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ON THE CONJECTUR OF M.GOLDBEJG

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Abstract:

we present in this paper that to each tournament T_n with n nodes there corresponds the automorphism group $G(T_n)$ consisting of all dominance preserving permutations, of the set of nodes .

الخلاصة :

وجدت انه لكل عدد صحيح n فان $g(n) \le \sqrt{3}^{\,n-1}$ حيث g(n) تمثل اكبر رتبة لزمرة التشاكل الذاتي للعلاقة الدورية. والنتائج السابقة التي توصلت إلى $g(n) \ge \sqrt{3}^{\,n-1}$ نحصل على حدسية

$\lim g(n)^{\frac{1}{n}} = \sqrt{3}$. M. Goldbejg and J. W Moon

Introduction:

In a recent paper [4] . M. Goldberg and J. W. Moon. consider the Maximum order g(n) which the group of a tournament with n nodes may have . Among other results they prove that:

(1) $g(n)^{\frac{1}{n}} \ge \sqrt{3}^{\,n-1}$ exists as $n \to \infty$ and ≤ 2.5

(2) $\lim g(n)^{\frac{1}{n}}$ for $n = 3^k$ $(k = 0,1,\ldots\ldots\ldots)$

Moreover ,they conjecture that

(3) $\lim g(n)^{\frac{1}{n}} = \sqrt{3}$

The object of the present paper is to prove.

Definitions:

Def 1

A Tournament T is a binary relation, When it is irreflexive , and that for all $x , y \in T$.
$T(x , y) \ne T(y , x)$, when $x \ne y$. [1]

Def 2

Let α denote a dominance – preserving permutation of the nodes of a given tournament Tn so that $\alpha(p) \rightarrow \alpha(q)$ if and only if $p \rightarrow q$. The set of all such permutations forms a group, the auto morphism group g(n) of Tn. [1]

THEOREM 1

For each positive integer n, $g(n) \le \sqrt{3}^{\,n-1}$ taken together with (2) this implies the truth of the conjecture (3).

In contrast to the graph theoretic approach of[4]. M. Goldberg and J. W. Moon. Our approach is via group theory . It takes as its starting point the addendum to[4]. M. Goldberg and J. W. Moon. Where it is shown that g(n). can be interpreted as being the largest order of permutation group of odd order and degree n. By the celebrated theorem of Feit and Thompson[3]. W. Feit and J. G. Thompson. Any groey of odd order is solvable . Thus Theorem, is equivalent to.

THEOREM1

Every solvable permutation group G of odd order and degree n has $|G| \le \sqrt{3}^{\,n-1}$.

we shall prove the result in this latter from. It would be interesting to know if the result is as deep as the use of the Feit- Thompson theorem suggests.

Proof

The main step in the proof is already contained in a previous paper of the author (see[2]). J. D. Dixon. It is shown there that we can use induction on n to reduce the problem to the case where G is primitive permutation group. In the latter case it is show that $G = AG_1$ with $A \cap G_1 = 1$, where A is a normal elementary abelian p-subgroup of order $p^k = n$ and G_1 is the stability subgroup of G fixing one symbol. Moreover. A equals its centralizer in G, and so G_1 is isomorphic to a subgroup of the group. Aut A of all automorphisms of A. Finally, since the order of Aut A for an elementary abelian p-group is known, we obtain.

(4) $|G| = |A|\ |G_1|$ divides $p^k (p^k - 1) (p^k - p)\ldots\ldots(p^k - p^{k-1})$
(see[2]). J. D. Dixon. Section 2 for details.)
The remaining step is to prove that (4) together with

the hypothesis that $|G|$ is odd implies $|G| \le \sqrt{3}^{\,n-1}$ with $n = p^k$. Direct calculation shows:

$$|G| \le p^k \left(p^k - 1\right)\ldots\ldots\ldots\left(p^k - p^{k-1}\right)\langle p^{k+k^2} \le \sqrt{3}^{\,p^k - 1}$$

unless $p^k = 3 , 3^2 , 5$ or 7. However, since $|G|$ is odd, we have in the exceptional cases :

$|G| \le 3 = \sqrt{3}^{3-1}$ if 3 = 3

$|G| \le 3^3 = \sqrt{3}^{9-1}$ if $n = 3^2$

$|G| \le 5 = \sqrt{3}^{5-1}$ if n = 5

$|G| \le 21 = \sqrt{3}^{7-1}$ if n = 7

Thus the inequality holds in all cases. And the theorem is proved.

Remarks

The inequality[3] can be proved in a direct group-theoretic manner by constructing imprimitive permutation groups of suitable order[2]. J. D. Dixon.

Theorem 1 shows that we actually have equality in [3] and a simple cheek of the inequalities in the proof above shows that. We cannot have equality expect when $n = 3^k$. Again a straight forward case- by- case analysis of the proof above gives an easy way of calculating the values of g(n).

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