Some New Properties of Convexly Compact Sets

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Abstract In this paper we study some new properties of convexly compact sets.

Key words compact sets and convexly compact sets.

1.Introduction. The genesis of the notation of compactness is connected with the Borel theorem (proved in 1984) stating that every countable open cover of a closed interval admits a finite subcover, and with the Lebesgue observation that the same holds for every open cover of a closed interval (in [1903] Borel generalization this result, in Lebesgue's seting, to all bounded closed subsets of Euclidean spaces) [2]. In 2010 Zitkovic [5] introduced the concept of convexly compact sets. A collection Ω of sets is said to have the finite intersection property (FIP) [2,4] if the intersection of each finite subcollection of Ω is non-empty. A subset C of a topological space X is said to be compact if every open cover of C admits a finite subcover,or equivalently, if and only if every collection of closed subsets of C with the finite intersection property admits non-empty intersection[2,4].A function from a topological space into topological space is continuous if and only if the inverse image of every open (closed) set is open (closed) set . A function from a topological space into topological space is said to be closed if the image of every closed set is closed set[2,4].

Let Λ be a non-empty set.The set $Fin(\Lambda)$ consisting of all non-empty finite subsets of A carries a natural structure of a partially ordered set when ordered by inclusion. Moreover, it is a directed set, since $D_1, D_2 \subseteq D_1 \cup D_2$ for any $D_1, D_2 \in Fin(\Lambda)$ [5].

Definition 1.1 [3] A topology τ on a vector space over afield Φ is called avector topology if the map $+ : X \times X \to X$ and $\bullet : \Phi \times X \to X$ are continuous.

A vector space endowed with a vector topology is called a topological vector space.

Definition 1.2 [1,3] A subset C of a vector space X is said to be convex set if $\lambda x + (1-\lambda) y \in C$ for every $x, y \in C$ and $0 \le \lambda \le 1$.

Remark 1.3 [3] let X be a vector space, then

(i) the empty set and the singleton set are convex set.

(ii) every intersection of convex sets is convex set .

(iii) The closure of every convex set is convex set.

Theorem 1.4 [1,3]

(i) The image of every convex set under a linear map is convex.

(ii) The inverse image of convex set under linear map is convex.

Definition 1.5 [2,4] Let X be any non-empty set. A filter on X is a non-empty collection $\mathscr{F}$ of a subsets of X satisfying the following axioms.

[F1] $\phi \notin \mathscr{F}$.

[F2] If $F \in \mathscr{F}$ and $H \supset F$, then $H \in \mathscr{F}$.

[F3] If $F \in \mathscr{F}$ and $H \in \mathscr{F}$, then $F \cap H \in \mathscr{F}$.

Remark 1.6 [2,4] Every filter on a non-empty set X admits the finite intersection property.

Definition 1.7 [2,4] Let (X,τ) be a topological space and let $\mathscr{F}$ be a filter on X. The point $x \in X$ is said to be a cluster point of $\mathscr{F}$ if and only if $x \in \overline{F}$ for all $F \in \mathscr{F}$.

2. Convexly compact sets

In this section we shall study some new properties of convexly compact sets.

Definition 2.1 [5] A convex subset C of a topological vector space X is said to be convexly compact if for any non-empty set A and any family $\{F_\alpha : \alpha \in A\}$ of closed and convex subsets of C, the condition

$$\forall\ D \in Fin(A),\ \bigcap_{\alpha \in D} F_\alpha \neq \phi \qquad (2.1)$$

implies

$$\bigcap_{\alpha \in A} F_\alpha \neq \phi, \qquad (2.2)$$

Without the additional restriction that the sets $\{F_\alpha : \alpha \in A\}$ be convex, Definition 2.1—postulating the finite-intersection property for families of closed and convex sets would be equivalent to the classical definition of compactness. It is, therefore, immediately clear that any convex and compact subset of a topological vector space is convexly compact [5]. The converse however is not true (see [5] Example 2.2).

Definition 2.2. A topological vector space X is said to be convexly compact if for any non-empty set A and any family $\{F_\alpha\}_{\alpha\in A}$ of closed and convex subsets of X,

the condition

$$\forall\ D \in Fin(A),\ \bigcap_{\alpha\in D} F_\alpha \neq \phi \qquad (2.1)^*$$

implies

$$\bigcap_{\alpha\in A} F_\alpha \neq \phi, \qquad (2.2)^*$$

Definition 2.3 A subset of a vector space X is said to be co-convex if its complement is convex.

