Differential Operator Of A Class Of Meromorphic Univalent Functions With Negative Coefficients

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Abstract: In the present paper, we have studied a class $\mathcal{A}(\lambda,\mu,\alpha,\tau)$ of analytic and meromorphic univalent functions defined by differential operator in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$ and obtain some sharp results including coefficient inequality, distortion theorem, radii of starlikeness and convexity, Hadamard product, closure theorems. We also obtain some results connected with (n,δ) – neighborhoods on $\mathcal{A}^{\sigma}(\lambda,\mu,\alpha,\tau)$ and integral operator.

Keywords: Meromorphic univalent function, Differential operator, Distortion theorem, Radii of starlikeness, Hadamard product, Neighborhood, Integral operator.

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1.Introduction:

Let $\sum$ denote the class of functions analytic and meromorphic univalent in the punctured unit disk $U^* = \{z \in \mathbb{C}: 0 < |z| < 1\} = U \setminus \{0\}$ and let $S(n)$ denote the subclass of $\sum$ consisting of functions of the form:

$$f(z) = z^{-1} - \sum_{n=1}^{\infty} a_n z^n, \quad (a_n \geq 0, n \in \mathbb{N} = \{1, 2, \dots\}), \tag{1}$$

which are analytic and meromorphic univalent in the punctured unit disk U^*. A function $f \in S(n)$ is said to be meromorphically starlike of order β if

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \beta, (z \in U = U^* \cup \{0\}, o \leq \beta < 1), \tag{2}$$

and a function $f \in S(n)$ is said to be meromorphically convex of order β if

$$Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} > \beta, (z \in U = U^* \cup \{0\}, o \leq \beta < 1). \tag{3}$$

We denote by $\delta^*(\beta), \delta(\beta)$, respectively, the classes of univalent meromorphic starlike functions of order β and univalent meromorphic convex functions of order β. Similar classes have been extensively studied by Clunie [7] and Miller [9] and Atshan [2, 5].

We shall use the differential operator (D_{λ,μ_1}) [11] defined as follows:

$$D_{\lambda,\mu_1} f(z) = z^{-1} - \sum_{n=1}^{\infty} a_n [\lambda(n+2)(\mu_1(n+1)+1) - \mu_1(n+2) + (1-\lambda+\mu_1)] z^n$$

$$= z^{-1} - \sum_{n=1}^{\infty} \emptyset(\lambda, \mu_1, n) a_n z^n, \tag{4}$$

where

$\emptyset(\lambda, \mu_1, n) = [\lambda(n+2)(\mu_1(n+1)+1) - \mu_1(n+2) + (1-\lambda+\mu_1)], 0 \leq \mu_1 \leq \lambda$, $n \in \mathbb{N} = \{1, 2, \ldots\}$.

Definition 1: A function $f \in S(n)$ is in the class $\mathcal{A}(\lambda, \mu_1, \alpha, \tau)$ if it satisfies the condition

$$\left| \frac{\dfrac{z^2\left(D_{\lambda,\mu_1} f(z)\right)''}{\left(D_{\lambda,\mu_1} f(z)\right)'} + 2z}{(2\tau-1)\dfrac{z^2\left(D_{\lambda,\mu_1} f(z)\right)''}{\left(D_{\lambda,\mu_1} f(z)\right)'} + (2\tau-2)z} \right| < \alpha, \qquad z \in U^* \tag{5}$$

for $0 < \alpha \leq 1$, $\frac{1}{2} < \tau \leq 1$.

2. Coefficient Inequality:

The following theorem gives a sufficient condition for a function to be in the class $\mathcal{A}(\lambda,\mu_1,\alpha,\tau)$.

Theorem 1: A function $f \in S(n)$ is in the class $\mathcal{A}(\lambda,\mu_1,\alpha,\tau)$ if and only if

$$\sum_{n=1}^{\infty} \emptyset(\lambda,\mu_1,n)\, n\big((n+1)+\alpha[(2\tau-1)n-1]\big)\, a_n \leq 2\alpha\tau, \tag{6}$$

where $0<\alpha\leq 1$, $\frac{1}{2}<\tau\leq 1$.

