$Semi\ S\alpha$ –compactness on bitopological spaces

Rasha Naser Majeed

Mathematics Department,College of Education – Ibn Al- Haitham,

Baghdad University,

Baghdad,

Iraq.

Abstract

In this paper we define a new kind of open sets in bitopological space which we called semi $S\alpha$ –open sets, which we lead to define a new type of compactness on bitopological spaces called " $semi\ S\alpha$ –compactness" and we study the properties of this spaces, also we define the continuous functions between these spaces.

التراص شبه – $S\alpha$ على الفضاءات التبولوجية الثنائية

رشا ناصر مجيد

قسم الرياضيات / كلية التربية ـ أبن الهيثم / جامعة بغداد

بغداد ـ العراق

المستخلص

قمنا في هذا البحث بتعريف نوع جديد من المجموعات المفتوحة على الفضاءات التبولوجية الثنائية والتي اسميناها المجموعات شبه المفتوحة – $S\alpha$. وبالتالي عرفنا نوع جديد من التراص على الفضاءات الثنائية والذي اسميناه التراص شبه – $S\alpha$ ، وقمنا بدراسة خواص هذا الفضاء وكذلك عرفنا الدوال المستمرة بين هذه الفضاءات .

1.Introduction

The concept of " bitopological space " was introduced by Kelly [1] in 1963 . A set equipped with two topologies is called a" bitopological space " and denote by (X , τ_1, τ_2) , where (X , τ_1), (X , τ_2) are two topological spaces. Since then many authors have contributed to the development of various bitopological properties. A subset A in bitopological space (X , τ_1, τ_2) is "S- open " if it is τ_1-open or τ_2- open . in 1996 Mrsevic and Reilly [2] defined a space (X , τ_1, τ_2) to be S-compact if and only if every S-open cover of X has a finite sub cover. And also they defined a space (X , τ_1, τ_2) to be pair-wise compact [2]. In this paper we introduced a new type of compactness on bitopological spaces namely " $semi\ S\alpha$ –compact" and we review some remarks , propositions, theorems and examples about it .

2. Preliminaries

In this section we introduce some definition, which is necessary for the paper.

Definition 2.1[1]:

Let X be a non-empty set , let τ_1, τ_2 be any two topologies on X , then (X , τ_1, τ_2) is called " ***bitopological space***".

Definition 2.2[3]:

A subset A of a topological space X is called ***"α-open set*** " if and only if $A \subseteq \overline{A^{\circ}}^{\circ}$. The family of all α-open sets is denoted by τ_α.

Definition 2.3[3]:

The complement of α-open set is called ***"α-closed set*** " . The family of all α-closed sets is denoted by αC(X).

Definition 2.4[4]:

A subset A of a topological space X is called "***semi- α-open set***" if and only if there exists an α-open set U in X, such that $U \subseteq A \subseteq \overline{U}$. The family of all semi- α-open sets of X is denoted by SαO(X).

Definition 2.5[4]:

The complement of semi- α-open set is called " ***semi- α-closed set***" . The family of all semi- α-closed sets of X is denoted by SαC(X).

Proposition 2.6[5]:

(i) Every open set is semi-α-open set .

(ii) Every closed set is semi- α-closed set .

Definition 2.7[6]:

Let (X, τ) be a topological space**,** $A \subseteq X$ a family W of subsets of X is said to be a " ***semi -α-open cover of A*** " if and only if W covers A and W is a subfamily of SαO(X).

Definition 2.8:

Let W be any semi-α-open cover of X, a subfamily V of W is said to be an " ***semi - α-open subcover of W*** " if and only if it's cover X.

Definition 2.9[6]:

A topological space (X, τ) is said to be " ***semi-α-compact*** " if and only if every semi- α-open cover of X has a finite subcover.

Proposition 2.10[6]:

Every semi- α-compact space is compact.

3. $\boldsymbol{Semi\ \mathcal{S}\alpha}$ −compactness

In this section, we will define a new type of covers in bitopological spaces, in order to define a new kind of compactness on bitopological spaces called " $semi\ \mathcal{S}\alpha$ −compactness" .

First we begin to the definition of $semi\ \mathcal{S}\alpha$ −open set in bitopological space .

