Solvability of Linear Perturbed Sylvester Dynamical System in Infinite Dimensional Space Using Perturbed Composite Semigroup Approach

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Abstract

The perturbed linear dynamical system of Sylvester type in infinite dimensional space has been considered . The solvability of this class of equations by using the perturbed composite semigroup of bounded linear operator is presented and developed . The necessary dynamical properties have also been presented and proved.

1. Introduction

The theory of one parameter semigroup of linear operator on Banach spaces started in 1948 with the Hill-Yoside generation theorem, and attained its first apex with the 1957 edition of semigroup and functional analysis by E. Hille and R. S. Phillips, in 1970's and 80's.The theory reached a certain state of perfection, which is well represented in the monograph by [6], [3], [5] and others. Today, the situation is characterized by manifold applications of this theory not only to the traditional areas, such as partial differential equations or stochastic processes. Semigroup has become important tools for integro-differential equations and functional differential equations in quantum mechanics or in infinite-dimensional control theory.

This paper introduces to the concept of a composite semigroup and applications to the analysis of the operator differential Sylvester equation. This equation arises in various control problems on finite time horizon [0,t], for linear infinite-dimensional systems with unbounded input / output operator.

The work of [4] introduced to the concept of a composite semigroup and its application to the analysis of the operator differential Sylvester equation. This equation arises in various control problems on finite time horizon [0, t], $t \in [0, \infty)$, for linear infinite-dimensional systems with bounded input or operators.

The solvability of such system and the study of some of its dynamical properties, up to our knowledge are still a challenge for many researchers. So, the main aim of the following work is to define such dynamical properties ,as well as, the solvability using the concept of composite semigroup generated by some unbounded linear generators. Some preliminaries are then needed to understand the present approach.

The following problem have been presented an disused in this paper.

$\frac{d}{dt}Z(t) = (\mathbb{A} + \Delta\mathbb{A})Z(t),\ t > 0$

$Z(0) = Z_0$

where $\mathbb{A} + \Delta\mathbb{A} : D(\mathbb{A}+ \Delta\mathbb{A}) \subseteq L(\ L(H))$ is a linear unbounded operator. The operator $\mathbb{A}$ is the infinitesimal generator of a C_o- composite semigroup denoted by $\mathbb{T}(t)$, $t \geq 0$ and $D(\mathbb{A}) \subseteq L(\ L(H))$. $D(A_1) \subseteq D(\Delta A_1)$ and $D(A_2) \subseteq D(\Delta A_2)$. For $Z \in D(\mathbb{A} + \Delta\mathbb{A})$ and ΔA_1, $\Delta A_2 \in L(H)$.

The operator $\mathbb{A} + \Delta\mathbb{A}$ is the infinitesimal generator of a C_0-composite perturbation semigroup $\mathbb{S}(t)$, $t \geq 0$ and $D(\mathbb{A} + \Delta\mathbb{A}) \subseteq L(L(H))$.and $h \in D(A_1 + \Delta A_1)$, where the generator $\mathbb{A}$ is defined as :

$((\mathbb{A} + \Delta\mathbb{A})Z)h = (A_1 + \Delta A_1)Zh + Z(A_2 + \Delta A_2)h$

2. Some Mathematical Concepts

In this sction, some necessary mathematical concepts for usual semigroup theory are discussed .

Definition (2.1),[8]:

Let T be an unbounded linear operator on a Hilbert space H, with domain D(T) is dense in H. The adjoint operator T* is defined by:

$\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in D(T)$, $y \in D(T^*)$, where:

$D(T^*) = \{y\in H \mid \langle Tx,y\rangle = \langle x,z\rangle$, for some $z\in H$ and all $x\in D(T)\}$.

Definition(2.2), [1] :

A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space X is called a (one-parameter) semigroup on X if it satisfies the following conditions:

$T(t + s) = T(t)T(s),\ \forall\ t, s \geq 0$

$T(0) = I$, where I stand for identity operator.

Definition(2.3),[5]:

The linear operator A defined on the domain:

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exists}\} \text{ and} \tag{1}$$

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \left.\frac{d^+T(t)}{dt}\right|_{t=0} \quad \text{for } x \in D(A)$$

is the infinitesimal generator of the semigroup T(t), D(A) is the domain of A.

Definition(2.4), [6]:

A semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X is called strongly continuous semigroup of a bounded linear operators or (C_0-semigroup) if the map $t \longrightarrow T(t) \in L(X), t \in \mathbb{R}^+$ satisfies the following conditions:

1. $T(t+s) = T(t)T(s), \forall\, t, s \in \mathbb{R}^+$.
2. $T(0) = I$.where I stands for identity operator.
3. $\lim_{t\downarrow 0} \|T(t)x - x\|_X = 0$, for every $x \in X$.

Remark(2.5), [5]:

Let T(t) be C_o-semigroup generated by infinitesimal generator A on a Banach space X.Then

i- $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} T(s)x\, ds = T(t)x$, $h\in(0,t)$ and for $x \in X$

ii- For $x \in D(A)$, $T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax, \text{ for all } t \geq 0$$

iii- For $x \in D(A)$

$$T(t)x - T(s)x = \int_s^t T(r)Ax\, dr = \int_s^t AT(r)x\, dr\,.$$

v- For every $\lambda \in \mathbb{R}$, one can define a linear bounded operators:

$$R(\lambda; A)x = \int_0^\infty e^{-\lambda t}T(t)x\, dt, \text{ for all } x \in X, \text{Re } \lambda > W_0.$$

Definition(2.6), [7]:

The weakest topology on L(X, Y), such that $E_x : L(X, Y) \longrightarrow Y$ given by: $E_x(T) = Tx$ are continuous for all $x \in X$ is called the strong operator topology.

Remark(2.7),[5]:

A semigroup $\{\{T(t)\}_{t\geq 0}$ is called a continuous in the uniform operator topology, if:

(1) $\|T(t+\Delta)x - T(t)x\|_X \longrightarrow 0$, as $\Delta \longrightarrow 0$, $\forall\, x \in X$.

(2) $\|T(t)x - T(t-\Delta)x\|_X \longrightarrow 0$, as $\Delta \longrightarrow 0$, $\forall\, x \in X$

Remarks(2.8),[6]:

Suppose that $x(0) \in D(A)$ and the function f(t) with range in X is continuous differentiable in the open interval $(0, \tau)$ with continuous derivative in the closed interval $[0, \tau]$, then the (non-homogeneous) initial value problem:

$$\left.\begin{array}{l} \frac{d}{dt}x(t) = Ax(t) + f(t), \quad 0 < t < \tau \\ x(0) = x_0, \text{ given in the domain A} \end{array}\right\} \qquad (2)$$

has a unique solution satisfying:

i. $x(t)$ is absolutely continuous in $(0, \tau)$.

ii. $x(t) \in D(A)$, $t > 0$.

iii. $\|x(t) - x(0)\|_X \longrightarrow 0$ as $t \longrightarrow 0^+$.

Definition (2.9), [9]:

Let $x_0 \in D(A)$ and $f \in C([0, \tau] : H)$, then $x(.)$ defined by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)\, ds$$

is called a strong solution of (1.11) if:

$x(.) \in C([0, \tau]; D(A)) \cap C^1([0, \tau]; H)$ and satisfying (2) for all $t \in [0, \tau]$.

Theorem(2.10), [6]:

Let X be a Banach space and let A be the unbounded linear infinitesimal generator of a C_o-semigroup $T(t)$ on X, satisfying:

$\|T(t)\|_{L(X)} \leq Me^{wt}$.

If ΔA is a bounded linear operator on X, then $A+\Delta A$ with $D(A+\Delta A)=D(A)$ is the infinitesimal generator of a C_o-perturbation semigroup $S(t)$ on X, satisfying:

$\|S(t)\|_{L(X)} \leq Me^{(w+M\|\Delta A\|_{L(X)})t}$

for any $t \geq 0$, $w \geq 0$ and $M \geq 1$.