Theorem 2.4 If A is co-convex subset of a vector space X, then σA is co-convex for every $\sigma \in \Phi/\{0\}$.

Proof. we need to prove that if X/A is convex set, then $X/\sigma A$ is convex set. Let $x, y \in X/\sigma A$ and $0 \le \lambda \le 1$. Then

$$x, y \in X \ \text{ and } \ x, y \notin \sigma A$$

$$\Rightarrow\ x, y \in X \ \text{ and } \ \sigma^{-1}x, \sigma^{-1}y \notin A.$$

$\Rightarrow$ $\sigma^{-1}x, \sigma^{-1}y \in X$ (since X is a vector space and $\sigma^{-1} \in \Phi$) and $\sigma^{-1}x, \sigma^{-1}y \notin A$.Therefore $\sigma^{-1}x, \sigma^{-1}y \in X/A$.Since X/A is convex set then

$$\lambda(\sigma^{-1}x) + (1-\lambda)(\sigma^{-1}y) \in X/A$$

$$\Rightarrow \sigma^{-1}(\lambda x) + \sigma^{-1}((1-\lambda)y) \in X/A$$

$$\Rightarrow \sigma^{-1}(\lambda x+(1-\lambda)y) \in X / A$$

$$\Rightarrow \sigma^{-1}(\lambda x+(1-\lambda)y) \in X \text{ and } \sigma^{-1}(\lambda x+(1-\lambda)y) \notin A$$

$$\Rightarrow \lambda x+(1-\lambda)y \in X \text{ and } \lambda x+(1-\lambda)y \notin \sigma A$$

$$\Rightarrow \lambda x+(1-\lambda)y \in X / \sigma A .$$

Then $X / \sigma A$ is convex and this complete the proof.

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Remark 2.5 The above theorem dose not still true if $\sigma = 0$. For example, consider the real line and let A be any co-convex subset of the real line. Since $\sigma A = \{0\}$ then $(\sigma A)^c = (-\infty,0) \cup (0,\infty)$ is not convex . Indeed the line segment that joint -1 and 1(for example) dose not lie in $(\sigma A)^c$.

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Theorem 2.6 The arbitrary union of a collection of a co-convex set is co-convex.

Proof. Suppose that $\{A_\sigma : \sigma \in \Lambda\}$ be arbitrary collection of a co-convex subsets of a vector space X. Then The collection $\{A_\sigma{}^c : \sigma \in \Lambda\}$ is convex for every $\sigma \in \Lambda$. By Remark 1.3 (ii) we get $\bigcap_{\sigma \in \Lambda} A_\sigma^c$ is convex set. By De-Morgan Law we have

$$\bigcap_{\sigma \in \Lambda} A_\sigma^c = (\bigcup_{\sigma \in \Lambda} A_\sigma)^c .$$

Therefore $(\bigcup_{\sigma \in \Lambda} A_\sigma)^c$ is convex i.e. $\bigcup_{\sigma \in \Lambda} A_\sigma$ is co-convex and the proof is complete. ////

Theorem 2.7 A topological vector space X is convexly compact if and only if every co-convex open cover of X admits a finite co-convex open cover.

Proof. Suppose that X is convexly compact and let $=\{G_\alpha : \alpha \in \Lambda\}$ be a co-convex open cover of X, so that

$$X = \bigcup_{\alpha\in\Lambda} G_\alpha .$$

Then

$$\phi = \bigcap_{\alpha\in\Lambda} G_\alpha^c$$

Thus $\{G_\alpha^c : \alpha \in \Lambda\}$ be a collection of convex closed sets with empty intersection so by hypothesis there exists $D \in Fin(\Lambda)$ such that $\bigcap_{\alpha\in D} G_\alpha^c = \phi$.Thus $(\bigcup_{\alpha\in D} G_\alpha)^c = \phi \Rightarrow \bigcup_{\alpha\in D} G_\alpha = X$.

