One Significance Test Estimator for the Shape Parameter of Generalized Rayleigh Distribution

Abbas Najim Salman

Department of Mathematics-Ibn-Al-Haitham College of Education - University of Baghdad

Abstract:

This paper is concerned with preliminary test single stage Shrinkage estimator for the unknown shape parameter (α) of two parameter generalized Rayleigh (GR) distribution with known scale parameter (λ), when a prior knowledge about the shape parameter (α) is available as initial value (α_0), using shrinkage weight factor $\psi(\cdot)$ and pretest region R. This prior knowledge about unknown parameter may be obtaind from past experiences or from a quaintance with similar situation.

Expression for the Bias, Mean squared error [MSE] and Relative Efficiency [R.Eff($\cdot$)] for the proposed estimator are derived. Numerical results and conclusions of mentioned expressions are carried out to assess the effects of the considered estimator and to illustrate these results. Tables of these numerical results are demonstrated. Comparisons between introduced estimator with respect to classical estimator $\hat{\alpha}$ and with some existing studies in the sense of mean suqared error or Relative Efficiency are performed.

1. Introduction

Different forms of cumulative distribution functions for modeling lifetime data are introduced by [1]. Among those distributions, Burr Type X and Burr Type XII are the most popular ones. Several authors consider different aspects of Burr Type X and Bure Type XII distribution, see for example [2] and [3].

In [4], they considered different estimators and studied how the estimator of different unknown parameter behave for different sample size and different parameter value, they showed that the two parameters generalized Rayleigh distribution (Burr Type X) can be used quite effectively in modeling strength data and also modeling general lifetime data. It has application in the field of acousties, spatial statistics and random walks; [5].

In this paper we also prefer to call the two parameters Burr Type X distribution as the two parameter generalized Rayleigh (GR) distribution.

For $\alpha > 0$ and $\lambda > 0$, the two parameters GR distribution has the following distribution function

$$F(x;\alpha,\lambda) = (1-e^{-(\lambda x)^2})^{\alpha} \quad \text{for } x>0,\ \alpha>0,\ \lambda>0 \qquad \ldots(1)$$

Therefore, GR distribution has the density function

$$f(x;\alpha,\lambda) = \begin{cases} 2\alpha\lambda^2 x\, e^{-(\lambda x)^2}(1-e^{-(\lambda x)^2})^{\alpha-1} & \text{for } x>0,\alpha>0,\lambda>0 \\ 0 & \text{o.w.} \end{cases} \qquad \ldots(2)$$

Here, α and λ are the shape and scale parameters respectively.

In conventional notation, we write x~GR(α,λ).

In this paper we introduce the problem of estimating the shape parameter (α) of two parameter GR distribution with known scale parameter (λ) when some prior information (α_0) regarding the actual value (α) available. More specifically we assume that the prior information regarding due the following reasons; [6].

1. We belive that (α_0) is closer to the true value of α, or
2. We fear that (α_0) may be near the true value of (α), i.e.; Something bad happens if (α) approximately equal to (α_0) and we do not know about it.

In such a situation it is natural to start with classical estimator $\hat{\alpha}$ (MLE for example) of α and modify it by moving it closure to (α_0) using shrinkage weight factor $\psi_1(\cdot)$; $0 \le \psi_1(\cdot) \le 1$, so that the resulting linear combination estimator $\tilde{\alpha}$ though perhaps biased has a smaller mean suqared error [MSE] than that $\hat{\alpha}$ in some interval around (α_0), i.e.

$$\tilde{\alpha} = \psi_1(\cdot)\hat{\alpha} + (1-\psi_1(\cdot))\alpha_0 \qquad \ldots(3)$$

Preliminary test single stage shrinkage estimator (PTSSE) is introduce in this paper which is a testimator of level of significance (Δ) for test the hypothesis H_0: $\alpha = \alpha_0$ vs H_A: $\alpha \neq \alpha_0$ using test statistics $T(\hat{\alpha}/\alpha_0)$.

If H_0 accepted, the shrinkage estimator $\tilde{\alpha}$ which is defined in (3) will be utilize to estimate α.

