On Solution of Two Point Second Order Non Linear Boundary Value Problems using Semi-Analytic Method

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Abstract

In this paper a new method is proposed for the solution of two-point second order boundary-value problems(TPBVP) ,that is, we interested in constructing polynomial solutions of two points second order boundary value problems for ordinary differential equation.

A semi-analytic technique using two-point osculatory interpolation with the fit equal numbers of derivatives at the end points of an interval [0,1] is compared with conventional methods via a series of examples and is shown to be that seems to converge faster and more accurately than the conventional methods and generally superior, particularly for problems involving nonlinear equations and/or boundary conditions

.1. Introduction

The most general form of the problem to be considered is :

$$y'' = f(x, y, y'), \quad x \in [a, b],$$

with boundary conditions : $y(a) = A$, $y(b) = B$

There is no loss of generality in taking a = 0 and b = 1, and we will sometimes employ this slight simplification. We view f as a generally nonlinear function of y and y', but for the present, we will take f = f(x) only. For such a problem to have a solution it is generally necessary either that $f(x) \neq 0$ hold, or that $A \neq 0$ at one or both ends of the interval. When $f(x) \equiv 0$, and A = 0 , B =0 the BVP is said to be homogeneous and will in general have only the trivial solution, $y(x) \equiv 0$ [1].In this paper we introduce a new technique for the qualitative and quantitative analysis of non homogeneous linear TPBVP using two-point polynomial interpolation .

2. Approximation Theory

The primary aim of a general approximation is to represent non-arithmetic quantities by arithmetic ones so that the accuracy can be ascertained to a desired degree. Secondly, we are also concerned with the amount of computation required to achieve this accuracy. A complicated function f(x) usually is approximated by an easier function of the form $\varphi(x; a_0, \ldots, a_n)$ where $a_0, \ldots, a_n$ are parameters to be determined so as to characterize the best approximation of f.

In this paper, we shall consider only the interpolatory approximation. From Weierstrass Approximation Theorem ,it follows that one can always find a polynomial that is arbitrarily close to a given function on some finite interval. This means that the approximation error is bounded and can be reduced by the choice of the adequate polynomial. Unfortunately Weierstrass Approximation Theorem is not a constructive one, i.e. it does not present a way how to obtain such a polynomial. i.e. the interpolation problem can also be formulated in another way, viz. as the answer to the following question: How to find a .good. representative of a function that is not known explicitly, but only at some points of the domain of interest .In this paper we use Osculatory Interpolation since it has high order with the same given points in the domain .

2.1. Osculatory Interpolation[2]

Given $\{x_i\}$, i = 1, . . .k and values $f_i^{(0)}, \ldots, f_i^{(r_i)}$,where r_i are nonnegative integers and $f_i=f(x_i)$.We want to construct a polynomial P(x) such that $P^{(j)}(x_i) = f_i^{(j)}$ (1),

for i = 1, . . . , k and j = 0, . . . , r_i.

Such a polynomial is said to be an sculatory interpolating polynomial of a function f

Remark

The degree of P(x) is at most $\sum_{i=1}^{k}(r_i+1)-1$.

Essentially this is a generalization of interpolation using Taylor polynomials and for that reason osculatory interpolation is sometimes referred to as two-point Taylor interpolation. The idea is to approximate a function y(x) by a polynomial P(x) in which values of y(x) and any number of its derivatives at given points are equal by the corresponding function values and derivatives of P(x).

In this paper we are particularly concerned with fitting function values and derivatives at the two end points of a finite interval, say [0,1],wherein a useful and succinct way of writing a osculatory interpolant $P_{2n+1}(x)$ of degree 2n + 1 was given for example by Phillips [3] as :

$$P_{2n+1}(x) = \sum_{j=0}^{n} \{y^{(j)}(0)\, q_j(x) + (-1)^j\, y^{(j)}(1)\, q_j(1-x)\} \qquad (2)$$

$$q_j(x) = (x^j/j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s = Q_j(x)/j! \qquad (3)$$

so that (2) with (3) satisfy the conditions :

$$y^{(r)}(0) = P_{2n+1}^{(r)}(0)\ ,\quad y^{(r)}(1) = P_{2n+1}^{(r)}(1)\ ,\quad r=0,1,2,\ldots,n.$$

this means $P_{2n+1}(x)$ consistens with the appropriately truncated Taylor series for $y(x)$ about $x = 0$ and $x = 1$.The error on [0, 1] is given by :

$R_{2n+1} = y(x) - P_{2n+1}(x) = \dfrac{(-1)^{n+1} x^{(n+1)} (1-x)^{n+1} y^{(2n+2)}(\varepsilon)}{(2n+2)!}$ where $0 \langle \varepsilon \langle 1$ and $\mathbf{y}^{(2n+2)}$ is assumed to be continuous.

