Small Quasi-Dedekind Modules

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Abstract. Let R be a commutative ring with unity .A unitary R-module M is called a quasi-Dedekind module if $Hom(M/N, M) = 0$ for all nonzero submodules N of M. In this paper we introduce and study the concept of small quasi-Dedekind module as a generalization of quasi-Dedekind module . Where an R-module M is called small quasi-Dedekind if, for each nonzero homomorphisms f from M to M, implies $Kerf$ small in M ($Kerf \ll M$). And an R-submodule N of an R-module M is called a small submodule of M ($N \ll M$, for short) if , for all $K \leq M$ with $N+K = M$ implies $K = M$

Key Words : *quasi-Dedekind Modules ; small quasi-Dedekind Modules .*

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1. Introduction

Let R be a commutative ring with unity and M be a unitary R-module . Mijbass A.S in [8] introduced and studied the concept of quasi-Dedekind , where an R-module M is called quasi-Dedekind if $Hom(M/N, M) = 0$ for all nonzero submodules N of M . In this paper we introduce and study another generalization of the concept a quasi-Dedekind module namely " small quasi-Dedekind module " . Also in this paper, we investigate the basic properties and characterizations about this concept . At the start of this paper we give some of the basic properties and characterizations of small quasi-Dedekind modules . Recall that an R-module P is projective if and only if , for any two R-modules A ,B and for any epimorphism $f : A \longrightarrow B$ and for any homomorphism $g : P \longrightarrow B$, there exists a homomorphism $h : P \longrightarrow A$ such that foh = g [6 , p.117] . Among results we obtain in this paper , we prove that : Let M be an R-module such that M/U is projective for all $U \ll M$. If M is a small quasi-Dedekind R-module then M/N is a small quasi-Dedekind R-module for all $N \leq M$.

Recall that an R-module M is a quasi-Dedekind module if and only if for all $f \in End_R(M)$, $f \neq 0$ implies $Ker f = 0$, (see Th 1.5, P.26, 8) .

Now we shall give a generalization to quasi-Dedekind module namely " small quasi-Dedekind module " as follows .

Definition 1.1. An R-module M is called a small quasi-Dedekind module if, for all $f \in End_R(M)$, $f \neq 0$ implies $Ker f \ll M$ (i.e. $Ker f$ is a small submodule in M) .

Remarks and Examples 1.2.

1) It is clear that every quasi-Dedekind R-module is a small quasi-Dedekind R-module . But the converse is not true in general, for example: Z_4 as Z-module is small quasi-Dedekind but it is not quasi-Dedekind, also it is not essentially quasi-Dedekind .Where an R- module M is called essentially quasi-Dedekind if $Hom(M/N, M) = 0$ for all $N \leq_e M$ [4, def.1.2.1] .

2) Z_6 as Z-module is not small quasi-Dedekind , since there exists , $f : Z_6 \longrightarrow Z_6$ define by $f(\bar{x}) = 3\bar{x}$, $\bar{x} \in Z_6$. So $f \neq 0$, but $Ker f = \{\bar{x} \in Z_6 : f(\bar{x}) = \bar{0}\} = \{\bar{x} \in Z_6 : 3\bar{x} = \bar{0}\} = (\bar{2}) \not\ll Z_6$. However , Z_6 is an essentially quasi-Dedekind Z-module .

3) $Z \oplus Z$ is not a small quasi-Dedekind Z-module , since there exists $f : Z \oplus Z \longrightarrow Z \oplus Z$ such that $f(x, y) = (x, 0)$; $x, y \in Z$. So $f \neq 0$, but $Ker f = (0) \oplus Z \not\ll Z \oplus Z$.

4) If $M = 0$, it is clear that M is a small quasi-Dedekind module .

5) Every integral domain R is a small quasi-Dedekind R-module ,but the converse is not true in general , for example :
Z_4 as Z_4-module is small quasi-Dedekind , but it is not an integral domain .