The result is sharp for the function

$$f(z) = z^{-1} - \frac{2\alpha\tau}{\emptyset(\lambda,\mu_1,n)\, n\big((n+1)+\alpha[(2\tau-1)n-1]\big)} z^n, \qquad n \in \mathbb{N}. \tag{7}$$

Proof: For $|z|=1$, we have

$$\left|z^2\left(D_{\lambda,\mu_1}f(z)\right)'' + 2z\left(D_{\lambda,\mu_1}f(z)\right)'\right| - \alpha\left|(2\tau-1)z^2\left(D_{\lambda,\mu_1}f(z)\right)'' + (2\tau-2)z\left(D_{\lambda,\mu_1}f(z)\right)'\right|$$

$$= \left|-\sum_{n=1}^{\infty} n(n+1)\emptyset\big(\lambda,\mu_1,n\big)a_n z^n\right| - \alpha\left|2\tau z^{-1} - \sum_{n=1}^{\infty} n[(2\tau-1)n-1]\emptyset\big(\lambda,\mu_1,n\big)a_n z^n\right|$$

$$\leq \sum_{n=1}^{\infty} n(n+1)\emptyset\big(\lambda,\mu_1,n\big)a_n - 2\alpha\tau + \sum_{n=1}^{\infty} n\alpha[(2\tau-1)n-1]\emptyset\big(\lambda,\mu_1,n\big)a_n$$

$$= \sum_{n=1}^{\infty} \emptyset\big(\lambda,\mu_1,n\big)n\big((n+1)+\alpha[(2\tau-1)n-1]\big)\, a_n \leq 0 .$$

By hypothesis. Thus by maximum modulus theorem $f \in \mathcal{A}(\lambda,\mu_1,\alpha,\tau)$.

Conversely, assume that

$$\left|\frac{\dfrac{z^2\left(D_{\lambda,\mu_1}f(z)\right)''}{\left(D_{\lambda,\mu_1}f(z)\right)'} + 2z}{(2\tau-1)\dfrac{z^2\left(D_{\lambda,\mu_1}f(z)\right)''}{\left(D_{\lambda,\mu_1}f(z)\right)'} + (2\tau-2)z}\right| = \left|\frac{z^2\left(D_{\lambda,\mu_1}f(z)\right)'' + 2z\left(D_{\lambda,\mu_1}f(z)\right)'}{(2\tau-1)z^2\left(D_{\lambda,\mu_1}f(z)\right)'' + (2\tau-2)z\left(D_{\lambda,\mu_1}f(z)\right)'}\right|$$

$$= \left| \frac{-\sum_{n=1}^{\infty} n(n+1)\emptyset(\lambda,\mu_1,n)a_n z^n}{2\tau z^{-1} - \sum_{n=1}^{\infty} n[(2\tau-1)n-1]\emptyset(\lambda,\mu_1,n)a_n z^n} \right| < \alpha .$$

Since $|Re(z)| \leq |z|$ for all z, we have

$$Re\left\{ \frac{\sum_{n=1}^{\infty} n(n+1)\emptyset(\lambda,\mu_1,n)a_n z^n}{2\tau z^{-1} - \sum_{n=1}^{\infty} n[(2\tau-1)n-1]\emptyset(\lambda,\mu_1,n)a_n z^n} \right\} < \alpha. \tag{8}$$

Now, choosing values of z on the real axis and allowing $z \to 1$ from the left through real values, the inequality (8) immediately yields the desired condition in (6). Finally, it is observed that the result is sharp for the function is given by (7).

Theorem 1 immediately yields the following result.

Corollary 1: Let $f \in \mathcal{A}(\lambda,\mu_1,\alpha,\tau)$. Then

$$a_n \leq \frac{2\alpha\tau}{\emptyset(\lambda,\mu_1,n)\, n\big((n+1)+\alpha[(2\tau-1)n-1]\big)}, \mathrm{n} = 1, 2, \ldots . \tag{9}$$

The equality in (9) is attained for the function f given by (7).

3. Distortion Theorem:

We now state the following distortion inequality for the class $\mathcal{A}(\lambda,\mu_1,\alpha,\tau)$.