The osculatory interpolant for $P_{2n+1}(x)$ may converge to $y(x)$ in [0,1] irrespective of whether the intervals of convergence of the constituent series intersect or are disjoint .The important consideration here is whether $R_{2n+1} \to 0$ as $n\to\infty$ for all x in [0,1]. In the application to the boundary value problems in this paper such convergence with n is always confirmed numerically .We observe that (2) fits an equal number of derivatives at each end point but it is possible and indeed sometimes desirable to use polynomials which fit different numbers of derivatives at the end points of an interval. As an example of a two-point osculatory interpolant we may take n = 2 so that (2) with (3) become the quintic :

$$P_5(x)= (1-x)^3(1+3x+6x^2)y(0) + x^3(10-15x+6x^2)y(1) + x(1-x)^3(1+3x)y'(0) - x^3(1-x)(4-3x)\,y'(1) + 1/2x^2(1-x)^3 y''(0) + 1/2x^3(1-x)^2 y''(1)$$

Satisfying :

$P_5(0)=y(0)$, $P'_5(0)= y'(0)$, $P''_5(0)= y''(0)$.

$P_5(1)=y(1)$, $P'_5(1)= y'(1)$, $P''_5(1)= y''(1)$.

Finally we observe that (2) can be written directly in terms of the Taylor coefficients a_i and b_i about x = 0 and x = 1 respectively, as :

$$P_{2n+1}(x) = \sum_{j=0}^{n} \{ a_j Q_j(x) + (-1)^j b_j Q_j(1-x) \}. \ldots. \quad (4)$$

3. Solution Of Two Point Second-Order Boundary Value Problems

We consider the boundary value problem :

$$y'' + f(x, y, y') = 0 \qquad (5)$$

$$g_i(y(0), y(1), y'(0), y'(1)) = 0, \quad i = 1, 2 \qquad (6)$$

where f , g_1, g_2 are non linear functions of their arguments and g_1 and g_2 are given in three kinds [4] :

1- $y(0) = a_0$, $y(1) = b_0$ (6a), Dirichlet condition (value specified).

2- $y'(0) = a_1$,$y'(1) = b_1$ (6b) , Neumann condition (Derivative specified).

3- $c_0y'(0)+c_1y(0) = a$, $d_0y'(1) + d_1y(1) = b$ (6c), where c_0 ,c_1 ,d_0 ,d_1 are all positive constants not all are zero but c_1 , d_0 are equal to zero or c_0 , $d_1$are equal to zero Mixed condition (Gradient & value) .

The simple idea behind the use of two-point polynomials is to replace y(x) in problem (5)–(6), or an alternative formulation of it, by P_{2n+1} which enables any unknown boundary values or derivatives of y(x) to be computed . The first step, therefore, is to construct P_{2n+1} . To do this we need the Taylor coefficients of y(x) at x = 0 :

$$y = a_0 + a_1 x + \sum_{i=2}^{\infty} a_i x^i \qquad (7a)$$

into equation (5)and equate coefficients of powers of x. The resulting system of equations can be solved to obtain a_i for all $i \geq 2$. Also we need the Taylor coefficients of y (x) at x = 1. Using MATLAB throughout we simply insert the series form :

$$y = b_0 + b_1(x-1) + \sum_{i=2}^{\infty} b_i (x-1)^i \qquad (7b)$$

into (5) and equate coefficients of powers of (x−1). The resulting system of equations can be solved to obtain b_i for all $i \geq 2$.The notation implies that the coefficients depend only on the indicated unknowns a_0, a_1, b_0, b_1. The algebraic manipulations needed for this process to construct $P_{2n+1}(x)$ from (7) of the form (2) and use it as a replacement in the problem (5)–(6). Since we have only the four unknowns to compute for any n we only need to generate two equations from this procedure as two equations are already supplied by the boundary conditions (6). An obvious way to do this would be to satisfy the equation (5) itself at two selected points $x = c_1$, $x = c_2$ in [0,1] so that the two required equations become :

$$P''_{2n+1}(c_i) + f\{P_{2n+1}(c_i), P'_{2n+1}(c_i), c_i\} = 0, \quad i=1,2. \qquad (8)$$