3. Problem Formulation

Consider the linear initial value problem in finite state space:

$$\frac{d}{dt}Z(t) = (\mathbb{A} + \Delta\mathbb{A})Z(t),\ t > 0 \qquad (3)$$

$$Z(0) = Z_0\ ,$$

where $\mathbb{A} + \Delta\mathbb{A}$: $D(\mathbb{A}+ \Delta\mathbb{A}) \subseteq L(H) \longrightarrow L(H)$ is a linear operator, defined as follows:

1. The operator $\mathbb{A} .=A_1 . + . A_2$ is the infinitesimal generator of a C_0-composite semigroup denoted by $\mathbb{T}(t) =T_1(t)ZT_2(t)=$, $t \geq 0$ and $D(\mathbb{A}) \subseteq L(H)$.
2. $\|\mathbb{T}(t))\|_{L(H)} \leq M_1M_2\, e^{(w_1+w_2)t}$ where $M_1, M_2 \geq 1$, $w_1, w_2 \geq 0$.
3. The operator $\mathbb{A} + \Delta\mathbb{A}$ is the infinitesimal generator of a C_o-composite perturbation semigroup $\mathbb{S}(t)$, $t \geq 0$ and $D(\mathbb{A} + \Delta\mathbb{A}) \subseteq L(H)$.
4. $D(A_1) \subseteq D(\Delta A_1)$ and $D(A_2) \subseteq D(\Delta A_2)$.
5. For $Z \in D(\mathbb{A} + \Delta\mathbb{A})$ and $h \in D(A_1 + \Delta A_1)$, we have

   $$((\mathbb{A} + \Delta\mathbb{A})Z)h = (A_1 + \Delta A_1)Zh + Z(A_2 + \Delta A_2)h$$
6. There exists positive constants k_1 and k_2, such that

   $$\|\Delta A_1\|_{L(H)} \leq k_1 \text{ and } \|\Delta A_2\|_{L(H)} \leq k_2.$$

7. $\|\Delta\mathbb{A}\|_{L(H)} = \|\Delta A_1.+.\Delta A_2\|_{L(H)}$

$$\leq \|\Delta A_1.\|_{L(H)} + \|.\Delta A_2\|_{L(H)} \leq \|\Delta A_1\|_{L(H)}\|.\|_{L(H)} + \|.\|_{L(H)}\|\Delta A_2\|_{L(H)}$$

$$\leq (\|\Delta A_1\|_{L(H)} + \|\Delta A_2\|_{L(H)})\ \|.\|_{L(H)}\ .$$

Definition(3.1):

Let L(H) be a Banach space, a one-parameter family $\{\mathbb{S}(t)\}_{t\geq 0} \subset L(L((H))$, $t\in[0,\infty)$ of bounded linear operators defined by:

$$\mathbb{S}(t) = S_1(t)ZS_2(t)\ , \qquad (4)$$

for generator $\mathbb{A} + \Delta\mathbb{A}$, for any $Z\in L(H)$ and $t \in[0,\infty)$ is called composite perturbation semigroup, where $S_1(t)$,$S_2(t)$ are two perturbation semigroups defined from H into H for $(A_1+\Delta A_1)$ and$(A_2 + \Delta A_2)$ respectively.

Definition(3.2):

The infinitesimal generator $\mathbb{A} + \Delta\mathbb{A}$ of $\mathbb{S}(t)$ of problem formulation on a uniform operator topology defined as the limit:

$$(\mathbb{A} + \Delta\mathbb{A})Zh = \lim_{t\downarrow 0}\left\{\frac{\mathbb{S}(t)Zh-Zh}{t}\right\},\ Z \in D(\mathbb{A} + \Delta\mathbb{A}),\ h \in H$$

where $D(\mathbb{A} + \Delta\mathbb{A}) \subset L(H)$ is the domain of $\mathbb{A}+\Delta\mathbb{A}$ defined as follows:

$$D(\mathbb{A}+\Delta\mathbb{A})=\left\{Z \,\square\, L(H): \lim_{t\downarrow 0}\left\{\frac{\mathbb{S}(t)Zh-Zh}{t}\right\} \text{exist in } L(H)\right\}.$$

Concluding Remarks(3.3):

1-$\{L(H), \tau\}$ stands for L(H) equipped with the strong operator topology τ, i.e., topology induced by family of seminorms $\rho=\{\rho_h\}_{h\in H}$, where seminorms $\rho_h(Z) = \|Zh\|_H$, $Z \in L(H)$.

2. Let $D(A_1) \subseteq D(\Delta A_1)$, $D(A_2) \subseteq D(\Delta A_2)$ and $D(\mathbb{A}) \subseteq D(\Delta\mathbb{A})$.Therefore the following are concluded

a-The different between the usual strongly continuous semigroups of problem formulation and the composite perturbation semigroup (4) follows from the fact that in general for $Z \in L(H)$, the function $[0,\infty) \supseteq t \mapsto \mathbb{S}(t)Z \in L(H)$ is continuous in $\{L(H), \tau\}$, and which cannot be continuous in $\{L(H), \|.\|\}$ unless the semigroups $\{\mathbb{S}_1(t)\}_{t\geq 0}$, $\{\mathbb{S}_2(t)\}_{t\geq 0} \subset L(H)$ are uniformly continuous. However, this takes place case only if their generators $A_1 + \Delta A_1$, $A_2 + \Delta A_2$ are bounded operators on H.

b-The generator $\mathbb{A} + \Delta\mathbb{A}$ is densely defined only in $\{L(H), \tau\}$ and does not in $\{L(H), \|.\|\}$. This implies that $D(\overline{A+\Delta A})$ in L(H) is only a proper set and not the whole L(H).

c-The infinitesimal generator $\mathbb{A} + \Delta\mathbb{A}$ of problem formulation of the composite perturbation semigroup $\{\mathbb{S}(t)\}_{t\geq 0} \subset L(L((H))$ on a strong operator topology is defined as the following limit:

$$(\mathbb{A} + \Delta\mathbb{A})Z = \tau\text{ - }\lim_{t\downarrow 0}\left\{\frac{\mathbb{S}(t)Zh-Z}{t}\right\},\ Z \in D(\mathbb{A} + \Delta\mathbb{A}),$$

where $D(\mathbb{A} + \Delta\mathbb{A}) \subset L(H)$is the domain of $\mathbb{A} + \Delta\mathbb{A}$ and defined as follows:

$$D(\mathbb{A} + \Delta\mathbb{A}) = \left\{ Z\in L(H): \tau\text{- } \lim_{t\downarrow 0} \left\{\frac{\mathbb{S}(t)Zh - Z}{t}\right\} \text{ exist in } \{L(H),\tau\} \right\}.$$

In the following lemma some generalized results on $\mathbb{S}(t)$, $t \in [0,\infty)$ of [2], are developed.

Lemma(3.4) :

Consider the problem formulation, *let* $\mathbb{S}(t)= S_1(t)ZS_2(t)$, $t \geq 0$ be a composite perturbation semigroup defined on L(L(H)); $S_1(t)$ and $S_2(t)$ are, perturbation semigroups defined on L(H) then

a- The family $\{\mathbb{S}(t)\}_{t\geq 0} \subseteq L(H)$, $t \geq 0$ is a semigroup, i.e.,

1. $\mathbb{S}(0)Z = Z$, $\forall\ Z \in L(H)$
2. $\mathbb{S}(t + s)Z = \mathbb{S}(t)(\mathbb{S}(s)z)$

   $= \mathbb{S}(s)(\mathbb{S}(t))Z$

$Z \in L(H)$, $t, s \in [0, \infty)$.

b- $\|\mathbb{S}(t)\|_{L(H)} \leq M_1M_2\, e^{t(w_1+w_2)+M_1\|\Delta A_1\|_{L(H)}+M_2\|\Delta A_2\|_{L(H)}}$, for $t \in [0, \infty)$.

c- $\mathbb{S}(t) \in L(L(H))$ is a strong-operator and continuous at the origin, i.e.,

$$\tau\text{-}\lim_{t\downarrow 0}\|(\mathbb{S}(t)Z)h - (\mathbb{S}(0)Z)h\|_H = 0,\ h \in H,\ Z \in L(H).$$

Proof:

a- Let Z be an arbitrary element in L(H). By theorem(2.10) and definition(2.2)in chapter one,we have:

(i) $\mathbb{S}(0)Z = S_1(0)ZS_2(0) = IZI = Z$.

From definition (2.2) we get:

(ii) $\mathbb{S}(t + s)Z = S_1(t + s)zS_2(t + s)$

$= S_1(t)S_1(s)ZS_2(t)S_2(s)$.