Theorem 2: Let the function $f \in \mathcal{A}(\lambda,\mu_1,\alpha,\tau)$. Then

$$\frac{1}{|z|} - \frac{\alpha\tau}{\emptyset(\lambda,\mu_1,1)\big(1+\alpha(\tau-1)\big)}|z| \leq |f(z)| \leq \frac{1}{|z|} + \frac{\alpha\tau}{\emptyset(\lambda,\mu_1,1)\big(1+\alpha(\tau-1)\big)}|z| . \tag{10}$$

The result is sharp for the function

$$f(z) = z^{-1} - \frac{\alpha\tau}{\emptyset(\lambda,\mu_1,1)\ (1+\alpha(\tau-1))} z .$$

Proof: We have

$$f(z) = z^{-1} - \sum_{n=1}^{\infty} a_n z^n$$

$$|f(z)| \leq \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n |z|^n \leq \frac{1}{|z|} + \frac{\alpha\tau}{\emptyset(\lambda,\mu_1,1)(1+\alpha(\tau-1))}|z|. \tag{11}$$

Similarly

$$|f(z)| \geq \frac{1}{|z|} - \sum_{n=1}^{\infty} a_n |z|^n \geq \frac{1}{|z|} - \frac{2\alpha\tau}{\emptyset(\lambda,\mu_1,1)(1+\alpha(\tau-1))}|z|. \tag{12}$$

Combining (11) and (12), we get (10).□

4. Radii of starlikeness and convexity:

Theorem 3: Let $f \in \mathcal{A}(\lambda,\mu_1,\alpha,\tau)$. Then f is starlike of order $\beta, (0 \leq \beta < 1)$ in $|z| < r = r_1(\lambda,\mu_1,\alpha,\tau,n,\beta)$, where

$$r_1(\lambda,\mu_1,\alpha,\tau,n,\beta) = \inf_n \left\{ \frac{(1-\beta)n\big((n+1)+\alpha[(2\tau-1)n-1]\big)\emptyset(\lambda,\mu_1,n)}{2\alpha\tau(n+2-\beta)} \right\}^{\frac{1}{n+1}}, \tag{13}$$

$n = 1, 2, \dots$.

The bound for each $|z|$ is sharp for each n, with the extremal function being of the form (7).

Proof: Let $f \in \mathcal{A}(\lambda,\mu_1,\alpha,\tau)$ then by Theorem 1

$$\sum_{n=1}^{\infty} \frac{n\big((n+1)+\alpha[(2\tau-1)n-1]\big)}{2\alpha\tau}\emptyset(\lambda,\mu_1,n)\, a_n \leq 1. \tag{14}$$

For $0 \leq \beta < 1$, we need to show that

$$\left|\frac{zf'(z)}{f(z)} + 1\right| \leq 1-\beta,$$

we have to show that

$$\left|\frac{zf'(z)+f(z)}{f(z)}\right| \leq \left|\frac{-\sum_{n=1}^{\infty}(n+1)a_n z^{n+1}}{1-\sum_{n=1}^{\infty} a_n z^{n+1}}\right| \leq \frac{\sum_{n=1}^{\infty}(n+1)a_n |z|^{n+1}}{1-\sum_{n=1}^{\infty} a_n |z|^{n+1}} \leq 1-\beta.$$

Hence

$$\sum_{n=1}^{\infty}\left(\frac{n+2-\beta}{1-\beta}\right)a_n|z|^{n+1}\leq 1.$$

This is enough to consider

$$|z|^{n+1}\leq\frac{(1-\beta)n\big((n+1)+\alpha[(2\tau-1)n-1]\big)\emptyset(\lambda,\mu_1,n)}{2\alpha\tau(n+2-\beta)} .$$

Therefore

$$|z|\leq\left\{\frac{(1-\beta)n\big((n+1)+\alpha[(2\tau-1)n-1]\big)\emptyset(\lambda,\mu_1,n)}{2\alpha\tau(n+2-\beta)}\right\}^{\frac{1}{n+1}} . \qquad (15)$$

Setting $|z|=r_1(\lambda,\mu_1,\alpha,\tau,n,\beta)$ in (15), we get the radius of starlikeness, which completes the proof of Theorem 3.

Theorem 4: Let $f\in\mathcal{A}(\lambda,\mu_1,\alpha,\tau)$. Then f is convex of order $\beta,(0\leq\beta<1)$ in $|z|<r=r_2(\lambda,\mu_1,\alpha,\tau,n,\beta)$, where

$$r_2(\lambda,\mu_1,\alpha,\tau,n,\beta)=\inf_n\left\{\frac{(1-\beta)\big((n+1)+\alpha[(2\tau-1)n-1]\big)\emptyset(\lambda,\mu_1,n)}{2\alpha\tau(n+2-\beta)}\right\}^{\frac{1}{n+1}}, \qquad (16)$$

$n=1,2,\dots$.
The bound for each $|z|$ is sharp for each n, with the extremal function being of the form (7).