Conversely, suppose that for every co-convex open cover of X admits a finite co-convex open cover, and let $\{F_\alpha : \alpha \in \Lambda\}$ be a collection of convex closed subset of X such that condition (2.1)* holds, i.e.

$$\forall \;\; D \in Fin(A),\; \bigcap_{\alpha\in D} F_\alpha \neq \phi .$$

Suppose if possible, $\bigcap_{\alpha\in A} F_\alpha = \phi$. Then

$$X = (\bigcap_{\alpha\in\Lambda} F_\alpha)^c = \bigcup_{\alpha\in\Lambda} F_\alpha^c .$$

This means that $\{F_\alpha^c : \alpha \in \Lambda\}$ is a co-convex open cover of X. By hypothesis, there exists $D \in Fin(\Lambda)$ such that

$$\bigcup_{\alpha\in D} F_\alpha^c = X \Rightarrow (\bigcap_{\alpha\in D} F_\alpha)^c = X \Rightarrow \bigcap_{\alpha\in D} F_\alpha = \phi$$

But this contradicts the condition (2.1)*. Hence we must have $\bigcap_{\alpha\in\Lambda} F_\alpha \neq \phi$.

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Theorem 2.8 A topological vector space X is convexly compact if and only if any basic co-convex open cover of X admits a finite subcover.

Proof. Let X be a convexly compact space. Then by Theorem 2.7 every co-convex open cover of X admits a finite subcover. In particular, every basic co-convex open cover of X admit a finite subcover.

Conversely, suppose that every basic co-convex open cover of X admits a finite subcover. Let $\{C_\lambda : \lambda \in \Lambda\}$ be any co-convex open cover of X. If $\{B_\gamma : \gamma \in \Gamma\}$ be any co-convex open base for X, then each C_λ is a union of some members of B. That is there exists $\Gamma_\circ \subseteq \Gamma$ such that $C_\lambda = \bigcup_{\gamma \in \Gamma_\circ} B_\gamma$ for every $\lambda \in \Lambda$. (Note that by Theorem 2.6 we have the union of co-convex sets is also co-convex set). And the totality of all such members of B is evidently a basic co-convex open cover of X. By hypothesis this collection of members of B admits a finite subcover, say, $\{B_{\gamma_i} : i = 1,2,\ldots,n\}$. For each B_{γ_i} in this finite subcover, we can select a C_{λ_i} from Ω such that $B_{\gamma_i} \subseteq C_{\lambda_i}$. It follows that the finite subcollection $\{C_{\lambda_i} : i = 1,2,\ldots,n\}$ each arise in this way is a subcover of Ω. Hence X is convexly compact.

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Theorem 2.9 If X be a convexly compact topological vector space. Then every convex filter on X admit a cluster point.

Proof. Assume X be a convexly compact topological vector space and let Φ be any convex filter on X. Then Φ be a collection of convex subsets of X and by Remark 1.6 it has FIP. Since by Remark 1.3(iii) the closure of convex set is also convex and the closure of any set is closed, then $\{\overline{F} : F \in \Phi\}$ is a collection of convex closed subsets of X. Since Φ has FIP property, then so is $\{\overline{F} : F \in \Phi\}$. By hypothesis $\bigcap_{F \in \Phi} \{\overline{F}\} \neq \phi$, hence there exists at least one point $p \in \bigcap_{F \in \Phi} \{\overline{F}\}$. This implies that $p \in \overline{F}$ for every $F \in \Phi$. Hence p is a cluster point of Φ by Definition 1.7. ////

Theorem 2.10 The image of convexly compact set under a continuous bijective linear map is convexly compact.

Proof. Suppose that C be a convexly compact in a topological vector space X and let f be continuous linear map from X onto another topological vector space Y. Since C is convex in X and f is linear then by Theorem 1.4 (i) $f(C)$ is convex set in Y. To show that $f(C)$ is a convexly compact. Let $\{F_\alpha : \alpha \in \Lambda\}$ be a collection of closed and convex subsets of $f(C)$ with the property that