From (4),we have that

$= S_1(t)\, \mathbb{S}(s)ZS_2(t)$,

since $\mathbb{S}(s)Z\in L(H)$. Hence definition (3.1),implies that:

$\mathbb{S}\,(t+s)Z= \mathbb{S}(t)\mathbb{S}(s)Z = \mathbb{S}(s)\mathbb{S}(t)Z$

b- From (4) ,we have that

$\|\mathbb{S}(t)\|_{L(L(H))} = \|S_1(t)ZS_2(t)\|_{L(H)}$

$\leq \|S_1(t)\|_{L(H)}\|Z\|_{L(H)}\|S_2(t)\|_{L(H)}$, from {theorem (2.10)}

$\leq M_1\, e^{(w_1+M_1\|\Delta A_1\|_{L(H)})t}\, \|Z\|_{L(H)} M_2\, e^{(w_2+M_2\|\Delta A_2\|_{L(H)})t}$

$$\leq M_1 M_2 e^{((w_1+w_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)t} \|Z\|_{L(H)} .$$

c- $\tau\text{-}\lim_{t\downarrow 0} \|(\mathbb{S}(t)Z)h-(\mathbb{S}(0)Z)h\|_H = \lim_{t\downarrow 0} \|(S_1(t)ZS_2(t))h-S_1(0)ZS_2(0)h\|_H$

$$= \lim_{t\downarrow 0}\Big[\|(S_1(t)ZS_2(t))h - (S_1(t)ZS_2(0))h + (S_1(t)ZS_2(0))h - (S_1(0)ZS_2(0))h\|_H \Big]$$

$$= \lim_{t\downarrow 0}\Big[\|(S_1(t)Z)[S_2(t)h - S_2(0)h] + [S_1(t)ZS_2(0)h - S_1(0)ZS_2(0)h\|_H \Big],$$

By using dfintion(2.2)

$$\tau\text{-}\lim_{t\downarrow 0} \|(\mathbb{S}(t)Z)h-(\mathbb{S}(0)Z)h\|_H \leq \tau\text{-}\lim_{t\downarrow 0}\Big[\|S_1(t)Z\|_{L(H)}\|S_2(t)h - h\|_H + \|S_1(t)Zh-Zh\|_H \Big],$$

and from theorem (2.10),we have got

$$\tau\text{-}\lim_{t\downarrow 0} \|(\mathbb{S}(t)Z)h-(\mathbb{S}(0)Z)h\|_H \leq M_1 e^{(w_1+M_1\|\Delta A_1\|_{L(H)})t} \|Z\| \ \tau\text{-}\lim_{t\downarrow 0} \|S_2(t)h - h\|_H + \tau\text{-}\lim_{t\downarrow 0} \|S_1(t)Zh - Zh\|_H$$

Now, since $\{S_1(t)\}_{t\geq 0}$ and $\{S_2(t)\}_{t\geq 0}$ are a C_0-semigroup, thus:

$$\tau\text{-}\lim_{t\downarrow 0} \|S_2(t)h - h\|_H = 0 = \tau\text{-}\lim_{t\downarrow 0} \|S_1(t)Zh - Zh\|_H \text{ ,for any } h\in D(Z)$$

and $Z\in L(H)$.

Which implies that $\{\mathbb{S}(t)\}_{t\geq 0}$ is a strongly continuous perturbation semigroup. ■

Based on the previous results and references, the following generalization have been proposed.

Lemma(3.5) :

The operator $\mathbb{A} + \Delta\mathbb{A}$ of problem formulation is infinitesimal generator for $\mathbb{S}(t)$ defined on its domain $D(\mathbb{A}+\Delta\mathbb{A})$ satisfying the following properties:

(a) $D(\mathbb{A} + \Delta\mathbb{A})$ is strong-operator dense in L(H).

(b) $\mathbb{A} + \Delta\mathbb{A}$ is uniform-operator closed on L(H).

(c) For $Z \in L(H)$:

$$\int_0^t (\mathbb{S}(r)Z)\, dr \in D(\mathbb{A} + \Delta\mathbb{A}), \text{ and}$$

$(\mathbb{A} + \Delta\mathbb{A}) \left(\int_0^t \mathbb{S}(r)Z\,dr \right) = \mathbb{S}(t)Z - Z.$

(d) For $Z \in D(\mathbb{A})$:

$\mathbb{S}(t)Z \in D(\mathbb{A} + \Delta\mathbb{A})$, the function $t:[0,\infty) \mapsto \mathbb{S}(t)Z \in L(H)$ is continuously differentiable in $\{L(H),\tau\}$ and

$$\frac{d}{dt}(\mathbb{S}(t)Z) = (\mathbb{A} + \Delta\mathbb{A})(\mathbb{S}(t)Z) = \mathbb{S}(t)((\mathbb{A} + \Delta\mathbb{A})Z)$$

(e) For $Z \in D(\mathbb{A} + \Delta\mathbb{A})$ and $h \in D(A_1 + \Delta A_1)$

$((\mathbb{A} + \Delta\mathbb{A})Z)h = (A_1 + \Delta A_1)Zh + Z(A_2 + \Delta A_2)h$.

Proof:

(a) By lemma(3.4)(c), $\mathbb{S}(t)Z$ is integrable, so $Z_t \equiv \int_0^t \mathbb{S}(s)Z\,ds$,

for a fixed $Z \in L(H)$, and fixed $t > 0$. Thus:

$$\mathbb{S}(\Delta)Z_t - Z_t = \int_0^t [\mathbb{S}(s + \Delta)Z - \mathbb{S}(s)]ds\ , \Delta \in (0,t)\ . \tag{5}$$

Hence, equation (5) becomes:

$$\int_0^{\Delta+t} \mathbb{S}(s)Z\,ds - \int_0^{\Delta} \mathbb{S}(s)Z\,ds = \int_0^{\Delta} \mathbb{S}(s + t)Z\,ds - \int_0^{\Delta} \mathbb{S}(s)Z\,ds = \int_0^{\Delta} \mathbb{S}(s)\mathbb{S}(t)Z\,ds - \int_0^{\Delta} \mathbb{S}(s)Z\,ds\ .$$

Then from Remark (2.5)(i), we have that:

$$\tau-\lim_{t\downarrow 0} \frac{\mathbb{S}(t)-I}{t} Z_t = \mathbb{S}(t)Z - Z\ , \tag{6}$$

that implies to $Z_t \in D(\mathbb{A} + \Delta\mathbb{A})$ and $\{Z_t\}_{t>0}$ generates a linear space contained in $D(\mathbb{A}+\Delta\mathbb{A})$,which implies that $\{\frac{Z_t}{t} : t > 0\} \subseteq D(\mathbb{A}+\Delta\mathbb{A})$ and $\lim_{t\downarrow 0} \frac{Z_t}{t} = Z$, for arbitrary $Z \in L(H)$ .Hence $D(\mathbb{A} + \Delta\mathbb{A})$ is dense in $\{L(H),\tau\}$.

(b) Since $D(\mathbb{A} + \Delta\mathbb{A})$ is dense in $\{L(H),\tau\}$, one can define $\{Z_n\}_{n=0}^{\infty}$ to be a $\|.\|$ bounded sequence in $D(\mathbb{A}+\Delta\mathbb{A}) \subseteq L(H)$ and $Z_n \longrightarrow Z$ as $n \longrightarrow \infty$ in $\{L(H), \tau\}$,where Z is $\|.\|$ bounded.

Now, let $(\mathbb{A} + \Delta\mathbb{A})Z_n \longrightarrow Y$; by remark(2.5)(iii) we have that:

$$\tau\text{-}\lim_{t\downarrow 0} \frac{\mathbb{S}(s)-I}{t} Z_n = \tau\text{-}\lim_{t\downarrow 0} \frac{1}{t} \int_0^t \mathbb{S}(s)\ (\mathbb{A} + \Delta\mathbb{A})\ Z_n\,ds.$$

Now, as $n \longrightarrow \infty$, we get:

$$\tau\text{-}\lim_{t\downarrow 0} \frac{\mathbb{S}(s)-I}{t} Z = \tau\text{-}\lim_{t\downarrow 0} \frac{1}{t} \int_0^t \mathbb{S}(s)Y\,ds\ .$$

Then by concluding remark (2.3)(c) and Remark(2.5)(i), we get

$(\mathbb{A}+\Delta\mathbb{A})\, Z = Y.$

Hence $\mathbb{A}+\Delta\mathbb{A}$ is a closed linear operator.