Proof: Let $f\in\mathcal{A}(\lambda,\mu_1,\alpha,\tau)$ then by Theorem 1

$$\sum_{n=1}^{\infty}\frac{n\big((n+1)+\alpha[(2\tau-1)n-1]\big)}{2\alpha\tau}\emptyset(\lambda,\mu_1,n)\,a_n\leq 1. \qquad (17)$$

For $0\leq\beta<1$, we need to show that

$$\left|\frac{zf''(z)}{f'(z)}+2\right|\leq 1-\beta,$$

we have to show that

$$\left|\frac{zf''(z)+2f'(z)}{f'(z)}\right| \leq \left|\frac{-\sum_{n=1}^{\infty} n(n+1)a_n z^{n+1}}{-1-\sum_{n=1}^{\infty} na_n z^{n+1}}\right| \leq \frac{\sum_{n=1}^{\infty} n(n+1)a_n|z|^{n+1}}{1-\sum_{n=1}^{\infty} na_n|z|^{n+1}} \leq 1-\beta.$$

Hence

$$\sum_{n=1}^{\infty} \frac{n(n+2-\beta)}{1-\beta} a_n|z|^{n+1} \leq 1.$$

This is enough to consider

$$|z|^{n+1} \leq \frac{(1-\beta)\big((n+1)+\alpha[(2\tau-1)n-1]\big)\emptyset(\lambda,\mu_1,n)}{2\alpha\tau(n+2-\beta)} \ .$$

Therefore

$$|z| \leq \left\{\frac{(1-\beta)\big((n+1)+\alpha[(2\tau-1)n-1]\big)\emptyset(\lambda,\mu_1,n)}{2\alpha\tau(n+2-\beta)}\right\}^{\frac{1}{n+1}} . \qquad (18)$$

Setting $|z| = r_2(\lambda,\mu_1,\alpha,\tau,n,\beta)$ in (18), we get the radius of convexity, which completes the proof of Theorem 4.

5. Hadamard product:

Theorem 5: If

$$f(z) = z^{-1} - \sum_{n=1}^{\infty} a_n z^n \quad and \quad g(z) = z^{-1} - \sum_{n=1}^{\infty} b_n z^n$$

be in the class $\mathcal{A}(\lambda,\mu_1,\alpha,\tau)$, then

$$(f*g)(z) = z^{-1} - \sum_{n=1}^{\infty} a_n b_n z^n$$

is in the class $\mathcal{A}(\lambda,\mu_1,\eta,\tau)$, where

$$\eta = \frac{2\alpha^2\tau(n+1)}{\emptyset(\lambda,\mu_1,n)n\big((n+1)+\alpha[(2\tau-1)n-1]\big)^2 - 2\alpha^2\tau[(2\tau-1)n-1]} .$$

Proof: Suppose that $f, g \in \mathcal{A}(\lambda, \mu_1, \alpha, \tau)$.

By Theorem 1, we have

$$\sum_{n=1}^{\infty} \frac{n\big((n+1)+\alpha[(2\tau-1)n-1]\big)}{2\alpha\tau} \emptyset(\lambda, \mu_1, n)\, a_n \leq 1 \tag{19}$$

and

$$\sum_{n=1}^{\infty} \frac{n\big((n+1)+\alpha[(2\tau-1)n-1]\big)}{2\alpha\tau} \emptyset(\lambda, \mu_1, n)\, b_n \leq 1\,. \tag{20}$$

We have to find the largest value η, such that

$$\sum_{n=1}^{\infty} \frac{n\left((n+1)+\eta[(2\tau-1)n-1]\right)}{2\eta\tau} \emptyset(\lambda, \mu_1, n)\, a_n b_n \leq 1\,. \tag{21}$$

By Cauchy-Schwarz inequality, we have

$$\sum_{n=1}^{\infty} \frac{n\big((n+1)+\alpha[(2\tau-1)n-1]\big)}{2\alpha\tau} \emptyset(\lambda, \mu_1, n) \sqrt{a_n b_n} \leq 1\,. \tag{22}$$

