

On Uncertain Martingale

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Abstract

We introduce some a new properties of uncertain conditional expectation, also we give a new kind of martingale and study some theorems related with it.

Key words

uncertain measure, uncertain variable, conditional uncertain measure, uncertain conditional expectation and uncertain martingale.

1-Introduction

Probability theory often profitable to interpret results in terms of a gambling situation . for example ,if X_1, X_2, \dots is a sequence of random variables, we may think of X_n as our total winnings after n trials in a succession of games. Having survived the first n trial, our expected fortune after trial $n+1$ is $E(X_{n+1} | X_1, \dots, X_n)$. If equals X_n , the game is "**fair**" since the expected gain on trial $n+1$ is $E(X_{n+1} - X_n | X_1, \dots, X_n) = X_n - X_n = 0$. If $E(X_{n+1} - X_n | X_1, \dots, X_n) \geq X_n$.the game is "**favorable**" and $E(X_{n+1} - X_n | X_1, \dots, X_n) \leq X_n$, the game is "**unfavorable**"

"[2]. Uncertainty theory was founded by Liu [3] in 2007 and refined by Liu [5] in 2010. Let (Ω, F) be a measurable space, where Ω is a set and F is a σ -field on Ω . A subset A of Ω is called measurable (measurable with respect to the σ -field F), if $A \in F$, i.e., any member of F is called a measurable set [2]. A set function μ from F to $[0,1]$ is a real-valued set function μ defined on σ -field F is called an uncertain measure, if it satisfies the following four axioms:

Axiom 1. (Normality Axiom) $\mu(\Omega) = 1$.

Axiom 2. (Self-duality Axiom) $\mu(A) + \mu(A^c) = 1$ for any event A .

Axiom 3. (Countable subadditivity Axiom) For every countable sequence of events $\{A_i\}$, we have $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$ (1)

Axiom 4. (Product measure Axiom) Let Ω_k be a nonempty sets on which μ_k are uncertain measures, $k = 1, 2, \dots, n$, respectively. Then the product of uncertain measures μ_k is an uncertain measure μ on the product σ -field $\Omega_1 \times \Omega_2 \times \dots \times \Omega_n$ satisfying

$$\mu\left(\prod_{k=1}^n A_k\right) = \min_{1 \leq k \leq n} \mu_k(A_k) \quad (2)$$

That is, for each $A \in \Omega$, we have

$$\mu(A) = \begin{cases} \sup_{A_1 \times A_2 \times \dots \times A_n \subset A} \left\{ \min_{1 \leq k \leq n} \mu_k(A_k) \right\} & \text{if } \sup_{A_1 \times A_2 \times \dots \times A_n \subset A} \left\{ \min_{1 \leq k \leq n} \mu_k(A_k) \right\} > 0.5 \\ 1 - \sup_{A_1 \times A_2 \times \dots \times A_n \subset A^c} \left\{ \min_{1 \leq k \leq n} \mu_k(A_k) \right\} & \text{if } \sup_{A_1 \times A_2 \times \dots \times A_n \subset A^c} \left\{ \min_{1 \leq k \leq n} \mu_k(A_k) \right\} > 0.5 \\ 0.5, & \text{o.w} \end{cases} \quad (3)$$

Then the triplet (Ω, F, μ) is called an uncertainty space .

Remark (1-1)

The probability measure is not uncertain measure as shown by the following example:

Example (1-2)[4]

Let $g : R \rightarrow R$ is a nonnegative and integrable function such that $\int_R g(x) dx = 1$. Define $P : \beta(R) \rightarrow R$ by $P(A) = \int_A g(x) dx$, is a probability measure but not uncertain measure.