An alternative approach is to recast the problem in an integral form before doing the replacement. Extensive computations have shown that this generally provides a more accurate polynomial representation for a given n. We therefore use this alternative formulation throughout this although we should keep in mind that the procedure based on (8) is a viable option and shares many common features with the approach outlined below. Of the many ways , could provide an integral formulation we adopt the following. first integrate (5) to obtain :

$$y'(x) - a_1 + \int_0^x f(y(s), y'(s), s)\, ds = 0 \qquad (9)$$

and integrate (9) to obtain :

$$y(x) - a_0 - x\, a_1 + \int_0^x (x-s)\, f(y(s), y'(s), s)\, ds = 0 \qquad (10)$$

where $a_0 = y(0)$ and $a_1 = y'(0)$. Putting $x = 1$ in (9) and (10) then gives :

$$b_1 - a_1 + \int_0^1 f(y(s), y'(s), s)\, ds = 0 \qquad (11)$$

and

$$b_0 - a_0 - a_1 + \int_0^1 (1-s)\, f(y(s), y'(s), s)\, ds = 0 \qquad (12)$$

where $b_0 = y(1)$ and $b_1 = y'(1)$.

The precise way we make the replacement of $y(x)$ with a $P_{2n+1}(x)$ in (11) and (12) depends on the nature of $f(y, y', x)$ and will be explained in the examples which follow. In any event the important point to note is that once this replacement has been made, equations (6), (11) and (12) constitute the four equations , require to determine the set $\{a_0, b_0, a_1, b_1\}$. As shall see the fact that the number of unknowns is independent of the number of derivatives fitted represents perhaps the most important feature of the method.

And make the following points at this stage :

(i) In the majority of cases where the boundary conditions are simple enough the system of algebraic equations may be reduced a priori to a system in two unknowns, since the boundary condition can be substituted directly into the integral formulations (11) and (12), which MATLAB can be utilized to solve, that is, if we have the BC(6a), then we have only the unknown pair $\{a_1, b_1\}$.As is known

,the required polynomial can be constructed. Possible advantage that the reader of other examples in section 4 . If the BC(6b), then we have only the unknown pair $\{a_0, b_0\}$ It is known can build polynomial limits required. Also if, the BC(6c), then we have only the unknown pair $\{a_0, b_1\}$or $\{a_1, b_0\}$,where they can build polynomial limits required .

(ii) The method offers a certain amount of flexibility. For example choose to satisfy (9) and (10) at two internal points or use alternative integral formulations. The fact remains that whatever strategy we adopt produces a quickly convergent sequence of values of the set $\{a_0, a_1, b_0, b_1\}$ as n increases.

(iii) Throughout assess the accuracy of the procedure by examining the convergence with n. Using a symbolic computational facility such as MATLAB, computing the required convergent is not an issue. Where possible we can also run checks on our solutions using shooting with MATLAB codes.

(iv) compare our method with the other methods. Now consider a number of examples designed to illustrate the convergence, accuracy, implementation and utility of the method. In what follows the use of bold digits in the tables is intended to give a rough visual indication of the convergence.

Remark

1- All computations in the following examples were performed by MATLAB environment, Version 7, running on a Microsoft Windows 2003 Professional operating system .

2- In the following examples when analytical solutions are known so that we can measure the error of a solution.

4. Examples

In this section we introduce some examples to illustrates suggested method , start with the problem of nonlinear boundary conditions :

Example 1

$$y'' + y\,.\sin(x) - e^{x} \quad \text{with BC :} \quad y(1) = \{y(0)\}^{3} \;,\; y'(1) = \{y(0)\}^{3}$$

The results of solution given in the following table :

Table 1 : The result of the methods for n = 2, 3, 4 of example 1

	P_5	P_7	P_9
a_0	1.4490338457	1.4490399023	1.4490400275
b_1	1.2195784290	1.2195901622	1.2195904017
X	P_5	P_7	P_9
0	1.4490338457	1.4490399023	1.4490400275
0.2	1.7123179359	1.7122634359	1.7122627106
0.5	2.1710062974	2.1707847801	2.1707796799
1	3.0425350415	3.0425731924	3.0425739810

Then from table 1 and the relation (2) - (3) in the previous section we have :

P_5=.394116e-1x^5-.106204x^4-.592852e- x^3+.500000x^2+1.21958x+1.44903

P_7=-.852883e-3x^7+.181385e-1x^6-.990244e-2x^5-.586001e-1x^4-.748400e-1x^3 +.500000x^2+1.21959x+1.44904