(c) Let $Z \in D(\mathbb{A}+\Delta\mathbb{A})$.

On using simple calculation of semigroup, as follows, we get :

$$(\mathbb{A}+\Delta\mathbb{A})\int_0^t \mathbb{S}(s)Z\,ds = \tau\text{-}\lim_{t\downarrow 0} \frac{\mathbb{S}(\Delta)\int_0^t \mathbb{S}(s)Z\,ds - \int_0^t \mathbb{S}(s)Z\,ds}{\Delta} = \tau\text{-}\lim_{t\downarrow 0} \frac{\int_0^t \mathbb{S}(s+\Delta)Z\,ds - \int_0^t \mathbb{S}(s)Z\,ds}{\Delta}$$

$$= \tau\text{-}\lim_{t\downarrow 0} \frac{1}{\Delta}\int_0^\Delta \mathbb{S}(t)\mathbb{S}(s)Z\,ds - \tau\text{-}\lim_{t\downarrow 0} \frac{1}{\Delta}\int_0^\Delta \mathbb{S}(s)Z\,ds = \mathbb{S}(t)Z - Z$$

From Concluding Remark (3.3)(c), we obtain:

$$\int_0^t \mathbb{S}(s)Z\,ds \in D(\mathbb{A}+\Delta\mathbb{A}).$$

(d) One can show that $\mathbb{S}(t)Z \in D(\mathbb{A}+\Delta\mathbb{A})$.

Let $Z \in D(\mathbb{A}+\Delta\mathbb{A})$,such that

$$(\mathbb{A}+\Delta\mathbb{A})\mathbb{S}(t)Z = \tau\text{-}\lim_{t\downarrow 0} \frac{\mathbb{S}(\Delta)\mathbb{S}(t)Z - \mathbb{S}(t)Z}{\Delta} = \tau\text{-}\lim_{t\downarrow 0} \mathbb{S}(t)\left[\frac{\mathbb{S}(\Delta)Z - Z}{\Delta}\right] = \mathbb{S}(t)(\mathbb{A}+\Delta\mathbb{A})Z\,,$$

implies that $\mathbb{S}(t)Z \in D(\mathbb{A}+\Delta\mathbb{A})$, and the right derivation exists in $\{L(H),\tau\}$.Hence,

$$(\mathbb{A}+\Delta\mathbb{A})\mathbb{S}(t)Z = \mathbb{S}(t)(\mathbb{A}+\Delta\mathbb{A})Z.$$

Now, one can show that the following left derivative for $\Delta > 0$ exists.

$$\tau\text{-}\lim_{t\downarrow 0}\left[\frac{\mathbb{S}(t)Z - \mathbb{S}(t-\Delta)Z}{\Delta} - \mathbb{S}(t)(\mathbb{A}+\Delta\mathbb{A})Z\right] \tag{7}$$

By adding $(\mathbb{A}+\Delta\mathbb{A})\mathbb{S}(t\text{-}\Delta)$ to (7),we obtain:= $\tau\text{-}\lim_{t\downarrow 0} \mathbb{S}(t-\Delta)\left[\frac{\mathbb{S}(\Delta)Z - Z}{\Delta} - (\mathbb{A}+\Delta\mathbb{A})Z\right]$

$$+\tau\text{-}\lim_{t\downarrow 0}[\mathbb{S}(t-\Delta)(\mathbb{A}+\Delta\mathbb{A})Z - \mathbb{S}(t)(\mathbb{A}+\Delta\mathbb{A})Z]. \tag{8}$$

Hence: $\tau\text{-}\lim_{t\downarrow 0}\|\mathbb{S}(t-\Delta)\left[\frac{\mathbb{S}(\Delta)Z - Z}{\Delta} - (\mathbb{A}+\Delta\mathbb{A})Z\right] + \mathbb{S}(t-\Delta)(\mathbb{A}+\Delta\mathbb{A})Z - \mathbb{S}(t)(\mathbb{A}+\Delta\mathbb{A})Z\|_{L(H)} \le \tau\text{-}\lim_{t\downarrow 0}\|\mathbb{S}(t-\Delta)\| \lim_{t\downarrow 0}\left\|\frac{\mathbb{S}(\Delta)Z - Z}{\Delta} - (\mathbb{A}+\Delta\mathbb{A})Z\right\|_{L(H)}$

$$+\tau\text{-}\lim_{t\downarrow 0}\|\mathbb{S}(t-\Delta)(\mathbb{A}+\Delta\mathbb{A})Z - \mathbb{S}(t)(\mathbb{A}+\Delta\mathbb{A})Z\|_{L(H)}.$$

Now, from Concluding Remark(3.3)(c) and the strongly continuous of $\{\mathbb{S}(t)\}_{t\geq 0}$, we get:

$$\tau\text{-}\lim_{t\downarrow 0}\frac{\mathbb{S}(t)Z - \mathbb{S}(t-\Delta)Z}{\Delta} = \mathbb{S}(t-\Delta)(\mathbb{A}+\Delta\mathbb{A})Z\,,$$

for any $Z \in D(\mathbb{A}+\Delta\mathbb{A})$.

(e) For $Z \in D(\mathbb{A} + \Delta\mathbb{A})$, $h \in D(A_1 + \Delta A_1)$ and $g \in D((A_1 + \Delta A_1)^*)$, we get:

$$<(\mathbb{A} + \Delta\mathbb{A})Zh, g>_H = <\tau\text{-}\lim_{t\downarrow 0} \frac{\mathbb{S}(t)Zh - Zh}{t}, g>_H = \tau\text{-}\lim_{t\downarrow 0} \frac{1}{t} < \mathbb{S}(t)Zh - Zh, g>_H .$$

By using (2.6),we have that:

$$\tau\text{-}\lim_{t\downarrow 0} \frac{1}{t} <\mathbb{S}(t)Zh - Zh, g>_H = \lim_{t\downarrow 0} \frac{1}{t} <S_1(t)ZS_2(t)h - Zh, g>_H \quad (9)$$

Adding $(S_1(t)Zh + ZS_2(t)h+Zh)$ to (9), we obtain ;

$$= \lim_{t\downarrow 0} \frac{1}{t} <S_1(t)ZS_2(t)h - S_1(t)Zh - ZS_2(t)h + S_1(t)Zh + ZS_2(t)h - Zh + Zh - Zh, g>_H = \lim_{t\downarrow 0} \frac{1}{t} <(S_1(t) - I)Z(S_2(t) - I)h, g>_H + \lim_{t\downarrow 0} \frac{1}{t} <ZS_2(t)h-Zh, g>_H + \lim_{t\downarrow 0} <S_1(t)Zh-Zh, g>_H . \quad (10)$$

Since $\{S_1(t)\}_{t\geq 0}$ is a family of bounded operators, the relation (10) becomes:

$$= \lim_{t\downarrow 0} \frac{1}{t} <Z(S_2(t) - I)h, (S_1(t) - I)^*g>_H + \lim_{t\downarrow 0} \frac{1}{t} <Z(S_2(t) - I)h, g>_H + \lim_{t\downarrow 0} \frac{1}{t} <Zh, (S_1(t) - I)^*g>_H = <\lim_{t\downarrow 0} Z(S_2(t) - I)h, \lim_{t\downarrow 0} \frac{1}{t}(S^*_1(t) - I)g>H + <\lim_{t\downarrow 0} \frac{1}{t} Z(S_2(t) - I)h, g>_H + <Zh, \lim_{t\downarrow 0} (S^*_1(t) - I), g>_H . \quad (11)$$

By using definition (2.3) of infinitesimal generator, (11) becomes:

$$<(\mathbb{A} + \Delta\mathbb{A})Zh, g>_H = <0, (A_1 + \Delta A_1)^*g>_H + <Z(A_2 + \Delta A_2)h, g>_H + <Zh, (A_1 + \Delta A_1)^*g>_H . \quad (12)$$

From definition(2.1) of unbounded adjoint operator of a Hilbert space, we have that

$$<(\mathbb{A} + \Delta\mathbb{A})Zh, g>_H = <(A_1 + \Delta A_1)Zh + Z(A_2 + \Delta A_2)h, g>_H$$

for all $g \in D(A_1 + \Delta A_1)^*)$. Thus:

$(\mathbb{A}+\Delta\mathbb{A})Zh = (A_1+\Delta A_1)Zh + Z(A_2+\Delta A_2)h$ and $(\mathbb{A}+\Delta\mathbb{A})Zh \in H$. ■

The following theorem presents some properties of the unbounded perturbed operator $\mathbb{A} + \Delta\mathbb{A}$, of problem formulation.