6) If M is a semisimple R-module , then it is not necessarily small quasi- Dedekind, (see Rem.and.Ex 1.2(2)) .

7) Every semisimple small quasi-Dedekind R-module M is a quasi-Dedekind R-module .

Proof : Let $f \in End_R(M)$, $f \neq 0$. Since M is small quasi-Dedekind, then $Kerf \ll M$. But M is semisimple , so $Kerf = 0$ .Thus M is a quasi-Dedekind R-module . □

The following theorem is a characterization of small quasi-Dedekind modules .

Theorem 1.3. Let M be an R-module. Then M is small quasi-Dedekind if and only if $Hom(M/N, M) = 0$ for all $N \not\ll M$.

Proof :

$\Rightarrow$) Suppose that there exists $N \not\ll M$ such that $Hom(M/N, M) \neq 0$,then there exists $\phi : M/N \longrightarrow M$, $\phi \neq 0$. Hence $\phi o \pi \in End_R(M)$, where π is the canonical projection ,and $\phi o \pi \neq 0$ which implies $Ker(\phi o \pi) \ll M$, but $N \subseteq Ker(\phi o \pi)$, so $N \ll M$ which is a contradiction .

$\Leftarrow$) Suppose that there exists $f : M \longrightarrow M$, $f \neq 0$ such that $Kerf \not\ll M$, define $g : M/Kerf \longrightarrow M$ by $g\ (m + Kerf) = f\ (m)$, for all $m \in M$. So g is well-defined and $g \neq 0$. Hence $Hom(M/Kerf, M) \neq 0$ which is a contradiction . □

Proposition 1.4. Let M be an R-module and let $\overline{R} = R/J$, where J is an ideal of R such that $J \subseteq ann_R(M)$. Then M is a small quasi-Dedekind R-module if and only if M is a small quasi-Dedekind $\overline{R}$-module .

Proof :

$\Rightarrow$) We have $Hom_R(M/K, M) = Hom_{\overline{R}}(M/K, M)$, for all $K \leq M$,by [6 , p.51]. Thus, if M is a small quasi-Dedekind R-module , then $Hom_R(M/K, M) = 0$ for all $K \not\ll M$, so $Hom_{\overline{R}}(M/K, M) = 0$ for all $K \not\ll M$, thus M is a small quasi-Dedekind $\overline{R}$-module .

$\Leftarrow$) The proof of the converse is similarly . □

Proposition 1.5. Let M_1 , M_2 be R-modules such that $M_1 \cong M_2$. M_1 is a small quasi-Dedekind R-module if and only if M_2 is a small quasi-Dedekind R-module .

Proof : $\Rightarrow$) Let $f: M_2 \longrightarrow M_2$, $f \neq 0$. To prove $Kerf \ll M_2$. Since $M_1 \cong M_2$, there exists an isomorphism $g: M_1 \longrightarrow M_2$. Consider the following :
$M_1 \xrightarrow{g} M_2 \xrightarrow{f} M_2 \xrightarrow{g^{-1}} M_1$. Hence $h = g^{-1}ofog \in End_R(M_1)$, $h \neq 0$. So $Kerh \ll M_1$ (since M_1 is small quasi-Dedekind) , then $g(Kerh) \ll M_2$ by [6, lemma 5.1.3, p.108] . But we can show that $g(Kerh) = Kerf$ as follows : let $y \in g(Kerh)$, so $y = g(x)$, $x \in Kerh$. Hence $h(x) = 0$; that is $g^{-1}ofog(x) = 0$, then $g^{-1}of(y) = 0$, so $g^{-1}(f(y)) = 0$ and hence $f(y) = 0$, since g^{-1} is monomorphism , so that $y \in Kerf$, hence $g(Kerh) \subseteq Kerf$. Now , Let $y \in Kerf$, then $f(y) = 0,$ but $y \in M_2$, so there exists an $x \in M_1$ such that $y = g(x)$, since g is onto . Thus $f(g(x)) = 0$ and so $g^{-1}(f(g(x))) = 0$; that is $h(x) = 0$. Hence $x \in Kerh$. This implies $y = g(x) \in g(Kerh)$, thus $Kerf = g(Kerh) \ll M_2$, hence $Kerf \ll M_2$.