However if H_0 rejected, we consider shrinkage estimator via another shrinkage weight factor $\psi_2(\cdot)$; $0 \le \psi_2(\cdot) \le 1$, so the shrinkage estimator will be:

$$\tilde{\alpha} = \psi_2(\cdot)\hat{\alpha} + (1-\psi_2(\cdot))\alpha_0 \qquad \ldots(4)$$

Thus, the general form of preliminary test single stage Shrinkage estimator (PTSSE) for the shape parameter (α) will be:

$$\tilde{\alpha} = \begin{cases} \psi_1(\hat{\alpha})\hat{\alpha} + (1-\psi_1(\hat{\alpha}))\alpha_0 & ,\text{if } \hat{\alpha} \in R \\ \psi_2(\hat{\alpha})\hat{\alpha} + (1-\psi_2(\hat{\alpha}))\alpha_0 & ,\text{if } \hat{\alpha} \notin R \end{cases} \qquad \ldots(5)$$

where $\psi_i(\hat{\alpha}), 0 \le \psi_i(\hat{\alpha}) \le 1$, i = 1, 2 is a shrinkage weight factor specifying the belief in ($\hat{\alpha}$) while ($1 - \psi_i(\cdot)$) specifying the belief in (α_0), and $\psi_i(\hat{\alpha})$ may be a function of $\hat{\alpha}$ or may be a constant (ad hoc basis).

The prior information may be incorporated in the estimation process using a preliminary test estimator; see for example [7], thus improving the estimation process.

Several authors have been studied (PTSSE) defined in (5) for various parameters of different distributions and for estimate the parameters of regression models, for example see [6], [8], [9] and [10].

The aim of this paper is to study the effect of (PTSSE) defined in (5) for the unknown shape parameter (α) of two parameters GR distribution with known scale parameter (λ) via study the performance of Bias, mean squared error and Efficiency expressions of the proposed estimator and make comparisons of the numerical results with $\hat{\alpha}$ and existing studies.

A numerical study is carried out to assess these effects of proposed estimators.

2. Maximum Likelihood Estimator ($\hat{\alpha}_{MLE}$)

In this section, we consider the maximum likelihood estimator (MLE) of two parameter GR(α,λ).

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from GR(α,λ) then the log-likelihood function L(α,λ) can be written as:

$$L(\alpha,\lambda) = c + n\ln\alpha + 2n\ln\lambda + \sum_{i=1}^{n}\ln x_i - \lambda^2\sum_{i=1}^{n} x_i^2 + (\alpha-1) + (\alpha-1)\sum_{i=1}^{n}\ln(1-e^{-(\lambda x_i)^2}) \ldots(6)$$

where c is constant.

without loss of generality, we assume that $\lambda = 1$ (λ is known).

So, $$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln(1-e^{-x^2}) = 0 \qquad \ldots(7)$$

the MLE of α, say $\hat{\alpha}_{MLE}$ is

$$\hat{\alpha}_{MLE} = -\frac{n}{\sum_{i=1}^{n} \ln(1-e^{-x_i^2})} \qquad \ldots(8)$$

Note that, if $x_i \overset{iid}{\sim} GR(\alpha,1)$, then $-\alpha\sum_{i=1}^{n}\ln(1-e^{-x_i^2})$ follows gamma distribution with shape parameter (n) and scale parameter 1; G(n,1), see [4].

i.e.; $E(\hat{\alpha}_{MLE}) = \frac{n}{n-1}\alpha$ and $var(\hat{\alpha}_{MLE}) = \frac{n^2\alpha^2}{(n-1)^2(n-2)}$.

Using (8), unbiased estimate of α can be easily obtained as:

$$\hat{\alpha} = \frac{n-1}{n}\hat{\alpha}_{MLE} = -\frac{n-1}{\sum_{i=1}^{n}\ln(1-e^{-x_i^2})} \qquad \ldots(9)$$

i.e.; $E(\hat{\alpha}) = \alpha$ and $var(\hat{\alpha}) = MSE(\hat{\alpha}) = \frac{\alpha^2}{n-2}$ $\qquad \ldots(10)$

And the probability density function of $\hat{\alpha}$ is

$$f(\hat{\alpha},\alpha) = \begin{cases} \dfrac{\left[\dfrac{(n-1)\alpha}{\hat{\alpha}}\right]^{n+1} e^{-\frac{(n-1)\alpha}{\hat{\alpha}}}}{\Gamma(n)\ (n-1)\alpha} & \text{for } \hat{\alpha} > 0, \alpha > 0 \\ 0 & \text{o.w.} \end{cases} \qquad \ldots(11)$$

3. Preliminary Test Single Stage Shrinkage Estimator $\tilde{\alpha}$

In this section, we consider the (PTSSE) defined in (5) when $\psi_1(\hat{\alpha}) = 0$ and $\psi_2(\hat{\alpha}) = k$ (constant); $0 \le k \le 1$ for estimate the shape parameter α of two parameter GR distribution when $\lambda = 1$.