$$\bigcap_{\alpha \in D} F_\alpha \neq \phi, \ \forall \ D \in Fin(\Lambda).$$

Since f is continuous,and F_α is closed $\forall \alpha \in \Lambda$ then $\{f^{-1}(F_\alpha) : \alpha \in \Lambda\}$ is closed. Since f is linear,and F_α is convex $\forall \alpha \in \Lambda$ then by theorem 1.4(ii) $\{f^{-1}(F_\alpha) : \alpha \in \Lambda\}$ is convex. Since $F_\alpha \subset F(C)$ for every α, then $f^{-1}(F_\alpha) \subset C$ for every α .Hence $\{f^{-1}(F_\alpha) : \alpha \in \Lambda\}$ is collection of convex and closed subsets of C since $\bigcap_{\alpha \in D} F_\alpha \neq \phi$ and f is onto then $F^{-1}(\bigcap_{\alpha \in D} F_\alpha) \neq \phi \quad \forall \ \alpha \in D$.
Since f is bijective then $F^{-1}(\bigcap_{\alpha \in D} F_\alpha) = \bigcap_{\alpha \in D} f^{-1}(F_\alpha)$ and hence $\bigcap_{\alpha \in D} f^{-1}(F_\alpha) \neq \phi \ \forall \ D \in Fin(A)$ .Since C is convexly compact, then $\bigcap_{\alpha \in A} f^{-1}(F_\alpha) \neq \phi$ then $F(\bigcap_{\alpha \in A} f^{-1}(F_\alpha)) \neq \phi$ since $F(\bigcap_{\alpha \in A} f^{-1}(F_\alpha)) = \bigcap_{\alpha \in A} f(f^{-1}(F_\alpha)) = \bigcap_{\alpha \in A} F_\alpha$ [since f is bijective]. Thus $\bigcap_{\alpha \in A} F_\alpha \neq \phi$. $\Rightarrow f(C)$ is convexly compact.This complete the proof. ////

Theorem 2.11 The inverse image of every convexly compact set under closed bijective linear map is convexly compact.

Proof. Suppose that X and Y are two topological vector spaces and $f: X \to Y$ be a closed bijective linear map. Let C be a convexly compact set in Y. To show that $f^{-1}(C)$ is convexly compact in X. Since C is convex and f is linear then by Theorem 1.4 (ii) $f^{-1}(C)$ is convex set. Now,let $\{F_\alpha : \alpha \in \Lambda\}$ be a collection of convex closed subsets of $f^{-1}(C)$ with the property that $\bigcap_{\alpha \in D} F_\alpha \neq \phi$, for every $D \in Fin(\Lambda)$. Since $F_\alpha \subseteq f^{-1}(C)$ and f is onto, then $f(F_\alpha) \subset f(f^{-1}(C)) = C$. Since F_α is closed $\forall \alpha \in \Lambda$ and f is closed function , then $f(F_\alpha)$ is closed subset of C. Since F_α is convex$\forall \alpha \in \Lambda$ and f is linear then $f(F_\alpha)$ is convex set. Thus $\{f(F_\alpha) : \alpha \in \Lambda\}$ is the a collection of closed convex subsets of C. Since $\bigcap_{\alpha \in D} F_\alpha \neq \phi, \forall\ D \in Fin(\Lambda)$ then $f(\bigcap_{\alpha \in D} F_\alpha) \neq \phi$. But f is onto, thus $f(\bigcap_{\alpha \in D} F_\alpha) \subset \bigcap_{\alpha \in D} f(F_\alpha)$, therefore $\bigcap_{\alpha \in D} f(F_\alpha) \neq \phi$ i.e. $\{f(F_\alpha) : \alpha \in \Lambda\}$ admits finite intersection property. By convexly compactness of C, we have $\bigcap_{\alpha \in \Lambda} f(F_\alpha) \neq \phi$. Since f is onto, then $f^{-1}(\bigcap_{\alpha \in \Lambda} f(F_\alpha)) \neq \phi$ since f is one-one then $f^{-1}(\bigcap_{\alpha \in \Lambda} f(F_\alpha)) = \bigcap_{\alpha \in \Lambda} f^{-1}(f(F_\alpha)) = \bigcap_{\alpha \in \Lambda} F_\alpha$;hence $\bigcap_{\alpha \in \Lambda} F_\alpha \neq \phi$,i.e. $f^{-1}(C)$ is convexly compact, and this complete the proof.

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Theorem 2.12 If A is convexly compact subset of a topological vector space X. Then the set λA is convexly compact where $\lambda \in \Phi$.

Proof. Suppose that A is convexly compact subset of a topological vector space X and $\lambda \in \Phi$. Since the function $\bullet : \Phi \times X \to X$ which define by $(\lambda, x) = \lambda x$ is continuous by definition of a topological vector space, then by Theorem 2.10 the set λA is convexly compact and the proof is complete.

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References

[1] G. Zitkovic, " Convexly Compactness and its Applications" Math. Finan Econ 3(2010). 1-12.

[2] J.N.Sharma,"Topology" Published by Krishna Pracushna , Mandir , and printed at Mano, 1977.

[3] N.F. Al-Mayahi and A.H. Battor " *Introduction to Functional Analysis*" Iraq. Najef. 2005.

[4] R. Engelking " General Topology " polish. Scientific publishers. Warsawx 1989.

[5] W .Rudin ,"*Functional Analysis*", New York. 1973