+1}$ *defined by* $g_{0r+1}(x) = g_{0r}(x)$, $\forall x \in \mathcal{C}^{r+1}$.
$g_{1r+1} : \mathcal{C}^{r+1} \to \overline{\mathcal{C}}^{r+1}$ *defined by* $g_{1r+1}([x]) = [g_{1r}(x)]$, $\forall x \in \mathcal{C}^{r+1}$.

Now, let $\mathcal{F}_r : \mathfrak{Z}^r \to \overline{\mathfrak{Z}}^r$ *and* $\mathcal{F}_r^* : \overline{\mathfrak{Z}}^r \to \overline{\overline{\mathfrak{Z}}}^r$ are two morphisms , we define the composition of morphisms as following ;

$\mathcal{F}_r^* \circ \mathcal{F}_r = (f_r^* \circ f_r , g_r^* \circ g_r) : \mathfrak{Z}^r \to \overline{\overline{\mathfrak{Z}}}^r$

it is easy to show that $\mathcal{F}_r^* \circ \mathcal{F}_r$ is a homomorphism .

<u>Remark 1.5</u>

The homomorphisms f_{r+1}, g_{0r+1} *and* g_{1r+1} **are well defined .**

To prove f_{r+1} **is well defined ;**

Let $[z] \in \mathcal{E}^{r+1} = \ker.d^r / im.d^r$, $z \in \ker.d^r \subseteq \mathcal{E}^{r+1}$.

$$\bar{d}^r(f_r(z)) = \bar{j}^r \circ \bar{\partial}^r(f_r(z)) = \bar{j}^r(g_{0r} \circ \partial^r)(z)$$
$$= \bar{j}^r\left(g_{0r}(\partial^r(z))\right) = \bar{j}^r(g_{1r}(\partial^r(z)))$$
$$= (\bar{j}^r \circ g_{1r})(\partial^r(z))) = (f_r \circ j^r)(\partial^r(z))$$
$$= f_r(j^r \circ \partial^r(z)) = f_r(d^r(z)) = 0$$

$\Rightarrow \quad f_r(z) \in \ker.\bar{d}^r \quad \Rightarrow \quad [f_r(z)] \in \bar{\mathcal{E}}^{r+1}$ .x

Assume $[z_1] = [z_2]$

$\Rightarrow \; z_1 - z_2 \in im.d^r$ *that is* $d^r(y) = z_1 - z_2$ *for some* $y \in \mathcal{E}^r$

$$\Rightarrow f_r(z_1 - z_2) = f_r(d^r(y)) = f_r(j^r \circ \partial^r(y)) = (f_r \circ j^r)\partial^r(y)$$
$$= (\bar{j}^r \circ g_{1r})\partial^r(y) = \bar{j}^r\left(g_{1r}(\partial^r(y))\right) = \bar{j}^r(g_{0r}(\partial^r(y)))$$
$$= \bar{j}^r\left(\bar{\partial}^r \circ f_r(y)\right) = \bar{d}^r(f_r(y))$$

but $f_r(y) \in \bar{\mathcal{E}}^r \quad \Rightarrow \quad f_r(z_1 - z_2) \in im.\bar{d}^r$

and $\bar{\mathcal{E}}^{r+1} = \ker.\bar{d}^r / im.\bar{d}^r$, *that is* $[f_r(z_1)] = [f_r(z_2)]$.

To prove g_{0r+1} **is well defined ;**

Let $x \in \mathcal{C}^{r+1} = \ker.j^r = im.i^{r+1} \subseteq \mathcal{C}^r$

$$\bar{j}^r(g_{0r}(x)) = \bar{j}^r(g_{1r}(x)) = f_r \circ j^r(x) = f_r(0) = 0$$

$\Rightarrow \quad g_{0r}(x) \in \ker.\bar{j}^r \quad \Rightarrow \quad g_{0r+1}(x) \in \bar{\mathcal{C}}^{r+1}$

Assume $x_1 = x_2 \;\Rightarrow\; g_{0r}(x_1) = g_{0r}(x_2) \;\Rightarrow\; g_{0r+1}(x_1) = g_{1r+1}(x_2)$.

