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On The Topological Projective Group $(Z(P^{\infty}),\oplus)$

الزمرة الاسقاطية التبولوجية $z(P^{\infty})$

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Abstract

In this search obtain the following results on $(Z(P^{\infty}),\oplus)$ as a projective topological group **;**

1. A necessary and sufficient condition for $(Z(P^{\infty}),\oplus)$) to be a topological group has been established . From Proposition (2-4).
2. A necessary and sufficient condition for $(Z(P^{\infty}),\oplus)$) to be a topological free group has been established . From Proposition (2-5) .
3. A necessary and sufficient conditions for projectivity of $(Z(P^{\infty}),\oplus)$) has been established . From Proposition (2-6) .

المستخلص

في بحثنا هذا حصلنا على النتائج الاتية في البرهنة على ان الزمرة $(Z(P^{\infty}),\oplus)$ زمرة اسقاطية توبولوجية :

١. الشرط الكافي والضروري لكي تكون الزمرة $(Z(P^{\infty}),\oplus)$ زمرة توبولوجية قد تحقق من مبرهنة (٤-٢) .

٢. الشرط الكافي والضروري لكي تكون الزمرة $(Z(P^{\infty}),\oplus)$ زمرة حرة توبولوجية قد تحقق من مبرهنة (٥-٢).

٣.الشرط الكافي والضروري لكي تكون الزمرة $(Z(P^{\infty}),\oplus)$ زمرة اسقاطية توبولوجية قد تحقق من مبرهنة(٦-٢)

Introduction:

The study of a topological groups started in 1920, while the study of topological rings started in 1940 by Kaplanski. In this research a necessary and sufficient conditions for $(Z(P^{\infty}),\oplus)$ to be topological group has been established . The topology with $(Z(P^{\infty}),\oplus)$ is (p-adic) topology. This research contains two sections , section one contains some necessary definitions and section two contains the important propositions about $(Z(P^{\infty}),\oplus)$. references [3]and[4] are used to construct some definitions.

Section " ONE" Some Definitions:

In this section we introduce some necessary definitions .

Definition (1–1) [1]: A non empty set E is said to be a topological group if:
1. E is a group;
2. τ is a topology on E ;
3. A mappings $\pi : E \times E \to E$ and $\mu : E \to E$ is continues, where μ defined as $\mu(x) \cong x^{-1}$, and $E \times E$ is the product of two topological spaces.

Definition (1–2) [1]: Let E be a topological group .A subset M of E is said to be a topological subgroup of E if:
1. M is a subgroup of E;
2. M is a subspace of E.

Definition (1–3) [1]: A topological group E is said to be a topological finitely generated group if it is generated by a finite set. If it is generated by one element it is called a topological cyclic group.

Definition (1–4) [1]: Let E, E^* be two topological groups. a mapping $f : E \rightarrow E^*$ is said to be topological group morphism if :
1. f is a homomorphism;
2. f is continuous.

Definition (1–5) [1]: Let E, E' be two topological groups , $f : E \rightarrow E'$ is a topological morphism. A topological kernel of f is the set $Kerf \cong \{t \in E : f(t) \cong e'\}$, where e' the identity element of E'.

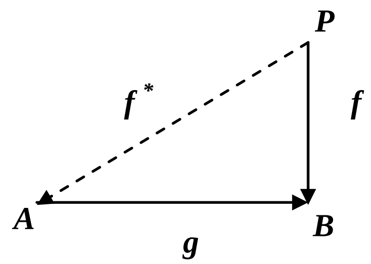
Definition (1–6) [1]: Let M be a topological subgroup of topological group E, the set E/M said to be topological quotient group if :
1. E/M is group;
2. E/M is topological group with quotient topology;
3. A mapping $\pi : E/M \rightarrow E/M$ is continuous.

Definition (1–7) [1]: A topological direct product group for a family of a topological groups $\{E_\lambda\}_{\lambda\in\Omega}$ is a topological group $E \cong \prod_{\lambda\in\Omega} E_\lambda$ with projection mappings π_i, and $E \cong \prod_{\lambda\in\Omega} E_\lambda \cong \bigoplus_{\lambda\in\Omega}$ where $\bigoplus_{\lambda\in\Omega} E_\lambda$ is a topological direct sum , and a mappings $\pi_i : \prod_{\lambda\in\Omega} E_\lambda \rightarrow E_i$ is continuous.