$\Leftarrow$) The proof of the converse is similarly . □

Remark 1.6. Let $N \leq M$, and $f \in End_R(M)$, $f \neq 0$. Note that if $f(N) \ll f(M)$, then it is not necessarily $N \ll M$. Consider the following example .

Example 1.7. Let $M = Z_6$ as Z-module , and let $N = (\bar{2}) \leq Z_6$. Let $f: Z_6 \longrightarrow Z_6$ define by $f(\bar{x}) = 3\bar{x}$, $\bar{x} \in Z_6$. So $f \neq 0$ and
$f(N) = f((\bar{2})) = \{\bar{0}\} \ll \{\bar{0}, \bar{3}\} = f(Z_6) = f(M)$, but $N = (\bar{2}) \not\ll Z_6 = M$.

In the following proposition we give a condition under which the remark (1.6) is true in general .

Proposition 1.8. Let M be a small quasi-Dedekind R-module and $f \in End_R(M)$, $f \neq 0$, $N \leq M$. If $f(N) \ll f(M)$ then $N \ll M$.

Proof : Let $B \leq M$ and $N + B = M$ then $f(N) + f(B) = f(M)$. But $f(N) \ll f(M)$ implies $f(B) = f(M)$. Now , we can show that $Kerf + B = M$. Let $m \in M$, hence $f(m) \in f(M) = f(B)$. So that there exists $b \in B$ such that $f(m) = f(b)$, hence $m - b \in Kerf$. It follows that $m = (m-b) + b$, thus $M \subseteq Kerf + B$. Thus $Kerf + B = M,$ but M is a small quasi-Dedekind R-module , so $Kerf \ll M$ which implies that $B = M$. Therefore $N \ll M$. □

Corollary 1.9. Let M be a small quasi-Dedekind R-module and $f \in End_R(M)$, f is surjective . Then $N \ll M$ if and only if $f(N) \ll M$.

Proof : $\Rightarrow$) It is clear by [6 , Lemma 5.1.3, p.108] .
$\Leftarrow$) It follows directly by (Prop 1.8) . □

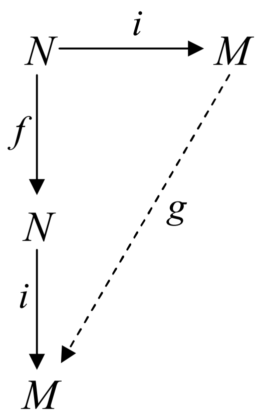
Proposition 1.10. Let M be a small quasi-Dedekind R-module , let $f \in End_R(M)$, $f \neq 0$, $N \leq M$. If $N \ll f(M)$ then $f^{-1}(N) \ll M$.

Proof : It is clear that $Kerf \subseteq f^{-1}(N)$. First we shall prove that $\frac{f^{-1}(N)}{Kerf} \ll \frac{M}{Kerf}$. Let $\frac{f^{-1}(N)}{Kerf} + \frac{L}{Kerf} = \frac{M}{Kerf}$, where $\frac{L}{Kerf} \leq \frac{M}{Kerf}$. Then $f^{-1}(N) + L = M$, hence $f(f^{-1}(N)) + f(L) = f(M)$ but $f(f^{-1}(N)) \subseteq N$, then $f(M) = f(f^{-1}(N)) + f(L) \subseteq N + f(L)$, also, we have $N \subseteq f(M)$ and $f(L) \subseteq f(M)$, so $N + f(L) \subseteq f(M)$ and thus $N + f(L) = f(M)$. Since $N \ll f(M)$, then $f(L) = f(M)$. We claim that $L = M$. Let $x \in M$, then $f(x) \in f(M) = f(L)$, hence $f(x) = f(l)$ for some $l \in L$. It follows that $x - l \in Kerf \subseteq L$ and hence $x \in L$, so $M \subseteq L$. Thus $M = L$ which implies $\frac{L}{Kerf} = \frac{M}{Kerf}$, so $\frac{f^{-1}(N)}{Kerf} \ll \frac{M}{Kerf}$. But $Kerf \ll M$, so by [1, Prop 1.1.2, p.10], $f^{-1}(N) \ll M$. □

Now we can give the following result .