To prove g_{1r+1} **is well defined ;**

Let $[x] \in \mathcal{C}^{r+1} = \mathcal{C}^r / \ker.i^r$, $x \in \mathcal{C}^r \quad \Rightarrow \quad g_{1r}(x) \in \bar{\mathcal{C}}^r$

$\Rightarrow \quad [g_{1r}(x)] \in \bar{\mathcal{C}}^{r+1} = \bar{\mathcal{C}}^r / \ker.\bar{i}^r$

Assume $[x_1] = [x_2] \quad \Rightarrow \quad x_1 - x_2 \in \ker.i^r \subseteq \bar{\mathcal{C}}^r$

$$\bar{i}^r(g_{1r}(x_1 - x_2)) = \bar{i}^r(g_{0r}(x_1 - x_2)) = g_{1r} \circ i^r(x_1 - x_2) = 0$$

$\Rightarrow \quad g_{1r}(x_1 - x_2) \in \ker.\bar{i}^r \quad \Rightarrow \quad [g_{1r}(x_1)] = [g_{1r}(x_2)]$ ■

We shall describe $\mathcal{F}_r : \mathfrak{Z}^r \to \bar{\mathfrak{Z}}^r$ **is an isomorphism , if and only if ,**

f_r, g_{0r} & g_{1r} are an isomorphisms . We shall describe $\mathfrak{Z}^r$ as isomorphic to $\bar{\mathfrak{Z}}^r$ and write $\mathfrak{Z}^r \cong \bar{\mathfrak{Z}}^r$, if and only if , there is an isomorphism $\mathcal{F}_r : \mathfrak{Z}^r \to \bar{\mathfrak{Z}}^r$.

Remark 1.6

The relation , $\cong$, is an equivalence relation .

Section 2 (Results and Conclusion)

From remark 1.3 , definition 1.4 and (theorem 3 in [D]), we have the following theorem

Theorem 2.1

The class of all exact couples and the homomorphisms between these couples forms a category $\mathfrak{T} = (\mathfrak{Z}^r, \mathcal{F}_r)$.

From remark 1.3 , definition 1.4 , (theorem6 in [D]) and (see [S]) we have the following theorem

Theorem 2.2

There is a functor from the category of $cw-complexes$ into $\mathfrak{T}$. Denote it by $\mathfrak{F}$, where $\mathfrak{F}(K) = \mathfrak{Z}^r(K)$ and $\mathfrak{F}(f) = \mathcal{F}_r$.

Theorem 2.3

In the second exact couple $\mathfrak{Z}^1 (= NES)$.
If $K\ is\ (n-1)-connected\ complex$. Then $\aleph_{n+1,q}(K)$ is an extension of $\mathcal{H}_{n+1,q}(K)$ by $\mathcal{L}_{n+1,q}(K) \big/ \vartheta_{n+2,q}(\mathcal{H}_{n+2,q}(K))$.

Proof

From our property of NES [D] , we have $\aleph_{n,q}(K) \cong \mathcal{H}_{n,q}(K)$ and $\mu_{n+1,q}$ $is\ onto$, we extract from $NES(= \mathfrak{Z}^1)$ the subsequence

$$\cdots \longrightarrow \mathcal{H}_{n+2,q}(K) \xrightarrow{\vartheta_{n+1,q}} \mathcal{L}_{n+1,q}(K) \xrightarrow{\theta_{n+1,q}} \aleph_{n+1,q}(K) \xrightarrow{\mu_{n+1,q}} \mathcal{H}_{n+1,q}(K) \longrightarrow 0$$

from exactness we have $ker.\mu_{n+1,q} = \theta_{n+1,q}(\mathcal{L}_{n+1,q}(K))$
but $\theta_{n+1,q}$ **maps** $\mathcal{L}_{n+1,q}(K)$ *onto* $\theta_{n+1,q}(\mathcal{L}_{n+1,q}(K))$
with $ker.\theta_{n+1,q} = \vartheta_{n+1,q}(\mathcal{H}_{n+2,q}(K))$
then from ***fundamental homomorphism theorem*** **, we have**

$$\theta_{n+1,q}\left(\mathcal{L}_{n+1,q}(K)\right) \cong \mathcal{L}_{n+1,q}(K) \big/ ker.\theta_{n+1,q}$$

$$= \mathcal{L}_{n+1,q}(K) \big/ \vartheta_{n+2,q}(\mathcal{H}_{n+2,q}(K))$$

but $\mu_{n+1,q}\left(\aleph_{n+1,q}(K)\right) = \mathcal{H}_{n+1,q}(K)$

Then

$$\mathcal{H}_{n+1,q}(K) \cong \aleph_{n+1,q}(K) \big/ ker.\mu_{n+1,q} = \aleph_{n+1,q}(K) \big/ \theta_{n+1,q}(\mathcal{L}_{n+1,q}(K))$$

Therefore

$$\mathcal{H}_{n+1,q}(K) \cong \frac{\aleph_{n+1,q}(K)}{\mathcal{L}_{n+1,q}(K) / \vartheta_{n+2,q}\left(\mathcal{H}_{n+2,q}(K)\right)} \qquad \blacksquare$$

Corollary 2.4

If $q = 0$, **then** $\mathcal{H}_{n+1}(K) \cong \dfrac{\pi_{n+1}(K)}{\mathcal{L}_{n+1}(K)/\vartheta_{n+2}(\mathcal{H}_{n+2}(K))}$.