Theorem(3.6) :

Let $\{\mathbb{S}(t)\}_{t\geq 0}$ be a family of a C_0-composite perturbation semigroup generated by unbounded linear operator $\mathbb{A} + \Delta\mathbb{A}$ satisfies:

$$\|\mathbb{S}(t)\|_{L(L(H))} \leq M_1 M_2 e^{((w_1+w_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)t}, \text{ for } M_1, M_2 \geq 1,$$

$w_1, w_2 \geq 0$ and $\Delta A_1, \Delta A_2 \in L(H)$.

Then the resolvent set $\rho(\mathbb{A} + \Delta\mathbb{A})$ contains the ray $(w_1+w_2 + M_1\|\Delta A_1\| + M_2\|\Delta A_2\|, \infty)$such that the resolvent operator of $\mathbb{A}+\Delta\mathbb{A}$ is estimated as:

$$\|\mathbb{R}(\lambda:\mathbb{A}+\Delta\mathbb{A})\| \le \frac{M_1M_2}{\mathrm{Re}\lambda-\left[(w_1+w_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|\right]},$$

for Re$\lambda > (w_1+w_2)+M_1\|\Delta A_1\|_{L(H)}+M_2\|\Delta A_2\|_{L(H)}$.

Proof:

From remark(2.5)(v),

$$\mathbb{R}(\lambda)Z=\int_0^\infty e^{-\lambda t}\,\mathbb{S}(t)Z\,dt,\text{ for }\lambda>0,\ Z\in L(H)\ \text{ and}$$

$$\|\mathbb{R}(\lambda)Z\|_{L(H)} \le \left\|\int_0^\infty e^{-\lambda t}\mathbb{S}(t)Z\,dt\right\|_{L(H)} \le \int_0^\infty e^{-\lambda t}\|\mathbb{S}(t)Z\|_{L(H)}\,dt .$$

From lemma (3.4)(b), we obtain:

$$\int_0^\infty e^{-\lambda t}\|\mathbb{S}(t)Z\|_{L(H)}dt=\int_0^\infty e^{-\lambda t}M\,e^{(w_1+w_2)+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|}\|Z\|_{L(H)}\,dt$$

$$=\int_0^\infty M_1M_2\,e^{-t\left(\lambda-M_1\|\Delta A_1\|_{L(H)}+M_2\|\Delta A_2\|_{L(H)}+(w_1+w_2)\right)}\|Z\|_{L(H)}\,dt$$

$$=\frac{M_1M_2\|Z\|_{L(H)}}{\lambda-\left[M_1\|\Delta A_1\|_{L(H)}+M_2\|\Delta A_2\|_{L(H)}+(w_1+w_2)\right]}.$$

for $\lambda>\left[M_1\|\Delta A_1\|_{L(H)}+M_2\|\Delta A_2\|_{L(H)}+(w_1+w_2)\right]$

Hence

$$\|\mathbb{R}(\lambda)Z\|_{L(H)}\le\frac{M_1M_2\|Z\|_{L(H)}}{\mathrm{Re}\lambda-\left[M_1\|\Delta A_1\|_{L(H)}+M_2\|\Delta A_2\|_{L(H)}+(w_1+w_2)\right]} \qquad (13)$$

Furthermore, for $h\in(0,\infty)$

$$\frac{\mathbb{S}(h)-I}{h}\mathbb{R}(\lambda)Z=\frac{1}{h}\int_0^\infty e^{-\lambda t}(\mathbb{S}(t+h)Z-\mathbb{S}(t)Z)\,dt$$

$$=\frac{1}{h}\int_0^\infty e^{-\lambda t}(\mathbb{S}(t+h)Z\,dt-\frac{1}{h}\int_0^\infty e^{-\lambda t}\mathbb{S}(t)Z\,dt=\frac{1}{h}\left[\int_0^\infty e^{-\lambda(t-h)}\mathbb{S}(t)Z\,dt-\int_0^\infty e^{-\lambda t}\mathbb{S}(t)Z\,dt\right]$$

$$=\frac{1}{h}\left[\int_0^\infty e^{-\lambda(t-h)}\mathbb{S}(t)Z\,dt-\int_0^\infty e^{-\lambda(t-h)}\mathbb{S}(t)Z\,dt-\int_0^\infty e^{-\lambda t}\mathbb{S}(t)Z\,dt\right]$$

$$= \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}\mathbb{S}(t)Z\,dt - \frac{e^{\lambda h}}{h}\int_0^\infty e^{-\lambda t}\mathbb{S}(t)Z\,dt, \tag{14}$$

as $h \uparrow 0$, the right hand side of (14) converges to $\lambda\mathbb{R}(\lambda)Z - Z$ in $\{L(H),\tau\}$. Hence:

$$(\mathbb{A} + \Delta\,\mathbb{A})\mathbb{R}(\lambda)Z = \lambda\mathbb{R}(\lambda)Z - Z .$$

Thus:

$$\lambda\mathbb{R}(\lambda)Z - (\mathbb{A} + \Delta\,\mathbb{A})\lambda\mathbb{R}(\lambda)Z = Z$$

$$(\lambda I - (\mathbb{A} + \Delta\,\mathbb{A}))\mathbb{R}(\lambda)Z = Z$$

$$\mathbb{R}(\lambda) = (\lambda - (\mathbb{A} + \Delta\mathbb{A}))^{-1}, \text{ for } \lambda\in\rho(\mathbb{A} + \Delta\mathbb{A}), \tag{15}$$

and from the fact that

$\|\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A})\| = \|(\lambda - (\mathbb{A} + \Delta\mathbb{A}))^{-1}\|$(see the definition of H_{-1}) and hence

$\|\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A})Z\| = \|Z\|_{H_{-1}} = \|(\lambda - (\mathbb{A} + \Delta\mathbb{A}))^{-1}\|$

From (13) and if $\lambda\in\sqcup$ with $\lambda\in\rho(\mathbb{A} + \Delta\mathbb{A})$,we gets

$$\|\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A})\| \le \frac{M_1M_2}{\mathrm{Re}\lambda - \left[(w_1 + w_2) + M_1\|\Delta A_1\| + M_2\|\Delta A_2\|\right]}$$

for $\mathrm{Re}\lambda > ((w_1 + w_2) + M_1\|\Delta A_1\|_{L(H)} + M_2\|\Delta A_2\|_{L(H)})$. ■

Corollary(3.7) :

Let the condition of theorem(3.6) be satisfied, then:

a-$\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A})\,\mathbb{S}(t) = \mathbb{S}(t)\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A})$.

b-$\mathbb{R}(\lambda)(\mathbb{A} + \Delta\mathbb{A})Z = (\mathbb{A} + \Delta\,\mathbb{A})\mathbb{R}(\lambda)Z$, for $Z \in D(\mathbb{A} + \Delta\mathbb{A})$.