Ans:

Step1: we prove the normality .i.e., $P(R) = 1$.

$$P(R) = \int_R g(x)dx = 1$$

Step2: we prove the self-duality .i.e., $P(A) + P(A^c) = 1$.

$$\text{Since } P(A \cup A^c) = \int_{A \cup A^c} g(x)dx = \int_R g(x)dx = 1, \text{ and } P(A) + P(A^c) = P(A \cup A^c)$$

$$\text{Thus, } P(A) + P(A^c) = P(A \cup A^c) = 1$$

Step3: Let $\{A_i\}$ be a sequence of sets in $\beta(R)$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \int_{\bigcup_{i=1}^{\infty} A_i} g(x)dx = \int_R g(x)dx = 1, \text{ and } \sum_{i=1}^{\infty} P(A_i) = 1$$

Then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$, and then P is not uncertain measure.

Example(1-3)[3,4,5,6]:

Suppose that $g : R \rightarrow R$ is nonnegative satisfying $\sup\{g(x) + g(y) : x \neq y\} = 1$, define $\mu : \beta(R) \rightarrow R$, the set function

$$\mu(A) = \begin{cases} \sup\{g(x) : x \in A\}, & \sup\{g(x) : x \in A\} < 0.5 \\ 1 - \sup\{g(x) : x \in A^c\}, & \sup\{g(x) : x \in A\} \geq 0.5 \end{cases} \quad (4)$$

for all $A \in \beta(R)$ is an uncertain measure.

Definition (1-4) [2]

Let (Ω, F) and (Ω', F') be two measurable spaces. A function $g : \Omega \rightarrow \Omega'$ is said to be a measurable with respect to F and F' , if $g^{-1}(A) \in F$ for all $A \in F'$.

Definition (1-5) [5,6]

an uncertain variable is a measurable function from an uncertainty space (Ω, F, μ) to the set of real numbers, i.e., for any Borel set B of real numbers. The set $\{X \in B\} = \{x \in \Omega \mid X(x) \in B\}$ is an event.

Definition (1-6) [2]

Let $A \subseteq \Omega$. A function $I_A : \Omega \rightarrow R$ defined by :

$$I_A(\omega) = \begin{cases} 1 & , \omega \in A \\ 0 & , \omega \notin A \end{cases} \quad (5)$$

Is called indicator function or (characteristic function) of A .

Definition (1-7)

Let μ and λ be two uncertain measures on measurable space (Ω, F) . We say that λ is absolute continuous with respect to μ (written $\lambda \ll \mu$) if $\lambda(A) = 0$ for every $A \in F$ with $\mu(A) = 0$, i.e., for all $A \in F$ with $\mu(A) = 0$, we have $\lambda(A) = 0$.

Definition (1-9) [1]

A filtration $\{F_n, n \in N\}$ is a sequence of sub σ -fields of F such that for all $n \in N$, $F_n \subset F_{n+1}$.

Definition (1-10)[6]

Let X be an uncertain variable. Then the expected value of X is defined by

$$E(X) = \int_0^{+\infty} \mu(X \geq r) dr - \int_{-\infty}^0 \mu(X \leq r) dr \quad (6)$$

Provided that at least one of the two integrals is finite.

Lemma (1-11) (Jensen's Inequality) [5,6]

Let X be an uncertain variable and $f : R \rightarrow R$ a convex function . If $E(X)$ and $E(f(X))$ are finite, then

$$f(E(X)) \leq E(f(X)) \quad (7)$$

Especially, when $f(X) = |x|^p$ and $p \geq 1$, we have $|E(X)|^p \leq E(|X|^p)$.

Proof:

Since f is a convex function, for each y , there exists a number K such that $f(x) - f(y) \geq K(x - y)$. Replacing x with X and y with $E(X)$, we obtain

$$f(X) - f(E(X)) \geq K(X - E(X)).$$

Taking the expected value on both sides, we have

$$E(f(X)) - f(E(X)) \geq K(E(X) - E(X)) = 0.$$

Which proves the inequality.