Proof

From $NES(=3^1)$ **, we have** $\aleph_{p,0} \cong \pi_p(K)$**, and from above theorem (Th. 2.3) the corollary is hold** $\blacksquare$

From remark 1.3 , definition 1.4 and (theorem 7 in [D]), we have the following lemma

Lemma 2.5

Let $f, f^* : K \longrightarrow L$ be homotopic $(f \approx f^*)$, then
$\mathfrak{F}(f) = \mathfrak{F}(f^*) : \mathfrak{Z}^r(K) \longrightarrow \mathfrak{Z}^r(L)$ $\quad (i.e.\ \mathcal{F}_r = \mathcal{F}_r^*)$.

Theorem 2.6

If $K \equiv L$, *then* $\mathfrak{Z}^r(K) \cong \mathfrak{Z}^r(L) \quad \forall r, r = 0, 1, 2, \cdots$.

proof

From [D], we have $\mathfrak{Z}^1(K) \cong \mathfrak{Z}^1(L)$, but $\mathfrak{Z}^{r+1}$ derivative from $\mathfrak{Z}^r$, $\forall r, r = 0, 1, 2 \cdots$. Therefore $\mathfrak{Z}^r(K) \cong \mathfrak{Z}^r(L) \quad \forall r, r = 0, 1, 2, \cdots$ ■

Theorem 2.7

If $\mathcal{C}^1_{p,q} \cong \overline{\mathcal{C}}^1_{p,q}$, $\forall p, q$ *in* $\mathfrak{Z}^1$. *Then* $\mathfrak{Z}^r \cong \overline{\mathfrak{Z}}^r \ \forall r, r = 1, 2, \cdots$.

Proof

From ***five lemma theorem***
(In a commutative diagram of abelian groups,

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \varepsilon \\ A^* & \xrightarrow{i^*} & B^* & \xrightarrow{j^*} & C^* & \xrightarrow{k^*} & D^* & \xrightarrow{l^*} & E^* \end{array}$$

if the two rows are exact and α, β, δ *and* ε are isomorphisms,
then γ is an isomorphism also) see [H]. ■

Theorem 2.8

If K is a $cw-complex$ such that K^{n-1} consists of a single $0-cell\ e^0$. Then

(1) $\mathcal{C}^r_{p,q} = 0$, $\forall r, r = 0, 1, 2, \cdots$, $\forall p, p \leq n-1$,

(2) $\mathcal{C}^r_{n,q} \cong \mathcal{E}^r_{n,q}$, $\forall r, r = 0, 1, 2, \cdots$.

Proof

Consider the second exact couple (NES) of K ;

$$\cdots \to \aleph_{n+1,q} \to \mathcal{H}_{n+1,q} \to \mathcal{L}_{n,q} \to \aleph_{n,q} \to \mathcal{H}_{n,q} \to \mathcal{L}_{n-1,q} \to \aleph_{n-1,q} \to \cdots$$

We have , (see [D])

$$\mathcal{L}_{p,q} = \aleph_{p,q} = 0 \quad if \quad p \leq n-1,$$

and $$\mathcal{L}_{n,q} = 0 \quad , \quad \aleph_{n,q} \cong \mathcal{H}_{n,q} \ .$$

Then we have the following exact sequence ;

$$\cdots \xrightarrow{\partial^1} \mathcal{L}_{n+1,q} \xrightarrow{i^1} \aleph_{n+1,q} \xrightarrow{j^1} \mathcal{H}_{n+1,q} \xrightarrow{\partial^1} 0 \xrightarrow{i^1} \aleph_{n,q} \xrightarrow{\cong} \mathcal{H}_{n,q} \to 0$$

return to second exact couple , we have

$$\mathcal{C}^1_{p,q} = \mathcal{L}_{p,q} = 0 \quad if \ p \leq n,$$

$$\mathcal{C}^1_{p,q} = \aleph_{p,q} = 0 \quad if \ p \leq n-1,$$

and $$\mathcal{C}^1_{n,q} = \aleph_{n,q} \cong \mathcal{H}_{n,q} = \mathcal{E}^1_{n,q} \ .$$