Proof:

a- By using the identity $(\lambda I - (\mathbb{A} + \Delta\mathbb{A}))\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A})$, we have that

$$\mathbb{S}(t) = \mathbb{S}(t)(\lambda I - (\mathbb{A} + \Delta\mathbb{A}))\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A}) .$$

$$= (\lambda\mathbb{S}(t) - \mathbb{S}(t)(\mathbb{A} + \Delta\mathbb{A}))\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A}) . \tag{16}$$

By Lemma (3.5)(d), yields:

$$\mathbb{S}(t) = (\lambda I - (\mathbb{A} + \Delta\mathbb{A}))\mathbb{S}(t)\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A}) \tag{17}$$

By multiplying both sides of (17) by $\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A})$, we get:

$$\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\,\mathbb{A})\mathbb{S}(t) = \mathbb{S}(t)\,\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A}).$$

b- Since $\mathbb{R}(\lambda)(\mathbb{A} + \Delta\,\mathbb{A})Z = \int_0^\infty e^{-\lambda t}\,\mathbb{S}(t)(\mathbb{A} + \Delta\mathbb{A})Z\,dt$, for $\lambda\in\rho(\mathbb{A} + \Delta\mathbb{A})$

$$= \int_0^\infty e^{-\lambda t}(\mathbb{A} + \Delta\mathbb{A})\mathbb{S}(t)Z\,dt = (\mathbb{A} + \Delta\mathbb{A})\left(\int_0^\infty e^{-\lambda t}\mathbb{S}(t)Z\,dt\right) = (\mathbb{A} + \Delta\mathbb{A})\mathbb{R}(\lambda)Z,$$

then (15) implies that:

$$\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A})(\mathbb{A} + \Delta\mathbb{A})Z = (\mathbb{A} + \Delta\mathbb{A})\mathbb{R}(\lambda{:}\mathbb{A} + \Delta\mathbb{A})Z. \quad ■$$

Definition (3.8):

The continuous function $Z(.) \in D(\mathbb{A} + \Delta\mathbb{A})$ given by:

$Z(t) = \mathbb{S}(t)\, Z_0$ for any $Z_0 \in L(H)$ and $t \geq 0$

which is strong operator differentiable in L(H), is called a strong solution to the linear perturbation initial value problem (3).

Concluding Remark(3.9) :

The necessary and sufficient conditions for any $Z \in D(\mathbb{A} + \Delta\mathbb{A})$ is that restriction Z to $D(A_2 + \Delta A_2)$ belong to $L(D(A_2 + \Delta A_2), D(A_1 + \Delta A_1))$, i.e., $D(\mathbb{A} + \Delta\mathbb{A}) \subset L(H) \cap L(D(A_2 + \Delta A_2), D(A_1 + \Delta A_1))$ and an extension of $(\mathbb{A} + \Delta\mathbb{A})\, Z \in L(D(A_2 + \Delta A_2), H)$ to H belong to L(H).

We are now interested in the relation between the semigroup $\mathbb{T}(t)$ generated by $\mathbb{A}$ and $\mathbb{S}(t)$ generated by $\mathbb{A} + \Delta\mathbb{A}$. By condition (a),(b), $\mathbb{T}(t)$ and $\mathbb{S}(t)$ are C_0-semigroups generated by the linear operators $\mathbb{A}$ and $\mathbb{A} + \Delta\mathbb{A}$ respectively and let $Z(.) \in D(\mathbb{A} + \Delta\mathbb{A})$. Then by remark (2.5)(ii) , we have $\mathbb{T}(t - s)\mathbb{S}(s)Z$ is differentiable, that implies the L(H)-value function, let:

$H(s) = \mathbb{T}(t - s)\mathbb{S}(s)Z$ for $0 < s < t$;

and from lemma(2.2)(d),

$$\frac{d}{ds} H(s)Z = \mathbb{T}(t - s)\frac{d}{ds}\mathbb{S}(s)Z + \frac{d}{ds}\mathbb{T}(t - s)\mathbb{S}(s)Z,$$

$$= \mathbb{T}(t - s)(\mathbb{A} + \Delta\mathbb{A})\,\mathbb{S}(s)Z - \mathbb{A}\,\mathbb{T}(t - s)\,\mathbb{S}(s)Z$$

$$= \mathbb{T}(t - s)\Delta\mathbb{A}\mathbb{S}(s)Z .$$

Integrating $\frac{dH(s)}{ds} Z$ from 0 to t, yields:

$$\int_0^t \frac{d}{ds} H(s) Z\, ds = \int_0^t \mathbb{T}(t - s)\Delta\mathbb{A}\mathbb{S}(s)Z\, ds$$

$$\mathbb{S}(t)Z = \mathbb{T}(t)Z + \int_0^t \mathbb{T}(t - s)\Delta\mathbb{A}\mathbb{S}(t)Z\, ds, \text{ for } Z \in D(\mathbb{A}+\Delta\mathbb{A}). \tag{18}$$

Since the operator on both sides of (18) are bounded,then (18) holds for every $Z \in L(H)$.

Theorem(3.10):

Let $\mathbb{T}(t)$ be a C_o-Composite semigroup of problem formulation satisfying:

$\| \mathbb{T}(t)\|_{L(H)} \leq M_1 M_2\, e^{(w_1 + w_2)t}$.

Let $\Delta\mathbb{A}$ be a bounded linear operator on a Banach space L(H). Then there exist a unique family $\mathbb{S}(t)$, $t \geq 0$ of bounded operators on L(H) such that (18) is continuous on $[0, \infty)$, for every $Z \in L(H)$.

Proof:

The main steps of the proof is as follows:

Set :

$$\mathbb{S}_0(t) = \mathbb{T}(t) \tag{19}$$

and define $\mathbb{S}_n(t)$ inductively by:

$$\mathbb{S}_{n+1}(t)Z = \int_0^t \mathbb{T}(t-s)\Delta\,\mathbb{A}\mathbb{S}_n(t)Z\,ds\,, \tag{20}$$

for $Z \in L(H)$, and $n \geq 0$.

We shall prove, by induction, that $\{\mathbb{S}_n(t)\}_{t\geq 0}$ is continuous family.

Now, for $n = 0$, we have from condition(a) of problem formulation, that $\mathbb{S}_0(t)Z = \mathbb{T}(t)Z$ is continuous for $t \geq 0$ and $Z \in L(H)$.

We assume that:

$$\lim_{n\to\infty} \|\mathbb{S}_n(t)Z - \mathbb{S}_n(t)Z_1\|_{L(H)} = 0, \tag{21}$$

for $Z_1 \in L(H)$ and for all $Z \in L(H)$.

Now:

$$\|\mathbb{S}_{n+1}(t)Z - \mathbb{S}_{n+1}(t)Z_1\| = \left\|\int_0^t \mathbb{T}(t-s)\Delta\,\mathbb{A}(\mathbb{S}_n(t)Z - \mathbb{S}_n(t)Z_1)\,ds\right\|_{L(H)}$$

$$\leq \int_0^t \|\mathbb{T}(t-s)\|_{L(H)}\|\Delta\mathbb{A}\|_{L(H)}\,\|\mathbb{S}_n(t)Z - \mathbb{S}_n(t)Z_1\|_{L(H)}\,ds\,.$$

From (21), we get:

$$\lim_{n\to\infty} \|\mathbb{S}_{n+1}(t)Z - \mathbb{S}_{n+1}(t)Z_1\|_{L(H)} = 0.$$

Thus $t \longrightarrow \mathbb{S}_{n+1}(t)Z$ is continuous, for $Z \in L(H)$, $t \geq 0$ and every $n \geq 0$.

From above, we have :

$$\|\mathbb{S}_n(t)\|_{L(H)} \leq \frac{M_1M_2e^{(w_1+w_2)t}}{n!}M_1^nM_2^n\|\Delta\mathbb{A}\|^n t^n\,. \tag{22}$$

For $n = 0$, we have:

$$\|\mathbb{S}_0(t)\|_{L(H)} = \|\pi(t)\|_{L(H)} \leq M_1M_2\,e^{(w_1+w_2)t}\,.$$

Assume that (22) holds for any $n \in N$.Then by (20), we get:

$$\|\mathbb{S}_{n+1}(t)Z\|_{L(H)} = \left\|\int_0^t \mathbb{T}(t-s)\Delta\mathbb{A}\mathbb{S}_n(s)Z\,ds\right\|_{L(H)}$$

$$\leq \int_0^t \| \mathbb{T}(t-s)\|_{L(H)} \|\Delta \mathbb{A}\|_{L(H)} \| \mathbb{S}_n(s)\|_{L(H)} \|Z\|_{L(H)}\, ds \leq \int_0^t M_1M_2\, e^{(w_1+w_2)(t-s)} \|\Delta\mathbb{A}\|_{L(H)} M_1M_2\, e^{(w_1+w_2)s} \frac{M_1^n M_2^n \|\Delta\mathbb{A}\|^n s^n \|Z\|_{L(H)}}{n!} ds$$

$$= M_1M_2\, e^{(w_1+w_2)t} \frac{M_1^{n+1} M_2^{n+1} e^{(w_1+w_2)s}}{(n+1)!} \|\Delta\mathbb{A}\|^{n+1} s^{n+1} \|Z\|_{L(H)}\, ds$$

for $n \geq 0$ and $Z \in L(H)$.