Proposition 1.11. Let M be a small quasi-Dedekind and quasi-injective R-module , let $N \leq M$ such that for all $U \leq N$, $U \ll M$ implies $U \ll N$. Then N is a small quasi-Dedekind R-module .

Proof : Let $f : N \longrightarrow N$, $f \neq 0$. To prove that $Kerf \ll N$. Since M is a quasi-injective R-module , there exists $g : M \longrightarrow M$ such that $goi = iof$, where i is the inclusion mapping .



Then $g(N) = f(N) \neq 0$; that is $g \neq 0$. So that $Kerg \ll M$, since M is small quasi-Dedekind . But $Kerf \subseteq Kerg$, hence $Kerf \ll M$ .On the other hand $Kerf \leq N$, so by hypothesis $Kerf \ll N$. Thus N is a small quasi-Dedekind R-module . □

We are now in a position to recall the definition of coclosed submodule which was introduced by Golan [5] . Recall that an R-submodule N of M is coclosed in M, if whenever $N/K \ll M/K$ then $N = K$ for all submodules K of M contained in N.

And let U be a submodule of M, a submodule V of M is called a supplement (or addition complement) of U in M if V is a minimal element in the set of all submodules L of M with $U + L = M$ .V is called a supplement submodule of M if, V is a supplement of some submodule of M, [7] .

Corollary 1.12. Let M be a small quasi-Dedekind and quasi-injective R-module , let $N \leq M$. If N is a supplement (or coclosed) submodule , then N is a small quasi-Dedekind R-module .

Proof : By [1, Prop 1.2.6] , N is supplement then N is coclosed, and hence for all $U \leq N$, $U \ll M$ implies $U \ll N$. So the result follows by (Prop 1.11). □

An R-module M is called a quasi-injective R-module if for each monomorphism $f: N \longrightarrow M$, $N \leq M$ and any homomorphism $g: N \longrightarrow M$, there exists a homomorphism $h: M \longrightarrow M$ such that hof = g . A quasi-injective R-module $\overline{M}$ is called a quasi-injective hull (a quasi-injective envelope) of an R-module M if there is a monomorphism $f: M \longrightarrow \overline{M}$ such that $\operatorname{Im} f \leq_e \overline{M}$.

Corollary 1.13. Let M be an R-module such that $\overline{M}$ is a small quasi-Dedekind R-module, and for all $U \leq M$,$U \ll \overline{M}$ implies $U \ll M$. Then M is a small quasi-Dedekind R-module .

proof : Since $\overline{M}$ is a small quasi-Dedekind and quasi-injective R-module , so by (Prop 1.11) , M is a small quasi-Dedekind R-module . □

Proposition 1.14. Let M be a small quasi-Dedekind R-module .Then for all $N \not\ll M$ $ann_R(N) = ann_R(M)$.

Proof : Since M is a small quasi-Dedekind R-module, so by (Th 1.3), $Hom(M/N, M) = 0$ for all $N \not\ll M$ which implies N is a quasi-invertible submodule for all $N \not\ll M$. Thus by (8, Prop1.4, P.7) , for all $N \not\ll M$ $ann_R(N) = ann_R(M)$. □

Remark (1.15) Let $N \leq M$. If M/N is a small quasi-Dedekind R-module , then it is not necessarily that M is a small quasi-Dedekind R-module, for example : If $M = Z_6$ as Z-module , and let $N = (\bar{2}) \leq M = Z_6$, then $Z_6/(\bar{2}) \cong Z_2$ which is a small quasi-Dedekind Z-module . But $M = Z_6$ as Z-module is not small quasi-Dedekind .