P_9=-.314200e-4x^9-.124115e-2x^8+.442680e-2x^7+.100612e-1x^6-.446597e-2x^5-.599659e-1x^4-.748400e-1x^3+.500000x^2+1.21959x+1.44904

Next we consider the nonlinear equation :

Example 2

$y'' + c\,y^2 = e^x$ With boundary conditions: $y(0) = 2$, $y(1) = 1$

Here (11) and (12) become :

$$F(a_1,b_1,c) \equiv 1+ b_1 - a_1 - e + c\int_0^1 y^2(s)\,ds = 0, \qquad (13)$$

$$G(a_1,b_1,c) \equiv 1- a_1 - e + c\int_0^1 (1-s)\,y^2(s)\,ds = 0, \qquad (14)$$

And the coefficients in (7) are :

$a_0 = 1$, $a_2 =(1-c)/2$, $a_3 = (1-2ca_1)/6$, $a_4 = (1-2c\,a_1^2-2c+2c^2)/24$,........

$b_0 = 2$,$b_2 = (e - 4c)/2$, $b_3 = (e - 4cb_1)/6$, $b_4 = (e + 16c^2 - 4ec - 2cb_1^2)/24$, ...

Taking c = 1 , obtain the results presented in Table 2. Only two real solutions are identified for each value of n represented by the two solutions to the problem for this value of c . As a further refinement use the resulting computed values of $\{a_1, b_1\}$ to compute a final two-point polynomial for the solution. A comparison between the two solutions of the suggested method and the solutions of other methods in [5] given in table 3 .

Continue to analyze this problem using Osculatory interpolation. If repeat the calculations for c = 2 , find that there are no real root for a_1 and b_1 , this suggests that there exists $c = c^*$ such that for $c > c^*$ there are no solution for the boundary value problem while for $c < c^*$ there are two. This of course is a well – known feature of the problem. What we do now is to compute the threshold value c^* using our two-point method. Essentially this involves finding double root of (13) and (14) for $\{a_1, b_1\}$.

Thus solve (13) and (14) together with : $\frac{\partial F}{\partial a_1}\frac{\partial G}{\partial b_1} - \frac{\partial G}{\partial a_1}\frac{\partial F}{\partial b_1} = 0$

For the unknowns a_1, b_1 and c . The results for n = 2 shown in Table 4 rounded to 9 decimal places. We thus conclude that c = 1.850291233 to 9 decimal places. We note there is no difficulty in taking higher values of n if we wished to refine this value .

TABLE 2 : Results of the methods for n = 4 of example 2 with c =1 .

no	Pa	P_5	P_7	P_9
1	a_1	28.19481976	24.64262129	23.93028383
	b_1	-28.01985541	-24.30190664	-23.55540623
	X	P_5	P_7	P_9
	0.2	6.283790124	5.761494443	5.64525806
	0.5	10.26351615	9.689317198	9.578566314
	0.8	7.133247594	6.60072879	6.482309834
2	a_1	1.301093133	1.301064016	1.301052744
	b_1	0.452402306	0.452409902	0.452421518
	X	P_5	P_7	P_9
	0.2	1.257965626	1.257933421	1.257928742
	0.5	1.612581095	1.612552924	1.612537243
	0.8	1.883145399	1.883159027	1.883155346

Then from table 2 and the relation (2) and (3) in previous section we have :

For the first solution :

$P5=4.83425x^5+15.7013x^4-47.7304x^3+28.1948x+1.$

$P7=-5.42264x^7-16.9735x^6+87.7933x^5-80.9923x^4-8.04754x^3+24.6426x+1.$

$P9=.265854e\text{-}2x^9-.149936e\text{-}1x^8+.159832e\text{-}1x^7+.255829e\text{-}1x^6+.361287e\text{-}1x^5-.993949e\text{-}1x^4-.267018x^3+1.30105x+1.$

For the second solution :

$P5=.986546e\text{-}1x^5-.142721x^4-.257027x^3+1.30109x+1.$

$P7=-.174054e\text{-}1x^7+.623549e\text{-}1x^6+.160872e\text{-}1x^5-.950793e\text{-}1x^4-.267021x^3+1.30106x+1.$

$P9=7.30227x^9+1.50618x^8-84.3887x^7+138.915x^6-30.7755x^5-47.6799x^4-7.81009x^3+23.9303x+1.$