The integral equation (18) is a Volterra integral equation of the second kind with continuous kernel of difference type $k(s, t) = \mathbb{T}(t-s)\Delta\mathbb{A}$.This equation has a solution may often appear as integral of the form :

$$\mathbb{S}(t) = \mathbb{S}_0(t) + \int_0^t \Gamma(t, \xi; 1)\mathbb{S}_0(\xi)\, d\xi , \tag{23}$$

where $\Gamma(t, \xi; 1)$ is called the resolvent bounded kernel of integral equation (23) and $k(t, \xi)$ and $\mathbb{S}_0(t)$ in (18) are both continuous.

It is easy to construct the resolvent $\Gamma(t, \xi; 1)$ for (23) as Numann series:

$$\Gamma(t, \xi; 1) = \sum_{n=0}^{\infty} k_{n+1}(t, \xi) ,$$

where $k_{n+1}(t, \xi)$, the iterated kernel, such that:

$$\Gamma(t, \xi; 1) = \sum_{n=0}^{\infty} k_{n+1}(t, \xi) ,$$

where $k_1(t, y) \equiv k(t, y)$. Thus:

$$\mathbb{S}(t) = \mathbb{S}_0(t) + \mathbb{S}_1(t) + \mathbb{S}_2(t) + \dots ,$$

where:

$$\mathbb{S}_0(t) = \mathbb{T}(t)$$

.

.

$$\mathbb{S}_n(t) = \int_0^t k(t, \xi)\, \mathbb{S}_{n-1}(\xi)\, d\xi$$

So:

$$\mathbb{S}(t) = \sum_{n=0}^{\infty} \mathbb{S}_n(t) \tag{24}$$

By using (22), yields:

$$\| \mathbb{S}(t)\|_{L(H)} = \left\| \sum_{n=0}^{\infty} \mathbb{S}_n(t) \right\|_{L(H)}$$

$$\leq M_1M_2 e^{(w_1+w_2)t} \sum_{n=0}^{\infty} \frac{M_1^n M_2^n \| \Delta\mathbb{A} \|^n \, t^n}{n!} .$$

The right part of inequality is convergent, the series (24) is uniformly convergence in the uniform operator topology on bounded interval, and $t \longrightarrow \mathbb{S}(t)Z$ is continuous for every $Z \in L(H)$.Therefore, $\mathbb{S}(t)Z \in C([0, t], L(H))$

To prove the uniqueness, let $U(t)$, $t \geq 0$ be a uniformly bounded operator for which

$t \longrightarrow U(t)Z$ is continuous for $Z \in L(H)$ and:

$$U(t)Z = \mathbb{T}(t)Z + \int_0^t \mathbb{T}(t-s)\Delta \mathbb{A} U(t)Z \, ds , \tag{25}$$

for $Z \in L(H)$.

By subtracting (24) from (18), yields:

$$\| \mathbb{S}(t)Z - U(t))Z\|_{L(H)} \leq \int_0^t M_1M_2 e^{(w_1+w_2)t} \|\Delta A\|_{L(H)} \|S(s) - U(s)\|_{L(H)} \|Z\|_{L(H)} \, ds$$

Hence from Gronwall's inequality, we get:

$\| \mathbb{S}(t) - U(t))Z\|_{L(H)} = 0$, for $t \geq 0$ and thus $\mathbb{S}(t) = U(t)$. ■

Corollary(3.11):

Let $\mathbb{A}$ be the infinitesimal generator of a C_0-composite semigroup $\mathbb{T}(t)$ satisfying $\| \mathbb{T}(t)\|_{L(H)} \leq M_1M_2 e^{(w_1+w_2)t}$. Let $\Delta\mathbb{A}$ be a bounded operator and let $\mathbb{S}(t)$ be the infinitesimal generator by $\mathbb{A} + \Delta\mathbb{A}$. Then:

$$\| \mathbb{S}(t) - \mathbb{T}(t)\|_{L(H)} \leq M_1M_2 e^{(w_1+w_2)t} \left(e^{(M_1\|\Delta A_1\|+M_2\|\Delta A_1\|)t} - 1 \right)$$

Proof:

On using (18) and conditions (b), (d), one gets:

$$\| \mathbb{S}(t)Z - \mathbb{T}(t)Z\|_{L(H)} = \| \mathbb{T}(t)Z + \int_0^t \mathbb{T}(t-s)\Delta \mathbb{A}\mathbb{S}(s)Z \, ds - \mathbb{T}(t)Z\|_{L(H)}$$

$$\leq \int_0^t \| \mathbb{T}(t-s)\| \, \|\Delta \mathbb{A}\|_{L(H)} \| \mathbb{S}(s)\|_{L(H)} \|Z\|_{L(H)} \, ds .$$

By using conditions (b),(e) of problem formulation , we obtain:

$$\| \mathbb{S}(t)Z - \mathbb{T}(t)Z\|_{L(H)} \leq \int_0^t M_1M_2 e^{(w_1+w_2)t} \|\Delta \mathbb{A}\|_{L(H)} M_1M_2$$

$$e^{s(w_1+w_2+M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)} \|Z\|_{L(H)}\, ds$$

$$= M_1^2 M_2^2 e^{(w_1+w_2)t} \|\Delta \mathbb{A}\|_{L(H)} \left[\frac{e^{(M_1\|\Delta A_1\|+M_2\|\Delta A_2\|)t}}{M_1\|\Delta A_1\|+M_2\|\Delta A_2\|} - \frac{1}{M_1\|\Delta A_1\|+M_2\|\Delta A_2\|} \right] \|Z\|_{L(H)} \ . \tag{26}$$

From condition (f) of problem formulation and the fact that $M_1, M_2 \geq 1$, we have that

$$\|\Delta\mathbb{A}.\|_{L(L(H))} = \|\Delta A_1. + .\Delta A_2\|_{L(H)} \leq \|\Delta A_1.\|_{L(H)} + \|.\Delta A_2\|_{L(H)}$$

$$\leq M_1\|\Delta A_1.\|_{L(H)} + M_2 \|.\Delta A_2\|_{L(H} \leq (M_1\|\Delta A_1\|_{L(H)} + M_2 \|\Delta A_2\|_{L(H)}) \|.\|_{L(H)} \ .$$

Hence (26), becomes:

$$\| \mathbb{S}(t) - \mathbb{T}(t)\|_{L(H)} \leq M_1^2 M_2^2 \ e^{(w_1+w_2)t} \left(e^{(M_1\|\Delta A_1\|+M_2\|\Delta A_1\|)t} - 1 \right) \|Z\|_{L(H)}$$

For M_1, $M_2 \geq 1$, w_1, $w_2 \geq 0$ and ΔA_1, $\Delta A_2 \in L(H)$.

Concluding Remark(3.12) :

The addition of a bounded linear operator $\Delta \mathbb{A}$ such that $D(\mathbb{A}) \subseteq D(\Delta\mathbb{A})$, to an infinitesimal generator $\mathbb{A}$ of a C_0-semigroup does not destroy the analytic and contraction properties, [6].

Theorem(3.13) :

Let $\mathbb{A}$ be the infinitesimal generator of a compact composite semigroup $\mathbb{T}(t)$ and, the resolvent operator for $\mathbb{A}$ satisfies

$$\|\mathbb{R}(\lambda:\mathbb{A})\|_{L(H)} \leq \frac{M_1M_2}{\lambda - (w_1 + w_2)}, \text{ for } \lambda > w_1 + w_2. \tag{27}$$

Let $\Delta\mathbb{A}$ be a bounded operator, then $\mathbb{A} + \Delta \mathbb{A}$ is the infinitesimal generator of a compact composite perturbation C_0-semigroup $\mathbb{S}(t)$.

Proof:

Assume that $\lambda > w_1+w_2+M_1M_2\|\Delta\mathbb{A}\|_{L(H)}$, for $M_1, M_2 \geq 1$, w_1, $w_2 \geq 0$.