2- Radon-Nikodym theorem on uncertain measure

Let μ and λ be two uncertain measure on the σ -field F of subsets of Ω . Assume that $\lambda \ll \mu$, then there is a Borel measurable function $g: \Omega \rightarrow R$ such that $\lambda(A) = \int_A g d\mu$, for all $A \in F$. g is called the Radon-Nikodym derivative of λ with respect to μ . It is sometimes denoted by

$$\left[\frac{d\lambda}{d\mu}\right], \text{ i.e. } g = \left[\frac{d\lambda}{d\mu}\right].$$

Proof:

Let $\{g_n\}$ be a bounded increasing sequence of nonnegative measurable function in F .

Since every monotonic and bounded sequence is converge.

Then there exists a unique function g in F such that $\lim_{n \rightarrow \infty} g_n = g$

Now, we must to show that $\lambda(A) = \int_A g d\mu$ for all $A \in F$.

Define $\lambda_1(A) = \lambda(A) - \int_A g d\mu$ for all $A \in F$, and $\lambda_1 \ll \mu$

Since $\mu(A) = 0$ for all $A \in F$, then $\lambda_1(A) = 0$. By absolute continuity thus

$$\lambda(A) = \int_A g d\mu \text{ for all } A \in F.$$

3- The conditional uncertain measure

Definition (3-1)

Let (Ω, F, μ) be an uncertainty space and $A, B \in F$, then the conditional uncertain measure of A given B is defined by

$$\mu_B(A) = \mu(A | B) = \begin{cases} \frac{\mu(A \cap B)}{\mu(B)}, & \text{if } \frac{\mu(A \cap B)}{\mu(B)} < 0.5 \\ 1 - \frac{\mu(A^c \cap B)}{\mu(B)}, & \text{if } \frac{\mu(A^c \cap B)}{\mu(B)} < 0.5 \\ 0.5, & \text{o.w} \end{cases} \quad (8)$$

Provided that $\mu(B) > 0$.

Definition (3-2)

Let (Ω, F, μ) be an uncertainty space and let G be a sub σ -field of F . Then the conditional expectation of X given G is any uncertain variable Z which satisfies the following two properties:

- (a) Z is G -measurable.
- (b) if $A \in G$, then $\int_A X d\mu = \int_A Z d\mu$.

We denote Z by $E(X | G)$.

Theorem (3-3)

If X, X_1, X_2, \dots be an uncertain variables on (Ω, F, μ) , a and b real numbers. Then

- 1- If $X = a$ a.e., then $E(X | G) = a$ a.e. $[\mu]$
- 2- If X is G -measurable, then $E(X | G) = X$ a.e. $[\mu]$
- 3- $E(E(X | G)) = E(X)$ a.e. $[\mu]$
- 4- $E(aX_1 + bX_2 | G) = aE(X_1 | G) + bE(X_2 | G)$ a.e. $[\mu]$
- 5- If $G = \{\emptyset, \Omega\}$, then $E(X | G) = E(X)$ a.e. $[\mu]$
- 6- If $X \geq 0$ a.e., then $E(X | G) \geq 0$ a.e. $[\mu]$
- 7- If $X_1 \leq X_2$ a.e., $E(X_1 | G) \leq E(X_2 | G)$ a.e. $[\mu]$
- 8- $|E(X | G)| \leq E(|X| | G)$ a.e. $[\mu]$

9- If Y is G -measurable and XY is integrable then $E(XY | G) = YE(X | G)$ a.e. $[\mu]$

10- If X_n and X are integrable, and if either $X_n \uparrow X$ or $X_n \downarrow X$, then

$$E(X_n | G) \rightarrow E(X | G) \text{ a.e. } [\mu]$$

Proof:

1- If $X = a$ a.e., then it is G -measurable and $\int_A X d\mu = \int_A a d\mu$ for all $A \in G$ and then $E(X | G) = a$ a.e. $[\mu]$

