A CLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY HADAMARD PRODUCT WITH RAFID - OPERATOR

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Abstract : In this paper, we study a class $WR(\lambda,\beta,\alpha,\mu,\theta)$, which consists of analytic and univalent functions with negative coefficients in the open unit disk $U=\{z\in C:|z|<1\}$ defined by Hadamard product (or convolution) with Rafid – Operator , we obtain coefficient bounds for this class. Also and some results for this class are obtained.

Key words: Univalent Function, Hadamard Product ,Rafid Operator, Integral Representation, Close – to – Convex, $(n,\tau)-$ Neighborhoods .

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1- Introduction :

Let R denote the class of functions of the form :

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n,(a_n\geq 0,n\in IN=\{1,2,3,...\}) \qquad (1)$$

which are analytic and univalent in the unit disk $U=\{z\in C:|z|<1\}$.
If $f\in R$ is given by (1) and $g\in R$ given by

$$g(z)=z-\sum_{n=2}^{\infty}b_n z^n,b_n\geq 0$$

then the Hadamard product (or convolution) (f_*g) of f and g is defined by

$$(f_*g)(z)=z-\sum_{n=2}^{\infty}a_n b_n z^n=(g_*f)(z) \qquad (2)$$

Lemma 1: The *Rafid –Operator* of $f \in R$ for $0 \le \mu < 1, 0 \le \theta \le 1$ is denoted by R_μ^θ and defined as following :

$$R_\mu^\theta(f(z)) = \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)}\int_0^\infty t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} f(zt)dt$$

$$= z - \sum_{n=2}^{\infty} K(n,\mu,\theta) a_n z^n, \tag{3}$$

where $K(n,\mu,\theta) = \dfrac{(1-\mu)^{n-1}\Gamma(\theta+n)}{\Gamma(\theta+1)}$.

Proof : $R_\mu^\theta(f(z)) = \dfrac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)}\displaystyle\int_0^\infty t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)} f(zt)dt$

$$= \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)}\int_0^\infty t^{\theta-1} e^{-\left(\frac{t}{1-\mu}\right)}\left[zt - \sum_{n=2}^{\infty} a_n (zt)^n\right]dt$$

$$= \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)}\left[z\int_o^\infty t^{\theta} e^{-\left(\frac{t}{1-\mu}\right)}dt - \sum_{n=2}^{\infty} a_n z^n \int_0^\infty t^{\theta-1+n} e^{-\left(\frac{t}{1-\mu}\right)}dt\right]$$

Let $x = \dfrac{t}{1-\mu}$, then if t= 0 , we get $x = 0$

t = ∞ , we get $x = \infty$

and $t = (1-\mu)x$, then $dt = (1-\mu)dx$.

Thus

$$R_\mu^\theta(f(z)) = \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)}\left[z\int_0^\infty (1-\mu)^{1+\theta} e^{-x} x^{\theta} dx - \sum_{n=2}^{\infty} a_n z^n \int_0^\infty (1-\mu)^{\theta+n} e^{-x} x^{\theta-1+n} dx\right]$$

$$= \frac{1}{(1-\mu)^{1+\theta}\Gamma(\theta+1)}\left[z(1-\mu)^{1+\theta}\Gamma(\theta+1) - \sum_{n=2}^{\infty} a_n z^n (1-\mu)^{\theta+n}\Gamma(\theta+n)\right]$$

$$= z - \sum_{n=2}^{\infty} \frac{(1-\mu)^{n-1}\Gamma(\theta+n)}{\Gamma(\theta+1)} a_n z^n$$

$$= z - \sum_{n=2}^{\infty} K(n,\mu,\theta) a_n z^n .$$

Definition 1: A function $f \in R, z \in U$ is said to be in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$ if and only if satisfies the inequality :

$$\operatorname{Re}\left\{\frac{z(R_\mu^\theta((f_* g)(z)))' + \lambda z^2 (R_\mu^\theta((f_* g)(z)))''}{(1-\lambda)R_\mu^\theta((f_* g)(z)) + \lambda z(R_\mu^\theta((f_* g)(z)))'}\right\} \ge$$

$$\beta\left\{\frac{z(R_\mu^\theta((f_*g)(z)))' + \lambda z^2 (R_\mu^\theta((f_*g)(z)))''}{(1-\lambda)R_\mu^\theta((f_*g)(z)) + \lambda z(R_\mu^\theta((f_*g)(z)))'} - 1\right\} + \alpha, \tag{4}$$

where $0 \le \alpha < 1$, $0 \le \lambda \le 1, \beta \ge 0, z \in U, 0 \le \mu < 1, 0 \le \theta \le 1$ and $g(z) \in R$ given by $g(z) = z - \sum_{n=2}^{\infty} b_n z^n, b_n \ge 0$.