Thus

$$\frac{M_1M_2 \|\Delta\mathbb{A}\|}{\lambda - (w_1 + w_2)} < 1 \ ,$$

$$\| \Delta\mathbb{A}\mathbb{R}(\lambda;\mathbb{A})\|_{L(L(H))} \le \|\Delta \mathbb{A}\|_{L(L(H))}\|\mathbb{R}(\lambda;\mathbb{A})\|_{L(L(H))} \quad \le \frac{\| \Delta\mathbb{A} \|_{L(L(H))}\, M_1M_2}{\lambda-(w_1+w_2)} < 1 \tag{28}$$

Hence:

$$\| \Delta\mathbb{A}\mathbb{R}(\lambda;\mathbb{A})\|_{L(H)} < 1 \;. \tag{29}$$

From (29), we get $(I - \Delta \mathbb{A}\, \mathbb{R}(\lambda;\mathbb{A}))$ is invertible and bounded for $\lambda > w_1 + w_2 + M_1M_2\|\Delta\mathbb{A}\|_{L(L(H))}$, set

$$R = \mathbb{R}(\lambda;\mathbb{A})(I - \Delta \mathbb{A}\mathbb{R}(\lambda;\mathbb{A}))^{-1} \text{ [from } (I - \Delta \mathbb{A}\mathbb{R}(\lambda;\mathbb{A}))^{-1}=\mathbb{R}(\lambda;\mathbb{A}) \sum_{k=0}^{\infty}\left[\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\right]^k) . \tag{30}$$

We have to show that R is a resolvent operator of $\mathbb{A}+\Delta\mathbb{A}$ and for $\lambda > w_1 + w_2 + M_1M_2\|\Delta\mathbb{A}\|_{L(L(H))}$. Note that

$$\begin{aligned}(\lambda\mathbb{I} - (\mathbb{A} + \Delta\mathbb{A}))R &= (\lambda\mathbb{I} - (\mathbb{A} + \Delta\mathbb{A}))\mathbb{R}(\lambda;\mathbb{A})[(I - \Delta\mathbb{A}\mathbb{R}(\lambda;\mathbb{A}))]^{-1}\\ &= (\lambda I - \mathbb{A} - \Delta\mathbb{A})[\mathbb{R}(\lambda;\mathbb{A})[(I - \Delta \mathbb{A}\mathbb{R}(\lambda;\mathbb{A}))]^{-1}\\ &= (I - \Delta\mathbb{A}\mathbb{R}(\lambda;\mathbb{A})][I - \Delta\mathbb{A}\mathbb{R}(\lambda;\mathbb{A}))]^{-1} = I \;.\end{aligned}$$

Let $Z \in D(\mathbb{A} + \Delta\mathbb{A}) = D(\mathbb{A}))$, then:

$$R(\lambda\mathbb{I} - (\mathbb{A} + \Delta\mathbb{A}))Z = \mathbb{R}(\lambda;\mathbb{A})(\lambda I - (\mathbb{A} + \Delta\mathbb{A}))Z + \sum_{k=1}^{\infty}\mathbb{R}(\lambda:\mathbb{A})\left[\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\right]^k$$

$$(\lambda I - (\mathbb{A} + \Delta \mathbb{A}))Z. \tag{31}$$

$$= Z - \mathbb{R}(\lambda;\mathbb{A})\Delta\mathbb{A}\, Z + \sum_{k=1}^{\infty}\left[\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\right]^k Z - \sum_{k=2}^{\infty}\left[\mathbb{R}(\lambda:\mathbb{A})\Delta\mathbb{A}\right]^k .$$

Hence:

$$R(\lambda\mathbb{I} - (\mathbb{A} + \Delta\mathbb{A}))= \sum_{k=0}^{\infty}\mathbb{R}(\lambda:\mathbb{A})\left[\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\right]^k \tag{32}$$

Moreover

$$\begin{aligned}\|(\lambda I - \mathbb{A} - \Delta\mathbb{A})^{-1}\|_{L(L(H))} &= \left\|\sum_{k=0}^{\infty}\mathbb{R}(\lambda:\mathbb{A})\left[\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\right]^k\right\|_{L(L(H))}\\ &= \left\|\mathbb{R}(\lambda:\mathbb{A})\sum_{k=0}^{\infty}\left[\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\right]^k\right\|_{L(L(H))}\\ &\le \left\|\mathbb{R}(\lambda:\mathbb{A})\right\|_{L(H)}\sum_{k=0}^{\infty}\left\|\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\right\|^k_{L(L(H))}\end{aligned}$$

Since$\| \Delta\mathbb{A}\mathbb{R}(\lambda;\mathbb{A})\|_{LL(H))} < 1$, we get:

$$\sum_{k=0}^{\infty}\left\|\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\right\|^k_{L(L(H))} = \frac{1}{1-\left\|\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\right\|_{L(L(H))}},\ 1>\left\|\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\right\|_{L(L(H))}$$

together with

$\| \mathbb{R}(\lambda;\mathbb{A})\|_{L(L(H))} \leq \dfrac{M_1M_2}{\lambda-(w_1+w_2)}$, $\lambda > (w_1 + w_2)$.

We have:

$$\|(\lambda I - \mathbb{A} - \Delta\,\mathbb{A})^{-1}\|_{L(L(H))} \leq \frac{M_1M_2}{\lambda-(w_1+w_2)}\;\frac{1}{1-\|\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\|_{L(H)}}$$

$$=\frac{1}{\lambda-(w_1+w_2)-(\lambda-(w_1+w_2))\|\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\|}. \qquad (33)$$

From (28), one gets

$\lambda - (w_1 + w_2)\|\Delta\mathbb{A}\mathbb{R}(\lambda;\mathbb{A})\|_{L(L(H))} < \|\|\Delta\mathbb{A}\|_{L(L(H))}M_1M_2$

$\lambda - (w_1 + w_2) - (\lambda - (w_1 + w_2))\|\Delta\mathbb{A}\mathbb{R}(\lambda;\mathbb{A})\|_{L(L(H))} >$

$\lambda - (w_1 + w_2) - \|\Delta\mathbb{A}\|_{L(L(H))}M_1M_2$.

Thus:

$$\frac{1}{\lambda-(w_1+w_2)-(\lambda-(w_1+w_2))\|\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\|} \leq \frac{1}{\lambda-(w_1+w_2)-\|\Delta\mathbb{A}\|M_1M_2}. \qquad (34)$$

from condition (g)of problem formulation, the inequality (34) becomes:

$$\frac{1}{\lambda-(w_1+w_2)-(\lambda-(w_1+w_2))\|\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})\|} \leq$$

$$\frac{1}{(\lambda-(w_1+w_2))-(M_1M_2\,\|\,\Delta A_1\,\|_{L(L(H))}+M_1M_2\,\|\,\Delta A_2\,\|_{L(L(H))})}.$$

Now, for

$$\lambda > (w_1+w_2))-(M_1M_2\,\|\,\Delta A_1\,\|_{L(L(H))}+M_1M_2\,\|\,\Delta A_2\,\|_{L(L(H))} + 1,$$

we have:

$$\frac{1}{(\lambda-(w_1+w_2))-(M_1M_2\,\|\,\Delta A_1\,\|_{L(L(H))}+M_1M_2\,\|\,\Delta A_2\,\|_{L(L(H))})} < 1 ,$$

where $(\lambda-(w_1+w_2))-(M_1M_2\,\|\,\Delta A_1\,\|_{L(L(H))}+M_1M_2\,\|\,\Delta A_2\,\|_{L(L(H))}) \neq 0$

also $\left\|\sum_{k=0}^{\infty}\mathbb{R}(\lambda:\mathbb{A})[\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})]^k\right\|_{L(L(H))} \leq 1$.

Hence (32) is convergent in L({L(H), τ}).

Now, $\mathbb{R}(\lambda:\mathbb{A})$ and $\Delta\mathbb{A}\mathbb{R}(\lambda:\mathbb{A})$ are compact .

Hence $\mathbb{R}(\lambda:\mathbb{A} + \Delta\,\mathbb{A})$ is compact, for $\lambda > (w_1 + w_2) - (M_1M_2\|\Delta A_1\| + M_1M_2\|\Delta A_2\|) + 1$. ■

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قابلية الحل لنضام سلفستر الديناميكي الخطي القلق في فضاء ذات بعد غير منتهي باستخدام مفهوم شبه الزمرة القلق

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المستخلص

تم افتراض نظام ديناميكي خطي قلق من نوع سلفستر في فضاء ذات بعد منتهي .وكذلك تم عرض و تطوير قابلية الحل لهذا الصف من المعادلات باستخدام زمرة قلقة مركبة من المؤثرات الخطية المقيدة .وايضا عرض وبرهان الخواص الديناميكية ا لضرورية للنظام.