Thus it is enough to show that

$$\frac{n\left((n+1)+\eta[(2\tau-1)n-1]\right)}{2\eta\tau} \emptyset(\lambda, \mu_1, n) a_n b_n$$
$$\leq \frac{n\big((n+1)+\alpha[(2\tau-1)n-1]\big)}{2\alpha\tau} \emptyset(\lambda, \mu_1, n) \sqrt{a_n b_n}\ ,$$

that is

$$\sqrt{a_n b_n} \leq \frac{\big((n+1)+\alpha[(2\tau-1)n-1]\big)\eta}{\left((n+1)+\eta[(2\tau-1)n-1]\right)\alpha}\,. \tag{23}$$

From (22), we get

$$\sqrt{a_n b_n} \leq \frac{2\alpha\tau}{\emptyset(\lambda, \mu_1, n) n\big((n+1)+\alpha[(2\tau-1)n-1]\big)}\,. \tag{24}$$

Therefore, in view of (23) and (24) it is enough to show that

$$\frac{2\alpha\tau}{\varnothing(\lambda,\mu_1,n)n\big((n+1)+\alpha[(2\tau-1)n-1]\big)} \le \frac{\big((n+1)+\alpha[(2\tau-1)n-1]\big)\eta}{\Big((n+1)+\eta[(2\tau-1)n-1]\Big)\alpha} ,$$

which simplifies to

$$\eta \le \frac{2\alpha^2\tau(n+1)}{\varnothing(\lambda,\mu_1,n)n\big((n+1)+\alpha[(2\tau-1)n-1]\big)^2 - 2\alpha^2\tau[(2\tau-1)n-1]} . \ \square$$

6. Closure theorems:

In the following theorems, we will show the class $\mathcal{A}(\lambda,\mu_1,\alpha,\tau)$ is closed under linear combination .

Theorem 6: Let $f_i \in \mathcal{A}(\lambda,\mu_1,\alpha,\tau)$,where

$$f_i(z) = z^{-1} - \sum_{n=1}^{\infty} a_{n,i} z^n \ , \big(a_{n,i} \ge 0, i = 1,2\ \big).$$

Then

$$w(z) = t f_1(z) + (1-t) f_2(z), \qquad (0 \le t \le 1)$$

is also in the class $\mathcal{A}(\lambda,\mu_1,\alpha,\tau)$.

Proof: Since for $0 \le t \le 1$, we get

$$w(z) = z^{-1} - \sum_{n=1}^{\infty} (t\, a_{n,1} + (1-t)\ a_{n,2}) z^n ,$$

we observe that

$$\sum_{n=1}^{\infty} \varnothing(\lambda,\mu_1,n)n\big((n+1)+\alpha[(2\tau-1)n-1]\big)(t\, a_{n,1} + (1-t)\ a_{n,2})$$

$$= t\sum_{n=1}^{\infty} \varnothing(\lambda,\mu_1,n)n\big((n+1)+\alpha[(2\tau-1)n-1]\big)a_{n,1}$$

$$+(1-t)\sum_{n=1}^{\infty} \varnothing(\lambda,\mu_1,n)n\big((n+1)+\alpha[(2\tau-1)n-1]\big)a_{n,2} \ \le 2\alpha\tau.$$

By Theorem 1, $w \in \mathcal{A}(\lambda,\mu_1,\alpha,\tau)$. $\square$

Theorem 7: Let

$$f_j(z) = z^{-1} - \sum_{n=1}^{\infty} a_{n,j} z^n \in \mathcal{A}(\lambda, \mu_1, \alpha, \tau)\ ,\quad j \in \{1, 2, \dots, t\}\ and\ \ 0 < k_j < 1$$

such that

$$\sum_{j=1}^{t} k_j = 1\ .$$

Then the function H defined

$$H(z) = \sum_{j=1}^{t} k_j\, f_j(z)$$

is also in the class $\mathcal{A}(\lambda, \mu_1, \alpha, \tau)$.