Lemma 2[1] : Let $w = u + iv$. Then $\operatorname{Re} w \ge \sigma$ if and only if $|w - (1+\sigma)| \le |w + (1-\sigma)|$

Lemma 3[1] : Let $w = u + iv$ and σ, γ are real numbers. Then $\operatorname{Re} w > \sigma|w-1| + \gamma$ if and only if Re $\{w(1+\sigma e^{i\phi}) - \sigma e^{i\phi}\} > \gamma$.

Here ,we aim to study the coefficient bounds, Hadamard product of the class $WR(\lambda,\beta,\alpha,\mu,\theta)$,radius of close-to-convexity, integral representation for $R_\mu^\theta((f_*g)(z))$, inclusive properties of the class $WR(\lambda,\beta,\alpha,\mu,\theta)$, (n,τ)- neighborhoods on $WR^\ell(\lambda,\beta,\alpha,\mu,\theta)$.

2. Coefficient Bounds:

Here we obtain a necessary and sufficient condition and extreme points for the functions f in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$.

Theorem 1: The function f defined by (1) is in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$ if and only if

$$\sum_{n=2}^{\infty} (1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)a_n b_n \le 1-\alpha, \tag{5}$$

where $0 \le \alpha < 1, \beta \ge 0, 0 \le \lambda \le 1, 0 \le \mu < 1$ and $0 \le \theta \le 1$.

Proof: Let (5) holds true . Then we must to show that $f \in WR(\lambda,\beta,\alpha,\mu,\theta)$ .By Definition 1 , we have

$$\operatorname{Re}\left\{\frac{z(R_\mu^\theta((f_*g)(z)))' + \lambda z^2 (R_\mu^\theta((f_*g)(z)))''}{(1-\lambda)R_\mu^\theta((f_*g)(z)) + \lambda z(R_\mu^\theta((f_*g)(z)))'}\right\} \ge$$
$$\beta\left|\frac{z(R_\mu^\theta((f_*g)(z)))' + \lambda z^2 (R_\mu^\theta((f_*g)(z)))''}{(1-\lambda)R_\mu^\theta((f_*g)(z)) + \lambda z(R_\mu^\theta((f_*g)(z)))'} - 1\right| + \alpha.$$

Then by Lemma 3 , we have

$$\operatorname{Re}\left\{\frac{z(R_\mu^\theta((f_*g)(z)))' + \lambda z^2(R_\mu^\theta((f_*g)(z)))''}{(1-\lambda)R_\mu^\theta((f_*g)(z)) + \lambda z(R_\mu^\theta((f_*g)(z)))'}(1+\beta e^{i\phi}) - \beta e^{i\phi}\right\} \geq \alpha,$$

$$-\pi < \phi \leq \pi$$

or equivalently

$$\operatorname{Re}\left\{\frac{(z(R_\mu^\theta((f_*g)(z)))' + \lambda z^2(R_\mu^\theta((f_*g)(z)))''(1+\beta e^{i\phi})}{(1-\lambda)(R_\mu^\theta((f_*g)(z))) + \lambda z(R_\mu^\theta((f_*g)(z)))'}\right.$$

.

$$\left. - \frac{\beta e^{i\phi}((1-\lambda)R_\mu^\theta((f_*g)(z)) + \lambda z^2(R_\mu^\theta((f_*g)(z)))')}{(1-\lambda)R_\mu^\theta((f_*g)(z)) + \lambda z(R_\mu^\theta((f_*g)(z)))'}\right\} \geq \alpha. \tag{6}$$

Let $F(z) = [z(R_\mu^\theta((f_*g)(z)))' + \lambda z^2(R_\mu^\theta((f_*g)(z)))''] \quad (1+\beta e^{i\phi})$
$- \beta e^{i\phi}[(1-\lambda)(R_\mu^\theta((f_*g)(z))) + \lambda z(R_\mu^\theta((f_*g)(z)))'],$

and $E(z) = (1-\lambda)R_\mu^\theta((f_*g)(z)) + \lambda z(R_\mu^\theta((f_*g)(z)))'.$

By Lemma 2 , (6) is equivalent to

$|F(z) + (1-\alpha)E(z)| \geq |F(z) - (1+\alpha)E(z)|$ for $0 \leq \alpha < 1$.