Definition 3.1:

Let (X , τ_1, τ_2) be a bitopological space , then any collection of subsets of X which is contained $S_{1\alpha}O(X)$ and $S_{2\alpha}O(X)$ and it is forms a topology on X called "***the supermom topology on X*** " and is denoted by $S_{1\alpha}O(X) \vee S_{2\alpha}O(X)$. Where $S_{1\alpha}O(X)$ is the family of all semi- α-open sets in the space (X, τ_1) and $S_{2\alpha}O(X)$ is the family of all semi-α-open sets in the space (X, τ_2).

Definition 3.2:

A subset A of a bitopological space (X , τ_1, τ_2) is said to be an " ***semi*** $\boldsymbol{\mathcal{S}\alpha}$ ***−open set*** " if and only if it is open in the space (X, $S_{1\alpha}O(X) \vee S_{2\alpha}O(X)$), where $S_{1\alpha}O(X) \vee S_{2\alpha}O(X)$ is the supermom topology on X contains $S_{1\alpha}O(X)$ and $S_{2\alpha}O(X)$.

Definition 3.3:

The complement of semi $\mathcal{S}\alpha$ −open set in a bitopological space (X , τ_1, τ_2) is called " ***semi*** $\boldsymbol{\mathcal{S}\alpha}$ ***−closed set*** ".

Remark 3.4:

Let (X , τ_1, τ_2) be a bitopological space , then :

(1) Every semi-α-open set in (X, τ_1) or (X, τ_2) is semi $\mathcal{S}\alpha$ −open set in (X , τ_1, τ_2).

(2) Every semi-α-closed set in (X, τ_1) or (X, τ_2) is semi $\mathcal{S}\alpha$ −closed set in (X , τ_1, τ_2).

Note 3.5:

The opposite direction of Remark (3.4) is not true as the following example shows:

Example 1:

Let X={1,2,3}, τ_1={$\emptyset$, {1}, X}, and τ_2={$\emptyset$, {2,3}, X} then $S_{1\alpha}O(X)$ =$\tau_{1\alpha}$ = τ_1 ∪{{1,2},{1,3}}, and $S_{2\alpha}O(X) = \tau_{2\alpha} = \tau_2$. thus $S_{1\alpha}O(X) \vee S_{2\alpha}O(X)$ ={$\emptyset$, {1}, {1,2}, {1,3}, {2,3}, {2}, {3}, X} is the family of all $semi\ \mathcal{S}\alpha$ −open sets in (X , τ_1, τ_2).

{2} is semi $\mathcal{S}\alpha -$ open set in (X , τ_1, τ_2) but it is not semi- α-open set of both (X, τ_1) and (X, τ_2). So {1,3} is semi $\mathcal{S}\alpha$ −closed set in (X , τ_1, τ_2) which is not semi- α-closed in both (X, τ_1) and (X, τ_2).

Now we introduce the definition of $semi\ \mathcal{S}\alpha$ −opencover in bitopological space (X , τ_1, τ_2).

Definition 3.6:

Let (X , τ_1, τ_2) be a bitopological space , let A be a subset of X. a sub collection of the family $S_{1_\alpha}O(X) \vee S_{2_\alpha}O(X)$ is called an **"semi $\mathcal{S}\alpha$ −opencover of A"** if the union of members of this collection contains A.

Definition 3.7:

A bitopological space (X , τ_1, τ_2) is said to be **"semi $\mathcal{S}\alpha$ −compact space"** if and only if every $semi\ \mathcal{S}\alpha$ −opencover of X has a finite sub cover.

Theorem 3.8:

If (X , τ_1, τ_2) is semi $\mathcal{S}\alpha$ −compact space, then both (X, τ_1) and (X, τ_2) are $semi - \alpha$ −compact.

Proof:

To prove (X, τ_1) is $semi - \alpha$ −compact space, we must prove for any semi-α-open cover of X, has a finite sub cover .

Let $\{U_i\}i \in \Lambda$ be any semi-α-open cover of X, implies $\{U_i\}i \in \Lambda$ is a semi $\mathcal{S}\alpha$ −opencover of X (by Remark (3.4)) and since (X , τ_1, τ_2) is semi $\mathcal{S}\alpha$ −compact space, implies there exists a finite sub cover of X, so (X, τ_1) is semi- α −compact.