2- If X is G -measurable and $\int_A X d\mu = \int_A E(X | G) d\mu$ for all $A \in G$, then $E(X | G) = X$ a.e. $[\mu]$

3- Take $A = \Omega$ in condition (b) of definition (3-2), we have that $\int_{\Omega} X d\mu = \int_{\Omega} E(X | G) d\mu$ and this implies that $E(E(X | G)) = E(X)$ a.e. $[\mu]$

4- It's clearly that $E(aX_1 + bX_2 | G)$ is G -measurable and if $A \in G$, apply condition (b) of definition (3-2) to X_1 and X_2 to see that

$$\int_A (aE(X_1 | G) + bE(X_2 | G)) d\mu = a \int_A X_1 d\mu + b \int_A X_2 d\mu = \int_A (aX_1 + bX_2) d\mu.$$

5- Since $\int_A X d\mu = \int_A E(X) d\mu$ for all $A = \phi$ or $A = \Omega$, we have $E(X | G) = E(X)$ a.e. $[\mu]$

6- Take $A = \{E(X | G) < 0\} \in G$. then by condition (b) of definition (3-2), we have, $0 \geq \int_A E(X | G) d\mu = \int_A X d\mu \geq 0 \Rightarrow \mu(A) = 0$.

7- Since $X_1 \leq X_2$ a.e., then $Z = X_2 - X_1 \geq 0$ a.e., by (6) we have $E(X_2 | G) - E(X_1 | G) \geq 0$ a.e., thus $E(X_1 | G) \leq E(X_2 | G)$ a.e. $[\mu]$

8- Since $-|X| \leq X \leq |X|$, it follows from (7), that $|E(X | G)| \leq E(|X| | G)$ a.e. $[\mu]$

9- Let $Y = I_B$ (I_B is indicator function) for some $B \in G$. Then $\int_A YE(X | G) d\mu = \int_A I_B(X | G) d\mu = \int_{A \cap B} E(X | G) d\mu = \int_{A \cap B} X d\mu = \int_A YX d\mu$

Thus the condition (b) of definition (3-2), holds in this case.

10- It follows from definition (3-2), by letting $Z = E(X | G)$ and for all $A \in G$. We have $\int_A Z d\mu = \lim_{n \rightarrow \infty} \int_A Z_n d\mu = \int_A Z d\mu$ a.e. $[\mu]$

Thus Z satisfies both (a) and (b) of definition (3-2), and therefore equals $E(X | G)$

4-Uncertain martingale

Definition (4-1)

An uncertain stochastic process is a family of uncertain variables defined on the same uncertainty space.

Definition (4-2)

An uncertain stochastic process $X = \{X_n, n \in N\}$ is an uncertain adapted to the filtration $\{F_n, n \in N\}$ if for all n , X is F_n -measurable.

Definition (4-3)

An uncertain stochastic process $X = \{X_n, F_n, n \in N\}$ is said to be an uncertain martingale if it is satisfying the following conditions:

- (1) X is an uncertain adapted to filtration $\{F_n, n \in N\}$.
- (2) X_n is integrable for all $n \in N$.
- (3) $E\{X_{n+1} | F_n\} = X_n$ a.e. $[\mu]$, for all $n \in N$ (9)

Definition (4-4)

An uncertain stochastic process $X = \{X_n, F_n, n \in N\}$ is said to be an uncertain sub-martingale (resp, uncertain super-martingale) with respect to the filtration $\{F_n, n \in N\}$ if it is satisfying the following conditions:

- (1) X is an uncertain adapted to filtration $\{F_n, n \in N\}$.
- (2) X_n is integrable for all $n \in N$.
- (3) $E\{X_{n+1} | F_n\} \geq X_n$ a.e. $[\mu]$, (resp, $E\{X_{n+1} | F_n\} \leq X_n$ a.e. $[\mu]$) for all $n \in N$

(10)

Example (4-5)

An uncertain stochastic process $X = \{X_n, F_n, n \in N\}$ is an uncertain sub-martingale iff $-X = \{-X_n, F_n, n \in N\}$ is an uncertain super-martingale.