But $|F(z) + (1-\alpha)E(z)|$

$$= \left|\left[z - \sum_{n=2}^{\infty} K(n,\mu,\theta)a_n b_n z^n - \lambda\sum_{n=2}^{\infty} n(n-1)K(n,\mu,\theta)a_n b_n z^n\right](1+\beta e^{i\phi})\right.$$

$$- \beta e^{i\phi}\left[(1-\lambda)(z - \sum_{n=2}^{\infty} K(n,\mu,\theta)a_n b_n z^n) + \lambda z - \lambda\sum_{n=2}^{\infty} nK(n,\mu,\theta)a_n b_n z^n\right]$$

$$\left. + (1-\alpha)\left[z - \sum_{n=2}^{\infty}(1-\lambda+n\lambda)K(n,\mu,\theta)a_n b_n z^n\right]\right|$$

$$= \left|(2-\alpha)z - \sum_{n=2}^{\infty}\left[(n+\lambda n(n-1)) + (1-\alpha)(1-\lambda+n\lambda)\right]K(n,\mu,\theta)a_n b_n z^n\right.$$

$$\left. - \beta e^{i\phi}\sum_{n=2}^{\infty}\left[n+\lambda n(n-1) - (1-\lambda+n\lambda)\right]K(n,\mu,\theta)a_n b_n z^n\right|$$

$$\geq (2-\alpha)|z| - \sum_{n=2}^{\infty}\left[(n+\lambda n(n-1)) + (1-\alpha)(1-\lambda+\lambda n)\right]K(n,\mu,\theta)a_n b_n |z|^n$$

$$- \beta\sum_{n=2}^{\infty}\left[n+\lambda n(n-2) - 1 + \lambda\right]K(n,\mu,\theta)a_n b_n |z|^n .$$

Also $|F(z) - (1+\alpha)E(z)| = \left|\left[z - \sum_{n=2}^{\infty} nK(n,\mu,\theta)a_n b_n z^n\right.\right.$

$$-\lambda\sum_{n=2}^{\infty} n(n-1)K(n,\mu,\theta)a_n b_n z^n \Bigg]\left(1+\beta e^{i\phi}\right)$$

$$-\beta e^{i\phi}\left[z-(1-\lambda)\sum_{n=2}^{\infty}K(n,\mu,\theta)a_n b_n z^n-\lambda\sum_{n=2}^{\infty}nK(n,\mu,\theta)a_n b_n z^n\right]$$

$$-(1+\alpha)\left[z-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)K(n,\mu,\theta)a_n b_n z^n\right]\Bigg|$$

$$=\Bigg|-\alpha z-\sum_{n=2}^{\infty}\left[(n+\lambda n(n-1))-(1+\alpha)(1-\lambda+n\lambda)\right]K(n,\mu,\theta)a_n b_n z^n$$

$$-\beta e^{i\phi}\sum_{n=2}^{\infty}\left[n+n\lambda(n-1)-(1-\lambda+n\lambda)\right]K(n,\mu,\theta)a_n b_n z^n\Bigg|$$

$$\leq \alpha|z|+\sum_{n=2}^{\infty}\left[(n+n\lambda(n-1))-(1+\alpha)(1-\lambda+n\lambda)\right]K(n,\mu,\theta)a_n b_n |z|^n$$

$$+\beta\sum_{n=2}^{\infty}\left[n+n\lambda(n-1)-(1-\lambda+n\lambda)\right]K(n,\mu,\theta)a_n b_n |z|^n$$

and so $|F(z)+(1-\alpha)E(z)|-|F(z)-(1+\alpha)E(z)|\geq 2(1-\alpha)|z|$

$$-\sum_{n=2}^{\infty}\ \left[(2n+2n\lambda(n-1))-2\alpha(1-\lambda+n\lambda)-\beta(2n+2n\lambda(n-1)-\right.$$

$$\left.-2(1-\lambda+n\lambda))\right]K(n,\mu,\theta)a_n b_n |z|^n\geq 0$$

or

$$\sum_{n=2}^{\infty}\left[n(1+\beta)+n\lambda(n-1)(1+\beta)-(1-\lambda+n\lambda)(\alpha+\beta)\right]K(n,\mu,\theta)a_n b_n\leq 1-\alpha.$$