Proof: For every $j \in \{1, 2, \dots, t\}$, we obtain

$$\sum_{n=1}^{\infty} \frac{n\big((n+1) + \alpha[(2\tau - 1)n - 1]\big)}{2\alpha\tau} \varnothing(\lambda, \mu_1, n)\, a_{n,j} \le 1\ .$$

Since

$$H(z) = \sum_{j=1}^{t} k_j\, f_j(z) = \sum_{j=1}^{t} k_j \left(z^{-1} - \sum_{n=1}^{\infty} a_{n,j} z^n \right) = z^{-1} - \sum_{n=1}^{\infty} \left(\sum_{j=1}^{t} k_j\, a_{n,j} \right) z^n.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{n\big((n+1) + \alpha[(2\tau - 1)n - 1]\big)}{2\alpha\tau} \varnothing(\lambda, \mu_1, n) \left(\sum_{j=1}^{t} k_j\, a_{n,j} \right)$$

$$= \sum_{j=1}^{t} k_j \left(\sum_{n=1}^{\infty} \frac{n\big((n+1) + \alpha[(2\tau - 1)n - 1]\big)}{2\alpha\tau} \varnothing(\lambda, \mu_1, n) a_{n,j} \right)$$

$$\le \sum_{j=1}^{t} k_j = 1.$$

Hence $H \in \mathcal{A}(\lambda, \mu_1, \alpha, \tau)$ and the proof is complete.

7. Integral operator:

Theorem 8: Let $f \in \mathcal{A}(\lambda, \mu_1, \alpha, \tau)$. Then the integral operator

$$F(z) = c\int_0^1 u^c\ f(uz)\,du\,, (0 < u \le 1, 0 < c < \infty)$$

Is also in the class $\mathcal{A}(\lambda, \mu_1, \gamma, \tau)$, where

$$\gamma = \frac{c\alpha(n+1)}{(n+c+1)\big((n+1)+\alpha[(2\tau-1)n-1]\big) - c\alpha[(2\tau-1)n-1]}. \qquad (25)$$

Proof: Let $f \in \mathcal{A}(\lambda, \mu_1, \alpha, \tau)$, we have

$$F(z) = c\int_0^1 u^c\ f(uz)\ du\ = c\int_0^1 \left(u^{c-1}z^{-1} - \sum_{n=1}^{\infty} u^{n+c} a_n z^n\right) du$$

$$= z^{-1} - \sum_{n=1}^{\infty} \frac{c}{n+c+1} a_n z^n.$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{c\ n\big((n+1)+\gamma[(2\tau-1)n-1]\big)}{2\gamma\tau\,(n+c+1)} \varnothing(\lambda, \mu_1, n)\ a_n \le 1, \qquad (26)$$

since $f \in \mathcal{A}\big(\lambda, \mu_1, \alpha, \tau\big)$, we have

$$\sum_{n=1}^{\infty} \frac{n\big((n+1)+\alpha[(2\tau-1)n-1]\big)}{2\alpha\tau} \varnothing(\lambda, \mu_1, n)\ a_n \le 1\ .$$

Note that (26) it satisfied if

$$\frac{c\ n\big((n+1)+\gamma[(2\tau-1)n-1]\big)}{2\gamma\tau(n+c+1)} \varnothing(\lambda, \mu_1, n)$$
$$\le \frac{n\big((n+1)+\alpha[(2\tau-1)n-1]\big)}{2\alpha\tau} \varnothing(\lambda, \mu_1, n)\ .$$

Rewriting the inequality, we have

$$c\alpha\big((n+1)+\gamma[(2\tau-1)n-1]\big) \le (n+c+1)\gamma\big((n+1)+\alpha[(2\tau-1)n-1]\big)$$

solving for γ, we have

$$\gamma \leq \frac{c\alpha(n+1)}{(n+c+1)\big((n+1)+\alpha[(2\tau-1)n-1]\big)-c\alpha[(2\tau-1)n-1]}\,. \tag{27}$$

Since the right hand side of (27) is an increasing function of n . □

8. Neighborhood property:

The concept of neighborhood of analytic function was first introduced by Goodman [8] and Ruscheweyh [12] investigated this concept for the elements of several famous subclasses of analytic functions and Altintas and Owa [1] considered for a certain family of analytic functions with negative coefficients, also Liu and Srivastava [10], Atshan [2], Atshan and Buti [3], Atshan and Sulman [6] and Atshan and Joudah [4] extended this concept for a certain subclass of meromorphically multivalent or univalent functions.