Definition (1– 8) [2]: A topological group basis B^* for a topological group F is a family $\{F_n\}_{n\in N}$ of open topological cyclic subgroups of F satisfies:
1. $\cap_{n\in N} F_n \cong \{0\}$ (the identity element of F) .
2. $F \cong F_1 \supset F_2 \supset F_3 \supset \cdots$.
3. If $F \cong \bigoplus_{n\in N} F_n$ and $\pi_j : F \rightarrow F_j$ is continuous and open projection mappings, then $ker(\pi_j) \underset{j\neq n}{\cong} F_n$ and $F \cong ker\,\pi_j \oplus F_n$.

Definition (1–9) [2]: A topological group F is said to be a topological free group if it has a topological group basis .

Definition (1 –10) [2]: A topological group P is said to be a topological projective group if for each topological group morphism $f : P \rightarrow B$ and for each topological group epimorphism $g : A \rightarrow B$ there is a topological group morphism $f^* : P \rightarrow A$ for which $f \cong gof^*$.(or the following diagram commutes);



Definition (1 –11): $Z(P^{\infty})$ defined as $Z(P^{\infty})=\{\overline{a/b}\in Q/Z$ and $b=p^{i}$ for some $i\geq 0\}$ and $\oplus$ defined as $\forall x,y\in Z(P^{\infty})$ then $\bar{x}\oplus\bar{y}=\overline{x\oplus y}\in Z(P^{\infty})$.

Definition (1 –12) [٣]: The (P-adic Topology) is a topology each of it's open set G_i is a class of the form $\Psi=\prod_{i\in N}\{D_p : D_p\in G_i\}$ where D_p is an open disc with center $p\in G_i$.

Proposition (1–13) [2]: Every topological free group is topological projective group.

Section " TWO" Some Propositions:

In this section introduce some propositions about Q/Z and $Z(P^{\infty})$.

Proposition (2-1) : $(Q/Z,\oplus)$ is a group .Where $\forall x,y\in Q/Z$ then $\bar{x}\oplus\bar{y}=\overline{x\oplus y}\in Q/Z$.

Proof :

Let $a,b,c,\in Q/Z$ then $\bar{a},\bar{b},\bar{c}\in Q/Z$ and

(1) $\bar{a}\oplus\bar{b}=\overline{a\oplus b}\in Q/Z$ then $\oplus$ is closed .

(2) $\bar{a}\oplus(\bar{b}\oplus\bar{c})=\bar{a}\oplus(\overline{b\oplus c})$

$=(\overline{a\oplus b\oplus c})$

$=(\overline{a\oplus b})\oplus\bar{c}$

$=(\overline{a\oplus b\oplus c})$ then $\oplus$ is associative .

(3) $\bar{0}\oplus\bar{a}=\overline{0\oplus a}=\bar{a}$ then $\bar{0}$ is the identity element.

(4) $\forall\bar{a}\in Q/Z\ \exists\overline{-a}\in Q/Z$ such that $\bar{a}\oplus-\bar{a}=\overline{a\oplus -a}=\bar{0}$,then $\overline{-a}$ is the inverse of $\bar{a}$.

Then $(Q/Z,\oplus)$ is a group .

Proposition (2-2) : $(Z(P^{\infty}),\oplus)$ is a group .

Proof :

Let $a,b,c,\in Z(P^{\infty})$ then $\bar{a},\bar{b},\bar{c}\in Z(P^{\infty})$ where $\bar{a}=\overline{x/p^{i}},\bar{b}=\overline{y/p^{i}},\bar{c}=\overline{z/p^{i}}$ and

(1) $\bar{a}\oplus\bar{b}=\overline{x/p^{i}}\oplus\overline{y/p^{i}}=\overline{x\oplus y/p^{i}}\in Z(P^{\infty})$

(2) $\bar{a}\oplus(\bar{b}\oplus\bar{c})=\bar{a}\oplus(\overline{y/p^{i}}\oplus\overline{z/p^{i}})$

$=\overline{x/p^{i}}\oplus(\overline{y/p^{i}}\oplus\overline{z/p^{i}})$

$=\overline{x/p^{i}}\oplus(\overline{y\oplus z/p^{i}})$

$=(\overline{x\oplus y\oplus z/p^{i}})$

$$= (\overline{x \oplus y / p^i}) \oplus \overline{z / p^i}$$

$$= (\overline{x / p^i} \oplus \overline{y / p^i}) \oplus \overline{z / p^i}$$

$$\bar{a} \oplus (\bar{b} \oplus \bar{c}) = \bar{a} \oplus (\overline{b \oplus c})$$

Then $\oplus$ is associative .