This is equivalent to

$$\sum_{n=2}^{\infty}(1-\lambda+n\lambda)\left[n(1+\beta)-(\beta+\alpha)\right]K(n,\mu,\theta)a_n b_n\leq 1-\alpha.$$

Conversely , suppose that (5) holds. Then we must show

$$\operatorname{Re}\left\{\frac{\left[z(R_\mu^\phi((f_*g)(z)))'+\lambda z^2(R_\mu^\theta((f_*g)(z)))''\right](1+\beta e^{i\phi})}{(1-\lambda)R_\mu^\theta((f_*g)(z))+\lambda z(R_\mu^\theta((f_*g)(z)))'}\right.$$
$$\left.-\frac{\beta e^{i\phi}\left[(1-\lambda)R_\mu^\phi((f_*g)(z))+\lambda z(R_\mu^\phi((f_*g)(z)))'\right]}{(1-\lambda)R_\mu^\theta((f_*g)(z))+\lambda z(R_\mu^\theta((f_*g)(z)))'}\right\}\geq\alpha.$$

Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, the above inequality reduces to

$$\operatorname{Re}\left\{\frac{(1-\alpha)-\sum_{n=2}^{\infty}\left[n(1+\beta e^{i\phi})(1-\lambda+\lambda n)-(\alpha+\beta e^{i\phi})(1-\lambda+n\lambda)\right]K(n,\mu,\theta)a_n b_n r^{n-1}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)K(n,\mu,\theta)a_n b_n r^{n-1}}\right\}\geq 0.$$

Since $\operatorname{Re}\left(-e^{i\phi}\right)\geq -\left|e^{i\phi}\right|=-1,$ the above inequality reduces to

$$\operatorname{Re}\left\{\frac{(1-\alpha)-\sum_{n=2}^{\infty}\left[n(1+\beta)(1-\lambda+\lambda n)-(\alpha+\beta)(1-\lambda+n\lambda)\right]K(n,\mu,\theta)a_n b_n r^{n-1}}{1-\sum_{n=2}^{\infty}(1-\lambda+n\lambda)K(n,\mu,\theta)a_n b_n r^{n-1}}\right\}\geq 0.$$

Letting $r\to 1^-$, we get desired conclusion .

There are many authors who have studied the various interesting properties of the classes, Y. Komatu [4], W. G. Atshan and S. R. Kulkarni [2], S. Kanas and A. Wisniowska [3], Rosy et. al.[5] .

In the next discussion, we concentrate upon getting the radius of close – to–convexity .

Theorem 2: Let the function f defined by (1) be in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$.Then f is close – to–convex of order δ $(0\leq\delta<1)$ in $|z|<r(\lambda,\beta,\alpha,\mu,\theta,\delta),$ where

$$r(\lambda,\beta,\alpha,\mu,\theta,\delta)=\inf_n\left\{\frac{(1-\delta)(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]k(n,\mu,\theta)b_n}{n(1-\alpha)}\right\}^{\frac{1}{n-1}},n\geq 2.(7)$$

Proof: We must show that $|f'(z)-1|\leq 1-\delta$ for $|z|<r(\lambda,\beta,\alpha,\mu,\theta,\delta)$,

we have

$$|f'(z)-1|\leq\sum_{n=2}^{\infty} na_n|z|^{n-1}.$$

Thus $$|f'(z)-1|\leq 1-\delta \text{ if } \sum_{n=2}^{\infty}\left(\frac{n}{1-\delta}\right)a_n|z|^{n-1}\leq 1. \qquad (8)$$

According to Theorem 1, we have

$$\sum_{n=2}^{\infty}\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)}{(1-\alpha)}a_n b_n\leq 1. \qquad (9)$$

Hence (8) will be true if

$$\frac{n|z|^{n-1}}{1-\delta} \le \frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)b_n}{(1-\alpha)}$$

equivalently if

$$|z| \le \left\{\frac{(1-\delta)(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)b_n}{n(1-\alpha)}\right\}^{\frac{1}{n-1}}, n \ge 2. \tag{10}$$

The theorem follows from (١٠).

The following theorem provides integral representation for $R_\mu^\theta((f*g)(z)), z \in U$.