We define the (n,δ) –neighborhood of a function $f \in S(n)$ by

$$N_{n,\delta}(f)=\left\{g \in S(n): g(z)=z^{-1}-\sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|a_n-b_n| \leq \delta, 0 \leq \delta<1\right\}. \tag{28}$$

For the identity function $e(z)=z$, we have

$$N_{n,\delta}(e)=\left\{g \in S(n): g(z)=z^{-1}-\sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|b_n| \leq \delta\right\}.$$

Definition 2: A function $f \in S(n)$ defined by (1) is said to be in the class $\mathcal{A}^{\sigma}(\lambda,\mu_1,\alpha,\tau)$ if there exists a function $g \in \mathcal{A}(\lambda,\mu_1,\alpha,\tau)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right|<1-\sigma, \quad (z \in U, 0 \leq \sigma<1). \tag{29}$$

Theorem 9: Let $g \in \mathcal{A}^{\sigma}(\lambda,\mu_1,\alpha,\tau)$ and

$$\sigma=1-\frac{\delta\varnothing(\lambda,\mu_1,1)\big(1+\alpha(\tau-1)\big)}{\varnothing(\lambda,\mu_1,1)\big(1+\alpha(\tau-1)\big)-\alpha\tau}\,. \tag{30}$$

Then $N_{n,\delta}(g) \subset \mathcal{A}^{\sigma}(\lambda,\mu_1,\alpha,\tau)$.

Proof: Assume that $f \in N_{n,\delta}(g)$, then we get from (28) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, (n \in \mathbb{N}). \tag{31}$$

since $g \in \mathcal{A}(\lambda, \mu_1, \alpha, \tau)$, we have from Theorem 1

$$\sum_{n=1}^{\infty} b_n \leq \frac{\alpha\tau}{\emptyset(\lambda, \mu_1, 1)\big(1 + \alpha(\tau - 1)\big)} \; ,$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| \leq \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n} \leq \frac{\delta\emptyset(\lambda, \mu_1, 1)\big(1 + \alpha(\tau - 1)\big)}{\emptyset(\lambda, \mu_1, 1)\big(1 + \alpha(\tau - 1)\big) - \alpha\tau} = 1 - \sigma.$$

Thus, by Definition 2, $f \in \mathcal{A}^{\sigma}(\lambda, \mu_1, \alpha, \tau)$ for σ given by (30). □

References

[1] *O. Altintas and S. Owa,* ***Neighborhoods of certain analytic functions with negative coefficients****, Int. J. Math. Sci. ,19(1996), 797-800.*

[2] W. G. Atshan, ***Subclass of meromorphic functions with positive coefficients defined by Ruscheweyh derivative II****, Surveys in Mathematics and its Applications, 3(2008), 67-77.*

[3] W. G. Atshan and R. H. Buti, ***Some properties of a new subclass of meromorphic univalent functions with positive coefficients defined by Ruscheweyh derivative II****, AL-Qadisiya Journal for Computer Science and Mathematics, 2(1)(2010), 49-55.*

[4] W. G. Atshan and A. S. Joudah, ***Subclass of meromorphic univalent functions defined by Hadamard product with multiplier transformations****, International Mathematical Forum, 6(46)(2011), 2279-2292.*

[5] W. G. Atshan and S. R. Kulkarni, ***On a class of p-valent meromorphic functions defined by integral operator****, International J. of Math. Sci. & Engg. Appls. (IJMSEA), 1(1)(2007), 129-140.*

[6] W. G. Atshan and J. H. Sulman, ***On a class of meromorphic univalent functions defined by linear derivative operator****, International Mathematical Forum, 6(46)(2011), 2267-2278.*

[7] J. G. Clunie, ***On meromorphic schlicht functions****, J. London Math. Soc. ,34(1959), 215-216.*

[8] A. W. Goodman, ***Univalent functions and non-analytic curve****, Proc. Amer. Math. Soc. , 8(1957), 598-601.*

[9] J. E. Miller, ***Convex meromorphic mapping and related functions****, Proc. Amer. Math. Soc. , 25(1970), 220-228.*

[10] J. L. Liu and H. M. Srivastava, ***A linear operator and associated families of meromorphically multivalent functions****, J. Math. Anal. Appl. 259(2001), 566-581.*

[11] H. Orhan, D. Răducanu and E. Deniz, ***Subclass of meromorphically multivalent functions defined by differential operator****, Math. CV. 27(2010), 1-23.*

[12] S. Ruscheweyh, ***Neighborhoods of univalent functions****, Proc. Amer. Math. Soc. ,81 (1981), 521-527.*