And by the same way we prove (X, τ_2) is semi α −compact. ■

Corollary 3.9:

If (X , τ_1, τ_2) is semi $\mathcal{S}\alpha$ −compact space, then both (X, τ_1) and (X, τ_2) are compact.

Proof:

The proof is follows from theorem(3.8) and proposition (2.10). ■

Remark 3.10:

The converse of theorem (3.8) and it's corollary is not true, as the following example shows:

Example 2:

Let X={0,2}, τ_1=={$\emptyset$,{0},X}, and τ_2={$\emptyset$,{2},X} then $S_{1_\alpha}O(X) = \tau_{1\alpha} = \tau_1$ and $S_{2_\alpha}O(X) = \tau_{2\alpha} = \tau_2$.

Now, both (X, τ_1) and (X, τ_2) are $semi - \alpha$ −compact (compact) space, but (X , τ_1, τ_2) is not semi $\mathcal{S}\alpha$ −compact space since there is {{0},{2}} is semi $\mathcal{S}\alpha$ −opencover of X which has no finite sub cover.

The converse of theorem (3.8) becomes valid in a special case , when $S_{1_\alpha}O(X) \subset S_{2_\alpha}O(X)$, as the following proposition shows:

Proposition 3.11:

If $S_{1\alpha}O(X)$ is a subfamily of $S_{2\alpha}O(X)$,then (X , τ_1, τ_2) is semi $\mathcal{S}\alpha$ −compact space if and only if (X, τ_2) is semi- α −compact.

Proof:

The first direction follows from theorem (3.8).

Now, if (X, τ_2) is $semi - \alpha$ −compact, we must prove (X , τ_1, τ_2) is semi $\mathcal{S}\alpha$ −compact . since $S_{1\alpha}O(X) \subset S_{2\alpha}O(X)$, then $S_{1\alpha}O(X) \vee S_{2\alpha}O(X) = S_{2\alpha}O(X)$. So (X , τ_1, τ_2) is semi $\mathcal{S}\alpha$ −compact space. ■

Corollary 3.12:

let (X , τ) be a topological space, then the bitopological space (X, $\tau, \tau \vee S\alpha O(X)$) is semi $\mathcal{S}\alpha$ −compact space if and only if (X, $\tau \vee S_\alpha O(X)$) is semi $-\alpha$ −compact.

Proof:

($\Rightarrow$) *it is* clear from theorem (3.8).

($\Leftarrow$) since $\tau \vee S\alpha O(X)$ is a finer than τ, then by proposition (3.11) we have (X, $\tau, \tau \vee S\alpha O(X)$) is semi $\mathcal{S}\alpha$ −compact. ■

Proposition 3.13:

If A and B are two semi $\mathcal{S}\alpha$ −compact subsets of a bitopological space (X , τ_1, τ_2) then $A \cup B$ is semi $\mathcal{S}\alpha$ −compact subset of X .

Proof:

To prove $A \cup B$ is semi $\mathcal{S}\alpha$ −compact subset of X, we must prove for any *semi* $\mathcal{S}\alpha$ −opencover of $A \cup B$, it has a finite sub cover .

Let $\{U_i\}$ $i \in \Lambda$ be any *semi* $\mathcal{S}\alpha$ −opencover of $A \cup B$, then $A \cup B \subseteq \{\cup U_i, i \in \Lambda\}$ and therefore $A \subseteq \cup U_i$ and B $\subseteq \cup U_i$, implies $\{U_i\}$ $i \in \Lambda$ is an *semi* $\mathcal{S}\alpha$ −opencover of A and B .

But A and B are semi $\mathcal{S}\alpha$ −compact subsets, therefore there exists $i_1, i_2, \dots, i_n \in \Lambda$ and $i_1, i_2, \dots, i_m \in \Lambda$ such that $\{U_{i_1}, U_{i_2}, \dots, U_{i_n}\}$ and $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}\}$ is a finite sub cover of A and B respectively, then

$\{U_{i_1}, U_{i_2}, \dots, U_{i_n}\} \cup \{U_{i_1}, U_{i_2}, \dots, U_{i_m}\}$ is a finite sub cover of A $\cup$ B , therefore $A \cup B$ is an semi $\mathcal{S}\alpha$ −compact subset of X . ■

Theorem 3.15:

The *semi* $\mathcal{S}\alpha$ −closed subset of an semi $\mathcal{S}\alpha$ −compact space is semi $\mathcal{S}\alpha$ −compact.