<u>Theorem 3:</u> Let $f \in WR(0,\beta,\alpha,\mu,\theta)$. Then

$$R_\mu^\theta((f*g)(z)) = \exp\left(\int_0^z \frac{\beta-\psi(t)\alpha}{t(\beta-\psi(t))}dt\right), \ |\psi(t)|<1, \ z \in U.$$

<u>proof:</u> The case $\beta = 0$ is obvious . Let $\beta \ne 0$, for $f \in WR(0,\beta,\alpha,\mu,\theta)$ and

$$w = \frac{z(R_\mu^\theta((f*g)(z)))'}{R_\mu^\theta((f*g)(z))},$$

we have $\operatorname{Re} w > \beta|w-1| + \alpha.$ Therefore

$$\left|\frac{w-1}{w-\alpha}\right| < \frac{1}{\beta} \quad \text{or equivalently} \quad \frac{w-1}{w-\alpha} = \frac{\psi(z)}{\beta},$$

where $|\psi(z)|<1, \ z \in U.$

so $$\frac{(R_\mu^\theta((f*g)(z)))'}{R_\mu^\theta((f*g)(z))} = \frac{\beta-\psi(z)\alpha}{z(\beta-\psi(z))}$$

after integration , we get

$$\log(R_\mu^\theta((f*g)(z))) = \int_0^z \frac{\beta-\psi(t)\alpha}{t(\beta-\psi(t))}dt \ .$$

Thus $$(R_\mu^\theta((f*g)(z))) = \exp\left[\int_0^z \frac{\beta-\psi(t)\alpha}{t(\beta-\psi(t))}dt\right].$$

This completes the proof .

Theorem 4: Let $f(z)=z-\sum_{n=2}^{\infty}a_n z^n$, $g(z)=z-\sum_{n=2}^{\infty}b_n z^n$ belong to $WR(\lambda,\beta,\alpha,\mu,\theta)$. Then the Hadamard product of f and g given by $(f_*g)(z)=z-\sum_{n=2}^{\infty}a_n b_n z^n$ belongs to $WR(\lambda,\beta,\alpha,\mu,\theta)$.

Proof : Since f and $g\in WR(\lambda,\beta,\alpha,\mu,\theta)$, we have

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)b_n}{1-\alpha}\right]a_n \le 1$$

and

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)a_n}{1-\alpha}\right]b_n \le 1$$

and by applying the (Cauchy- Schwarz) inequality , we have

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)\sqrt{a_n b_n}}{1-\alpha}\right]\sqrt{a_n b_n}$$

$$\le\left(\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)b_n}{1-\alpha}\right]a_n\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)a_n}{1-\alpha}\right]b_n\right)^{\frac{1}{2}}.$$

However , we obtain

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)\sqrt{a_n b_n}}{1-\alpha}\right]\sqrt{a_n b_n}\le 1.$$

Now, we want to prove

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)}{1-\alpha}\right]a_n b_n \le 1$$

Since

$$\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)}{1-\alpha}\right]a_n b_n$$

$$=\sum_{n=2}^{\infty}\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)\sqrt{a_nb_n}}{1-\alpha}\right]\sqrt{a_nb_n}\ .$$

Hence, we get the required result.

Theorem 5: Let the function f defined by (1) and g given by

$g(z)=z-\sum_{n=2}^{\infty}b_nz^n$ be in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$.Then the function h defined

by $h(z)=z-\sum_{n=2}^{\infty}a_n^2b_n^2z^n$ is in the class $WR(\lambda,\beta,\gamma,\mu,\theta)$, where $0\le\lambda\le1, 0\le\alpha<1$,

$0\le\gamma<1, \beta\ge0, z\in U, 0\le\mu<1, 0\le\theta\le1$ and

$$\gamma\le1-\frac{(1-\alpha)^2(1+\beta)}{(1+\lambda)(2+\beta-\alpha)^2(1-\mu)(1+\theta)-(1-\alpha)^2}\ .$$

Proof: We must find the largest γ such that

$$\sum_{n=2}^{\infty}\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma)]K(n,\mu,\theta)}{1-\gamma}a_n^2b_n^2\le1. \tag{11}$$

Since f and g are in the class $WR(\lambda,\beta,\alpha,\mu,\theta)$, we see that

$$\sum_{n=2}^{\infty}\left\{\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)b_n}{1-\alpha}\right]a_n\right\}^2\le1 \tag{12}$$

and

$$\sum_{n=2}^{\infty}\left\{\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)a_n}{1-\alpha}\right]b_n\right\}^2\le1 \tag{13}$$