In third exact couple , we have

$$\mathcal{E}^2_{p,q} = {ker.d^1_{p,q}}\Big/{im.d^1_{p+1,q}} \quad , \ where \ \ d^1_{p,q} = j^1_{p,q} \circ \partial^1_{p,q}$$

$$\mathcal{C}^2_{p,q} = ker.j^1_{p,q} = im.i^1_{p,q}$$

$$\mathcal{C}^2_{p,q} = {\aleph_{p,q}}\Big/{im.\vartheta_{p+1,q}} = {\mathcal{C}^1_{p,q}}\Big/{ker.i^1_{p,q}}$$

Now

$$\mathcal{C}^2_{p,q} = ker.j^1_{p,q} = 0 \quad if \ p \leq n-1,$$

$$\mathcal{C}^2_{n,q} = ker.j^1_{n,q} = 0 \ ,$$

$$\mathcal{C}^2_{p,q} = {\mathcal{C}^1_{p,q}}\Big/{ker.i^1_{p,q}} = 0 \quad if \ p \leq n-1,$$

$$d^1_{n,q} = j^1_{n-1,q} \circ \partial^1_{n,q} = 0 \quad \Longrightarrow \quad ker.d^1_{n,q} = \mathcal{H}_{n,q} = \mathcal{E}^1_{n,q} \, ,$$

$$d^1_{n+1,q} = j^1_{n,q} \circ \partial^1_{n+1,q} = 0 \quad \Longrightarrow \quad im.d^1_{n+1,q} = 0 \, .$$

Then

$$\mathcal{E}^2_{n,q} = {ker.d^1_{n,q}}\Big/{im.d^1_{n+1,q}} = {\mathcal{H}_{n,q}}\big/{0} = \mathcal{E}^1_{n,q} \cong \mathcal{C}^1_{n,q}$$

But

$$\mathcal{C}^2_{n,q} = {\mathcal{C}^1_{n,q}}\Big/{ker.i^1_{n,q}} = {\mathcal{C}^1_{n,q}}\big/{0} = \mathcal{C}^1_{n,q}$$

Then

$$\mathcal{E}^2_{n,q} \cong \mathcal{C}^2_{n,q} \, .$$

Therefore the third exact couple is

$$\cdots \to \mathcal{C}^2_{n+1,q} \to \mathcal{C}^2_{n+1,q} \to \mathcal{E}^2_{n+1,q} \to 0 \to \mathcal{C}^2_{n,q} \overset{\cong}{\to} \mathcal{E}^2_{n,q} \to 0 .$$

Continue in this way , we have for each $r = 0, 1, 2, \cdots$ **,**

$$\cdots \to \mathcal{C}^r_{n+1,q} \to \mathcal{C}^r_{n+1,q} \to \mathcal{E}^r_{n+1,q} \to 0 \to \mathcal{C}^r_{n,q} \to \mathcal{E}^r_{n,q} \to 0 .$$

That is

$$\mathcal{C}^r_{p,q} = 0 \qquad \forall r\, , r = 0, 1, 2, \cdots , \forall p\, , p \le n-1 ,$$

and $$\mathcal{C}^r_{n,q} \cong \mathcal{E}^r_{n,q} \qquad \forall r\, , r = 0, 1, 2, \cdots \qquad \blacksquare$$

From above theorems (Th. 2.6 & Th. 2.8) , and two lemmas in [D] (Le. 2.14 & Le.2.15) , we have the following theorem

theorem 2.9

If K **be any** $(n-1)-connected\ complex$ **. Then**

(1) $\mathcal{C}^r_{p,q} = 0 \ , \forall r, r = 0, 1, 2, \cdots \ , \forall p, p \le n-1$,

(2) $\mathcal{C}^r_{n,q} \cong \mathcal{E}^r_{n,q} \ , \forall r, r = 0, 1, 2, \cdots$.

References

[B] H.J. Baues . On Homotopy Classification Problems of J.H.C. Whitehead , in Algebraic Topology Gottingen 1984 . Edited by L. Smith , Springer-Verlag , Berlin Heidelberg 1985.

[D] Dheia Al-Khafaji and Raymond Shekoury . A new exact sequence (a generalization of J. H. C. Whiteheads "certain exact sequence") . Journal of Al-Qadisiyah for Computer Science and Mathematics , vol(2) , no(2) , 2010 .

[H] A. Hatcher . Algebraic Topology .Cambridge University Press 2002 .(Indian edition 2003).

[S] R.M. Switzer .Algebraic Topology , Homotopy and homology , Springer-Verlag , New York 1975 .

[W3] J.H.C. Whitehead . A Certain Exact Sequence . Ann. Of Math.

52(1950),51-110 .