Table 3a : A Comparison between P_9 and other methods in [5] of example 3 for the first solution

x_i	P_9 by using Osculatory interpolation	P_9 by using Hermite interpolation	Φ_1 by another numerical solution	Φ_5 by using Chebyshev series for N=5
0.2	1.257928742	1.257929	1.257928	1.257930
0.5	1.612537243	1.612536	1.612536	1.612540
0.8	1.883155346	1.883155	1.883155	1.883155

Table 3b : A Comparison between P_9 and other methods in [5] of example 3 for the second solution

x_i	P_9 by using Osculatory interpolation	P_9 by using Hermite interpolation	Φ_1 by another numerical solution
0.2	5.64525806	5.645258	5.604135
0.5	9.578566314	9.578566	9.545713
0.8	6.482309834	6.482310	6.440038

Table 4 : Results of the methods for n = 2 of example 2 with $c = c^*$

no	Pa	P_5
1	a_1	5.500860385
	b_1	- 4.699286617
	c	1.850291233

Now, give the nonlinear problem with Neumann boundary conditions :

Example 3

$$y'' = y^3 - y\ y' \quad \text{with} \quad BC:\ y'(0) = -1\ ,\ y'(1) = -1/4$$

have the exact solution [6] : $y(x) = 1 / (x + 1)$

The result of method given in the following table :

Table 5 : The result of the methods for n = 2, 3, 4 of example 3

		P_5	P_7	P_9	
a_0		-2.0730192434	-1.9094622257	1.0084460394	
b_0		2.2375726588	2.0790662118	0.5109116568	
x	**Y**	P_5	P_7	P_9	$\lvert Y-P_9 \rvert$
0.25	0.8000000000	0.7788980098	0.7715765789	0.7715767601	0.028423239916215
0.5	0.6666666667	0.6509123549	0.6567887010	0.6618496652	0.004817001507875
0.75	0.5714285714	0.6070021412	0.6088950935	0.6025318078	0.031103236348971
S.S.E = 0.00179849538224153					

Then from table 5 and the relation (2) and (3) in the previous section we have :

$$P_9 = 21.1466\ x^9 - 95.0579\ x^8 + 163.2064\ x^7 - 127.9210\ x^6 + 39.2950\ x^5 - 1/6\ x^3 - x + 1.0084$$

The accuracy of the solution given in the following figure :

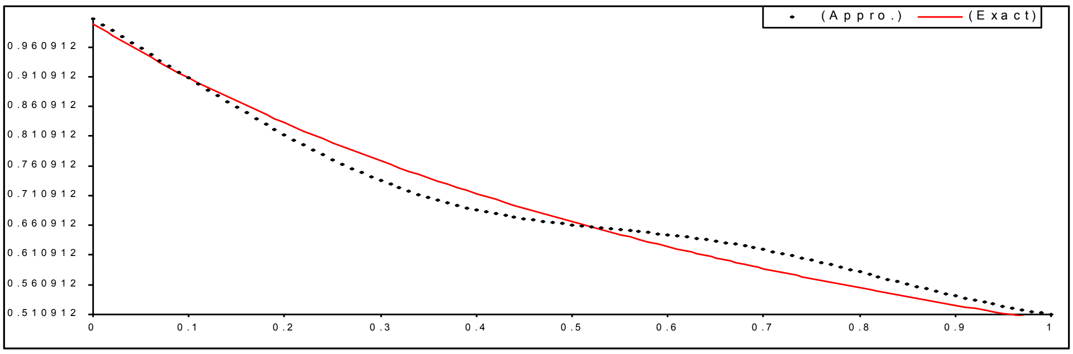

Figure1:A comparison between exact and approximate solution of example 3

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حول حـــل مسائـل القيم الحدودية غير الخطية من الرتبة الثانية ذات نقطتين باستخدام الطـريقة شـــبه التحـلـيلية

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المستخلص

يتضمن البحث طريقة جديدة لحل مسائل القيم الحدودية غير الخطية من الرتبة الثانية ذات نقطتين (TPBVP) حيث نقترح الحل كمتعددة حدود لحل مسائل القيم الحدودية الاعتيادية غير الخطية من الرتبة الثانية ذات نقطتين (TPBVP) .

أستخدمنا الإستراتيجية شبه التحليلية باستخدام نوع من الاندراج(two-point osculatory interpolation) لعدد من مشتقات نقطتي نهاية الفترة [0 , 1] وقورنت النتائج مع الحل باستخدام الطرق التقليدية الأخرى وأثبتنا من خلال الأمثلة بان الطريقة المقترحة هي الأسرع والأدق بالأخص عندما تكون المعادلة و / أو الشروط الحدودية غير خطية