Combining the inequalities (12) and (13) ,gives

$$\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)}{1-\alpha}\right\}^2a_n^2b_n^2\le1,$$

but $h\in WR(\lambda,\beta,\gamma,\mu,\theta)$, if and only if

$$\sum_{n=2}^{\infty}\left\{\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma)]K(n,\mu,\theta)}{1-\gamma}\right\}a_n^2b_n^2\le1\,. \tag{14}$$

The inequality (14) would obviously imply (11) if

$$\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma)]K(n,\mu,\theta)}{1-\gamma}\le\left[\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)}{1-\alpha}\right]^2=u^2,$$

then

$$\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma)]K(n,\mu,\theta)}{1-\gamma}\le u^2$$

or

$$\frac{1-\gamma}{1+\beta}\ge\frac{(1-\lambda+n\lambda)(n-1)K(n,\mu,\theta)}{u^2-(1-\lambda+n\lambda)K(n,\mu,\theta)}$$

The right hand is a decreasing function of n and its maximum if $n=2$.

Now

$$\frac{1-\gamma}{1+\beta}\ge\frac{(1-\lambda+n\lambda)(n-1)(1-\alpha)^2}{[(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma)]]^2K(n,\mu,\theta)-(1-\lambda+n\lambda)(1-\alpha)^2}. \tag{15}$$

Simplifying (15) we get

$$\frac{1-\gamma}{1+\beta}\ge\frac{(1-\alpha)^2}{(1+\lambda)(2+\beta-\alpha)^2(1-\mu)(\theta+1)-(1-\alpha)^2}$$

or

$$\gamma\le 1-\frac{(1-\alpha)^2(1+\beta)}{(1+\lambda)(2+\beta-\alpha)^2(1-\mu)(1+\theta)-(1-\alpha)^2}.$$

This completes the proof of theorem.

Next, we obtain the inclusive properties of the class $WR(\lambda,\beta,\alpha,\mu,\theta)$.

Theorem 6 : Let $\beta\ge 0, 0\le\alpha<1, 0\le\lambda\le 1,\gamma\ge 0, 0\le\mu<1$ and $0\le\theta\le 1$. Then $WR(\lambda,\beta,\alpha,\mu,\theta)\subseteq WR(0,\beta,\gamma,\mu,\theta)$, where

$$\gamma\le 1-\frac{(n-1)(1-\alpha)(1+\beta)K(n,\mu,\theta)}{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\gamma)]K(n,\mu,\theta)-(1-\alpha)K(n,\mu,\theta)},$$

$n\in IN, n\ge 2.$

Proof: Let $f\in WR(\lambda,\beta,\alpha,\mu,\theta)$ .Then in view of Theorem 1, we have

$$\sum_{n=2}^{\infty}\frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)}{1-\alpha}a_n b_n\le 1 \tag{16}$$

we wish to find the value γ such that

$$\sum_{n=2}^{\infty} \frac{[n(1+\beta)-(\beta+\gamma)]K(n,\mu,\theta)}{1-\gamma} a_n b_n \leq 1. \tag{17}$$

The inequality (16) would obviously imply (17) if

$$\frac{[n(1+\beta)-(\beta+\gamma)]K(n,\mu,\theta)}{1-\gamma} \leq \frac{(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)}{1-\alpha} = u.$$

Therefore

$$\frac{[n(1+\beta)-(\beta+\gamma)]K(n,\mu,\theta)}{1-\gamma} \leq u. \tag{18}$$

Now (18) gives on simplification

$$\frac{1-\gamma}{1+\beta} \geq \frac{(n-1)K(n,\mu,\theta)}{u-K(n,\mu,\theta)} \quad , \ (n \geq 2, n \in IN). \tag{19}$$

The right hand side of (19) decreases as n increases and so is maximum for $n=2$.

So (19) is satisfied provided

$$\frac{1-\gamma}{1+\beta} \geq \frac{(n-1)(1-\alpha)K(n,\mu,\theta)}{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)-(1-\alpha)K(n,\mu,\theta)} = d.$$

Obviously $d<1$ and

$$\gamma \leq 1-\frac{(n-1)(1-\alpha)(1+\beta)K(n,\mu,\theta)}{(1-\lambda+n\lambda)[n(1+\beta)-(\beta+\alpha)]K(n,\mu,\theta)-(1-\alpha)K(n,\mu,\theta)}.$$

This completes the proof of theorem.