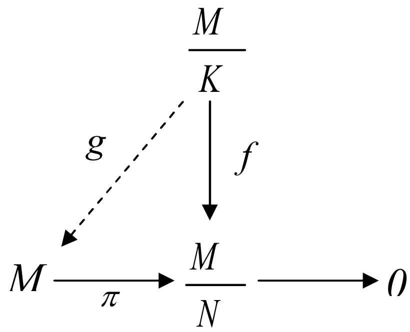
Remark 1.16. If M is a small quasi-Dedekind R-module , $N \leq M$. Then it is not necessarily that M/N is a small quasi-Dedekind R-module . Consider the following example .

Example 1.17. The Z-module $M=Z$ is small quasi-Dedekind .Let $N = 6Z \leq Z$, then $M/N = Z/6Z \cong Z_6$ is not a small quasi-Dedekind Z-module .

The following result shows that under certain condition , the module M/N is small quasi-Dedekind .

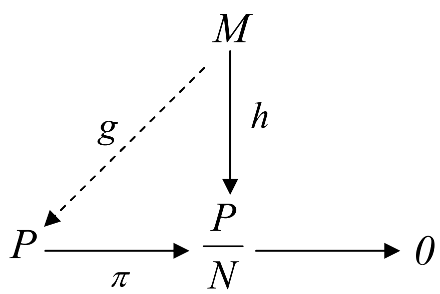
Proposition 1.18. Let M be an R-module such that M/U is projective for all $U \not\ll M$. If M is a small quasi-Dedekind R-module , then M/N is a small quasi-Dedekind R-module for all $N \leq M$.

Proof : Let $K/N \not\ll M/N$, so by [1, Prop 1.1.2 , p.10] , $K \not\ll M$.
Suppose that $Hom(\frac{M/N}{K/N}, \frac{M}{N}) \neq 0$, but $Hom(\frac{M/N}{K/N}, \frac{M}{N}) \cong Hom(\frac{M}{K}, \frac{M}{N})$, so there exists $f : M/K \longrightarrow M/N$, $f \neq 0$. Since M/K is projective , then there exists $g : M/K \longrightarrow M$ such that $\pi o g = f$, where π is the canonical projection .



Hence $\pi o g(M/K) = f(M/K) \neq 0$, so $g \neq 0$, but $g \in Hom(M/K, M)$, $K \not\ll M$. Thus $Hom(M/K, M) \neq 0$, $K \not\ll M$; that is M is not small quasi-Dedekind , which is a contradiction . Thus M/N is a small quasi-Dedekind R-module . □

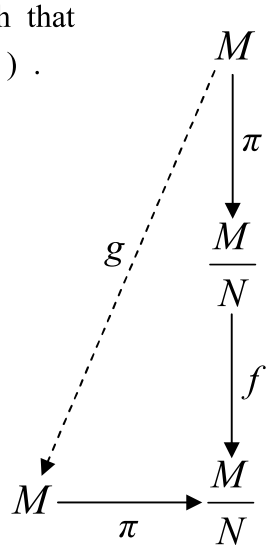
Let M and P be modules , then M is called P-projective in case for each $N \leq P$ and every homomorphism $h : M \longrightarrow P/N$, there exists a homomorphism $g : M \longrightarrow P$ such that $\pi o g = h$.(where π is the natural epimorphism) ; that is the following diagram is commutative , [2] .



An R-module M is called quasi-projective if, M is M- projective ; that is for each $N \leq M$ and every homomorphism $h: M \longrightarrow M/N$, there exists a homomorphism $g: M \longrightarrow M$ such that $\pi o g = h$. (where π is the natural epimorphism) , [9] .

Theorem 1.19. Let M be a quasi-projective R-module, let $N \leq M$ such that $g^{-1}(N) \ll M$ for each $g \in End_R(M)$, then M/N is a small quasi-Dedekind R-module .