Ans:

Since $E\{X_{n+1} | F_n\} \geq X_n$ a.e., iff $E\{-X_{n+1} | F_n\} \leq -X_n$ a.e. for all $n \in N$.

$\Rightarrow X = \{X_n, F_n, n \in N\}$ is an uncertain sub-martingale iff $-X = \{-X_n, F_n, n \in N\}$ is an uncertain super-martingale.

Example (4-6)

If $X = \{X_n, F_n, n \in N\}$ is an uncertain sub-martingale and K a constant, then $\max\{X, k\} = \{\max\{X_n, K\}, F_n, n \in N\}$ is an uncertain sub-martingale.

Ans:

$$\max\{X_{n+1}, K\} \geq X_{n+1} \Rightarrow E\{\max\{X_{n+1}, K\} | F_n\} \geq E\{X_{n+1} | F_n\}$$

$$\text{Since } E\{X_{n+1} | F_n\} \geq X_n \Rightarrow E\{\max\{X_{n+1}, K\} | F_n\} \geq X_n$$

And similarly

$$E\{\max\{X_{n+1}, K\} | F_n\} \geq K \Rightarrow E\{\max\{X_{n+1}, K\} | F_n\} \geq \max\{X_n, K\}$$

$$\Rightarrow \max\{X, K\} = \{\max\{X_n, K\}, F_n, n \in N\}$$

is an uncertain sub-martingale.

Example (4-7)

If $X = \{X_n, F_n, n \in N\}$ is an uncertain super-martingale and K a constant, then $\min\{X, k\} = \{\min\{X_n, K\}, F_n, n \in N\}$ is an uncertain super-martingale.

Ans:

By the same way we get the answer.

Theorem (4-8)

Let $X = \{X_n, F_n, n \in N\}$ is an uncertain martingale and $f : R \rightarrow R$ a convex function, such that $f(X_n)$ is integrable for all n , then $f(X) = \{f(X_n), F_n, n \in N\}$ is an uncertain sub-martingale.

Proof:

By Jensen's inequality for uncertain conditional expectations we have,

$E\{f(X_{n+1} | F_n)\} \geq f\{E(X_{n+1} | F_n)\}$. Since $X = \{X_n, F_n, n \in N\}$ is an uncertain martingale, it follows that $E\{X_{n+1} | F_n\} = X_n$

then $E\{f(X_{n+1} | F_n)\} \geq f\{E(X_{n+1} | F_n)\} = f(X_n)$

thus $f(X) = \{f(X_n), F_n, n \in N\}$ is an uncertain sub-martingale.

Theorem (4-9)

Let $X = \{X_n, F_n, n \in N\}$ is an uncertain sub- martingale and $f : R \rightarrow R$ a convex increasing function, such that $f(X_n)$ is integrable for all n , then $f(X) = \{f(X_n), F_n, n \in N\}$ is an uncertain sub-martingale.

Proof:

By Jensen's inequality for uncertain conditional expectations we have,

$$E\{f(X_{n+1} | F_n)\} \geq f\{E(X_{n+1} | F_n)\}$$

Since $X = \{X_n, F_n, n \in N\}$ is an uncertain

sub- martingale, it follows that $E\{X_{n+1} | F_n\} \geq X_n$ a.e.,

since is increasing function, then $f\{E(X_{n+1} | F_n)\} \geq f(X_n)$ and then

$$E\{f(X_{n+1} | F_n)\} \geq f(X_n)$$

Thus $f(X) = \{f(X_n), F_n, n \in N\}$ is an uncertain sub-martingale.