Proof:

Let (X , τ_1, τ_2) be semi $\mathcal{S}\alpha$ −compact space and let A be a semi $\mathcal{S}\alpha$ −closed subset of X. to show that A is semi $\mathcal{S}\alpha$ −compact set.

Let $\{U_i\}$ $i \in \Lambda$ be any *semi* $\mathcal{S}\alpha$ −opencover of A. Since A is *semi* $\mathcal{S}\alpha$ −closed subset of X, then X-A is a semi $\mathcal{S}\alpha$ −open subset of X , so {X-A}$\cup\{U_i;\ i \in \Lambda\}$ is a semi $\mathcal{S}\alpha$ −opencover of X , which is semi $\mathcal{S}\alpha$ −compact space.

Therefore, there exists $i_1, i_2, \dots, i_n \in \Lambda$ such that {X-A, $U_{i_1}, U_{i_2}, \dots, U_{i_n}$} is a finite sub cover of X .as A$\subseteq X$ and X-A covers no part of A , then $\{U_{i_1}, U_{i_2}, \dots, U_{i_n}\}$ is a finite sub cover of A . so A is semi $\mathcal{S}\alpha$ −compact set. ■

Definition 3.16:

A function $f: (X, \tau_1, \tau_2) \to (Y, \tau_1', \tau_2')$ is said to be " ***semi*** $\mathcal{S}\alpha$ ***−continuous function*** " if and only if the inverse image of each *semi* $\mathcal{S}\alpha$ −open subset of Y is a semi $\mathcal{S}\alpha$ −open subset of X.

Theorem 3.17:

The *semi* $\mathcal{S}\alpha$ *−continuous* image of a semi $\mathcal{S}\alpha$ −compact space is a semi $\mathcal{S}\alpha$ −compact space.

Proof:

Let (X, τ_1, τ_2) be a semi $\mathcal{S}\alpha$ −compact space, and let $f: (X, \tau_1, \tau_2) \to (Y, \tau_1', \tau_2')$ be a semi $\mathcal{S}\alpha$ −continuous, onto function. To show that (Y, τ_1', τ_2') is a semi $\mathcal{S}\alpha$ −compact space. Let $\{U_i;\ i \in \Lambda\}$ be a semi $\mathcal{S}\alpha$ −opencover of Y, then $\{f^{-1}(U_i)$; i$\in$ $\Lambda\}$ is a semi $\mathcal{S}\alpha$ −opencover of X , which is semi $\mathcal{S}\alpha$ −compact space.

So there exists $i_1, i_2, \dots, i_n \in \Lambda$, such that the family $\{f^{-1}\left(U_{i_j}\right); j = 1,2,\dots,n\}$ covers X and since f is onto, then $\{U_{i_j}; j = 1,2,\dots,n\}$ is s finite sub cover of Y.

Hence Y is a semi $\mathcal{S}\alpha$ −compact space. ■

References:

[1]. J.C.Kelly, " Bitopological Spaces ", Prok .London Math. Soc., Vol. 13(1963), 71-89.

[2]. Mrsevic and I.L.Reilly, "Covering and connectedness properties of a topological space and it is Associated topology of α −subsets", Indian J. pure apple. Math.,27(10):995-1004, Oct.(1996).

[3]. Olav Najasted," On some classes of nearly open sets ", pacific journal of math., vol.15, No.3, 961-970,(1965).

[4]. G.B.Navalagi, "Definition Bank in General Topology ", Department of math., G.H.college, Haveri-581110, Karanataka, India.

[5]. Nadia M.Ali, "On New Types of weakly open sets", M.Sc. Thesis , University of Baghdad , college of education, Ibn Al- Haitham,(2004) .

[6]. Ahmed I.Nasir, " Some kinds of strongly compact and pair-wise compact spaces", M.Sc. Thesis , University of Baghdad , college of education, Ibn Al- Haitham,(2005).