Proof : Let $f: M/N \longrightarrow M/N$ such that $f \neq 0$. Since M is quasi-projective , there exists a homomorphism $g: M \longrightarrow M$ such that $\pi o g = f o \pi$ (where π is the canonical projection) .



Let $Kerf = L/N = \{x+N : f(x+N) = N\} = \{x+N : fo\pi(x) = N\} =$
$\{x+N : \pi og(x) = N\} = \{x+N : g(x)+N = N\} = \{x+N : g(x) \in N\} =$
$\{x+N : x \in g^{-1}(N)\}$. Thus $Kerf = g^{-1}(N)/N$, but $g^{-1}(N) \ll M$,
so by [6, Lemma 5.1.3 , p.108] , $g^{-1}(N)/N \ll M/N$; that is $Kerf \ll M/N$. □

Corollary 1.20. Let M be a quasi-projective R-module such that for each $N \leq M$, $N \ll h(M)$ for all $h \in End_R(M)$. Then M is a small quasi-Dedekind R-module if and only if M/N is a small quasi-Dedekind R-module .

Proof : $\Leftarrow$) It is clear by taking $N = (0)$.

$\Rightarrow$) By (prop 1.10) , $N \ll h(M)$ implies $h^{-1}(N) \ll M$. Hence the result follows by the previous theorem . □

Recall that an R-submodule N of an R-module M is invariant if $f(N) \subseteq N$ for each $f \in End_R(M)$. Some authors called an invariant submodule, fully invariant submodule, by [3] .

Theorem 1.21. Let M be an R-module . Then M is small quasi-Dedekind if and only if there exists $N \ll M, N$ is fully invariant such that for each $f \in End_R(M)$, $f \neq 0$, $f(M) \not\subset N$ and M/N is small quasi-Dedekind .

Proof : $\Rightarrow$) Choose $N = (0)$ implies $N \ll M$ and N is fully invariant and for all $f \in End_R(M)$, $f \neq 0$,hence $f(M) \not\subset (0) = N$ and $M/N = M/(0) \cong M$ is small quasi-Dedekind .

$\Leftarrow$) If $N = 0$, then M is small quasi-Dedekind .Suppose that $N \neq (0)$, $N \ll M$. Let $f \in End_R(M)$, $f \neq 0$. To prove $Kerf \ll M$. Define $g : M/N \longrightarrow M/N$ by $g(m + N) = f(m) + N$ for all $m \in M$. g is well-defined , since if $m_1 + N = m_2 + N$ where $m_1, m_2 \in M$, then $m_1 - m_2 \in N$ and $f(m_1 - m_2) \in f(N) \subseteq N$, since N is fully invariant .This implies $f(m_1) - f(m_2) \in N$; that is $f(m_1) + N = f(m_2) + N$, thus $g(m_1 + N) = g(m_2 + N)$. $g \neq 0$, because if $g = 0$ then $g(M/N) = N = 0_{M/N}$. Hence $f(M) + N = N$, it follows that $f(M) \subseteq N$ which is a contradiction with the hypothesis . Thus $Kerg \ll M/N$, since M/N is a small quasi-Dedekind R-module . Let $Kerg = L/N \ll M/N$, but $N \ll M$, so by [1, Prop 1.1.2, p.10] , $L \ll M$. On the other hand it is easy to see that $Kerf \subseteq L$, so $Kerf \ll M$,thus M is a small quasi-Dedekind R-module . □

An R-module M is called multiplication if for each submodule N of M, $N = IM$ for some ideal I of R. Equivalently , M is a multiplication R-module if , for each submodule N of M , $N = [N : M].M$, where $[N : M] = \{r \in R : rM \subseteq N\}$.

Corollary 1.22. Let M be a multiplication R-module .Then M is small quasi-Dedekind if and only if there exists $N \ll M$ such that for all $f \in End_R(M)$, $f \neq 0$, $f(M) \not\subset N$ and M/N is small quasi-Dedekind .

Proof : Since M is a multiplication R-module , every proper submodule of M is fully invariant . Thus the result is obtained by (Th 1.21) . □

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