(3) $\overline{0 / p^i}$ is the identity element of $Z(P^\infty)$ since

$$\overline{x / p^i} \oplus \overline{0 / p^i} = \overline{x \oplus 0 / p^i} = \overline{x / p^i} \forall \overline{x / p^i} \in Z(P^\infty)$$

(4) $\forall \bar{a} = \overline{x / p^i} \exists \overline{-a} = \overline{-x / p^i}$ such that $\bar{a} \oplus \overline{-a} = \overline{x / p^i} \oplus \overline{-x / p^i} = \overline{x \oplus -x / p^i} = \overline{0 / p^i}$

Then $(Z(P^\infty), \oplus)$ is a group .

Proposition (2-3) : $(Z(P^\infty), \oplus)$ is a free group .

Proof :

The only proper subgroups of $Z(P^\infty)$ are the finite cyclic groups $C_n = \overline{\left\langle 1 / P^n \right\rangle}$ (n=0,1,2,….)

Furthermore, $\langle 0 \rangle = C_0 \subset C_1 \subset C_2 \subset C_3 \subset \cdots$.

A basis B^* for a group $Z(P^\infty)$ is a family $\{C_n\}_{n \in N}$ of cyclic subgroups of $Z(P^\infty)$ satisfies:

1. $\cup_{n \in N} C_n = Z(p^\infty)$
2. $Z(P^\infty) = \bigoplus_{n \in N} C_n$.

Since $(Z(P^\infty), \oplus)$)has basis $\{C_n\}_{n \in N}$, then $(Z(P^\infty), \oplus)$ is a free group .

Proposition (2-4) : $(Z(P^\infty), \oplus)$ is a topological group .

Proof :

(1) From Proposition (2-2) $(Z(P^\infty), \oplus)$ is a group .

(2) The topology on $(Z(P^\infty))$is (P-adic topology).

Then $(Z(P^\infty), \oplus))$ with (P-adic topology)is a topological group.

Proposition (2-5) : $(Z(P^\infty), \oplus)$ is a topological free group .

Proof :

The only proper subgroups of $Z(P^\infty)$ are the finite cyclic groups $C_n \cong \overline{\left\langle 1 / P^n \right\rangle}$ (n=0,1,2,….)

Furthermore, $\langle 0 \rangle \cong C_0 \subset C_1 \subset C_2 \subset C_3 \subset \cdots$.

A topological group basis B^* for a topological group $Z(P^\infty)$ with (P-adic) topology is a family $\{C_n\}_{n \in N}$ of open topological cyclic subgroups of $Z(P^\infty)$ satisfies:

1. $\cap_{n \in N} C_n \cong \{0\}$ (the identity element of $Z(P^\infty)$) .

2. $\langle 0\rangle \cong C_0 \subset C_1 \subset C_2 \subset C_3 \subset \cdots$.

3.If $\boldsymbol{Z(P^{\infty}) \cong \underset{n\in N}{\oplus} C_n}$ and $\pi_j : Z(P^{\infty}) \to C_j$ is continuous and open projection mappings, then $ker(\pi_j) \underset{j\neq n}{\cong} C_n$ and $ker\ Z(P^{\infty}) \cong \pi_j \oplus C_n$.

Then $(Z(P^{\infty}),\oplus)$ is a topological free group .

Proposition (2-6) : $(Z(P^{\infty}),\oplus)$) is a topological projective group .

Proof : Since $(Z(P^{\infty}),\oplus)$) is a topological free group (Proposition (2-5)).

Then from Proposition (1-11) $(Z(P^{\infty}),\oplus)$) is a topological projective group .

References:

[1].Higgins; *P,J* , (An Introduction to Topological Groups), London, mathematical society lecture note series 15, (1974).

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