Theorem (4-10) (Doop Decomposition)

Let $X = \{X_n, F_n, n \in N\}$ is an uncertain sub- martingale with respect to the filtration $\{F_n, n \in N\}$. Then there exists an uncertain martingale $M = \{M_n, F_n, n \in N\}$ and an uncertain process $A = \{A_n, n \in N\}$ such that

- 1- M is an uncertain martingale relative to $\{F_n, n \in N\}$.
- 2- A is an increasing uncertain process: $A_n \leq A_{n+1}$ a.e.
- 3- A_n is F_{n-1} -measurable for all $n \in N$.
- 4- $X_n = M_n + A_n$.

Proof:

1- Set $A_0 = 0$, and $A_n = A_{n-1} - (X_{n-1} - E(X_n | F_{n-1}))$ for all $n \geq 0$.

$\Rightarrow A_n = \sum_{k=0}^{n-1} (E(X_{k+1} | F_n) - X_k)$ for all $n \geq 0$ and $A_{n+1} - A_n = E(X_{n+1} | F_n) - X_n$ for all $n \geq 1$.

Since $X = \{X_n, F_n, n \in N\}$ is an uncertain sub- martingale

$$A_{n+1} - A_n = E(X_{n+1} | F_n) - X_n \Rightarrow E\{X_{n+1} | F_n\} \geq X_n$$

$$\Rightarrow E\{X_{n+1} | F_n\} - X_n \geq 0$$

$$\Rightarrow A_{n+1} - A_n \geq 0$$

$\Rightarrow \{A_n\}$ is increasing sequence of uncertain variables.

Take, $M_n = X_n - A_n$ for all n

$$\Rightarrow M_{n+1} - M_n = (X_{n+1} - A_{n+1}) - (X_n - A_n)$$

$$= X_{n+1} - X_n - (A_{n+1} - A_n)$$

$$= X_{n+1} - E(X_{n+1} | F_n)$$

$$\Rightarrow E(M_{n+1} - M_n | F_n) = E(X_{n+1} | F_n) - E(E(X_{n+1} | F_n) | F_n)$$

$$\Rightarrow E(M_{n+1} | F_n) - E(M_n | F_n) = E(X_{n+1} | F_n) - E(X_{n+1} | F_n)$$

$$\Rightarrow E(M_{n+1} | F_n) - M_n = 0$$

$$\Rightarrow E(M_{n+1} | F_n) = \mu_n$$

$\Rightarrow M = \{M_n, F_n, n \in N\}$ is an uncertain martingale.

2- Set $A_0 = 0$, and $A_n = A_{n-1} - (X_{n-1} - E(X_n | F_{n-1}))$ for all $n \geq 0$.

$\Rightarrow A_n = \sum_{k=0}^{n-1} (E(X_{k+1} | F_n) - X_k)$ for all $n \geq 0$ and $A_{n+1} - A_n = E(X_{n+1} | F_n) - X_n$ for all $n \geq 1$.

Since $X = \{X_n, F_n, n \in N\}$ is an uncertain sub- martingale

$$A_{n+1} - A_n = E(X_{n+1} | F_n) - X_n \Rightarrow E\{X_{n+1} | F_n\} \geq X_n$$

$$\Rightarrow E\{X_{n+1} | F_n\} - X_n \geq 0$$

$$\Rightarrow A_{n+1} - A_n \geq 0$$

$\Rightarrow \{A_n\}$ is increasing sequence of uncertain variables.

3- since A_n is F_n -measurable and $F_n \subseteq F_{n-1}$, then A_n is F_{n-1} -measurable for all $n \in N$.

4- Let $X = \{X_n, F_n, n \in N\}$ is an uncertain sub- martingale

Set $A_0 = 0$, and $A_n = A_{n-1} - (X_{n-1} - E(X_n | F_{n-1}))$ for all $n \geq 0$.

$\Rightarrow \{A_n\}$ is increasing sequence of uncertain variables.

Take, $M_n = X_n - A_n$ for all n

$\Rightarrow M = \{M_n, F_n, n \in N\}$ is an uncertain martingale.

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