Theorem 7 : Let $\beta \geq 0, 0 \leq \alpha < 1, \lambda_1 \geq \lambda_2 \geq 0, 0 \leq \theta \leq 1, 0 \leq \mu < 1$. Then

$$WR(\lambda_1,\beta,\alpha,\mu,\theta) \subseteq WR(\lambda_2,\beta,\alpha,\mu,\theta).$$

The proof of theorem follows also from Theorem 6 .

Now, we determine a set of inclusion relations involving (n,τ) – neighborhoods. Following [6], we define the (n,τ) – neighborhoods of a function $f \in R$ by

$$N_{n,\tau}(f) = \left\{ g \in R : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{and} \sum_{n=2}^{\infty} n|a_n - b_n| \le \tau, 0 \le \tau < 1 \right\} \tag{20}$$

We need the following definition:

Definition 2: The function f defined by (1) is said to be member of the class $WR^{\ell}(\lambda,\beta,\alpha,\mu,\theta)$ if there exists a function $g \in WR(\lambda,\beta,\alpha,\mu,\theta)$

such that $\left| \frac{f(z)}{g(z)} - 1 \right| \le 1 - \ell$, $(z \in U, 0 \le \ell < 1)$.

Theorem 8: Let $g \in WR(\lambda,\beta,\alpha,\mu,\theta)$ and

$$\ell = 1 - \frac{\tau(1+\lambda)(2+\beta-\alpha)(1-\mu)(\theta+1)a_2}{2\{(1+\lambda)(2+\beta-\alpha)(1-\mu)(\theta+1)a_2 - (1-\alpha)\}} \tag{21}$$

Then $N_{n,\tau}(g) \subset WR^{\ell}(\lambda,\beta,\alpha,\mu,\theta)$.

Proof: Let $f \in N_{n,\tau}(g)$ .Then we have from (20) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \le \tau ,$$

which readily implies the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \le \frac{\tau}{2} .$$

Also, since $g \in WR(\lambda,\beta,\alpha,\mu,\theta)$,we have from Theorem 1

$$\sum_{n=2}^{\infty} b_n \le \frac{(1-\alpha)}{(1+\lambda)(2+\beta-\lambda)(1-\mu)(\theta+1)a_2},$$

so that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{n=2}^{\infty} |a_n - b_n|}{1 - \sum_{n=2}^{\infty} b_n} \le \frac{\tau}{2} . \frac{(1+\lambda)(2+\beta-\alpha)(1-\mu)(\theta+1)a_2}{(1+\lambda)(2+\beta-\alpha)(1-\mu)(\theta+1)a_2 - (1-\alpha)}$$

$$= 1 - \ell .$$

Thus by definition , $f \in WR^{\ell}(\lambda,\beta,\alpha,\mu,\theta)$ for ℓ given by (21).

This completes the proof.

We further define the integral operator , in the following discussion :

Theorem 9: Let c be a real number such that $c > -1$. If $f \in WR(\lambda,\beta,\alpha,\mu,\theta)$.

Then the function F_c defined by

$$F_c(z) = \frac{c+1}{z^c}\int_0^z s^{c-1} f(s)ds \tag{22}$$

also belongs to $WR(\lambda,\beta,\alpha,\mu,\theta)$.

Proof: Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$. Then

$$F_c(z) = \frac{c+1}{z^c}\int_0^z s^{c-1}(s - \sum_{n=2}^{\infty} a_n s^n)ds$$

$$= \frac{c+1}{z^c}\int_0^z (s^c - \sum_{n=2}^{\infty} s^{c-1+n} a_n)ds$$

$$= \frac{c+1}{z^c}\left[\frac{s^{c+1}}{c+1} - \sum_{n=2}^{\infty} \frac{s^{c+n}}{c+n} a_n\right]_0^z$$

$$= z - \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n .$$

Hence $$F_c(z) = z - \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n .$$

Therefore ,

$$\sum_{n=2}^{\infty} \frac{(c+1)(1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta)}{(c+n)} a_n b_n$$

$$\le (1-\lambda+n\lambda)[n(1+\beta)-(\alpha+\beta)]K(n,\mu,\theta) a_n b_n \le 1-\alpha .$$

Hence $F_c \in WR(\lambda,\beta,\alpha,\mu,\theta)$.

This completes the proof.

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