Thus, PTSSE of α can be written as:

$$\tilde{\alpha} = \begin{cases} \alpha_0 & ,\text{if } \hat{\alpha} \in R \\ k(\hat{\alpha}-\alpha_0)+\alpha_0 & ,\text{if } \hat{\alpha} \notin R \end{cases} \qquad \ldots(12)$$

where R is a pretest region for testing the null hypothesis H_0: $\alpha = \alpha_0$ vs the alternative hypothesis H_A: $\alpha \ne \alpha_0$ with level of significance (Δ) using test statistic $T(\hat{\alpha}/\alpha_0) = \frac{2(n-1)}{\hat{\alpha}}\alpha_0$

i.e.; $R = \left[\frac{2(n-1)\alpha_0}{b}, \frac{2(n-1)\alpha_0}{a}\right]$ $\qquad \ldots(13)$

where $a = (X^2_{1-\Delta/2,2n})$ and $b = (X^2_{\Delta/2,2n})$, $\qquad \ldots(14)$

are respectively the lower and upper 100(Δ/2) percentile point of chi-square distribution with degree of freedom (2n).

The expression for Bias of PTSSE ($\tilde{\alpha}$) is defined as below

$$Bias(\tilde{\alpha}/\alpha,R) = E(\tilde{\alpha}-\alpha)$$
$$= \int_R (\alpha_0-\alpha)f(\hat{\alpha})d\hat{\alpha} + \int_{\bar{R}} (k\hat{\alpha}+(1-k)\alpha_0)f(\hat{\alpha})d\hat{\alpha}$$

where $\bar{R}$ is the complement region of R in real space and f($\hat{\alpha}$) is a p.d.f. defined in (11).

We conclude,

$$\text{Bias}(\tilde{\alpha}/\alpha,R)=\alpha\{(\zeta-1)J_0(a^*,b^*)+(1-k)(\zeta-1)-(n-1)kJ_1(a^*,b^*)-(1-k)\zeta J_0(a^*,b^*)+J_0(\cdot)\}\quad\ldots(15)$$

where $J_\ell(a^*,b^*)=\int_{a^*}^{b^*} y^{-\ell}\frac{y^{n-1}e^{-y}}{\Gamma(n)}dy;\ \ell=0,1,2$, ...(16)

also $\zeta=\frac{\alpha_0}{\alpha}, a^*=\zeta^{-1}\cdot a,\ b^*=\zeta^{-1}\cdot b$ and $y=\frac{(n-1)\alpha}{\hat{\alpha}}$ ...(17)

i.e.; $R^*=[\zeta^{-1}a^*, \zeta^{-1}b^*]$

The Bias ratio [B(·)] of PTSSE ($\tilde{\alpha}$) is defined as follows

$$\text{Bias}(\tilde{\alpha})=\frac{\text{Bias}(\tilde{\alpha}/\alpha,R)}{\alpha}\quad\ldots(18)$$

The expression of Mean squared error (MSE) of PTSSE ($\tilde{\alpha}$) given as follows:-

$$\text{MSE}(\tilde{\alpha}/\alpha,R)=E(\tilde{\alpha}-\alpha)^2$$

$$=\int_R[\alpha_0-\alpha]^2 f(\hat{\alpha})d\hat{\alpha}+\int_{\bar{R}}[k\hat{\alpha}+(1-k)\alpha_0-\alpha]^2 f(\hat{\alpha})d\hat{\alpha}$$

and by simple computations, one can get:

$$\text{MSE}(\tilde{\alpha}/\alpha,R)=\alpha^2\Big\{(\zeta-1)^2J_0(a^*,b^*)+\frac{k^2}{n-2}+(\zeta-1)^2(k-1)^2-k^2[(n-1)^2J_2(a^*,b^*)-(n-1)\zeta J_1(a^*,b^*)+$$
$$\zeta^2J_0(a^*,b^*)]-2k(\zeta-1)[(n-1)J_1(a^*,b^*)-\zeta J_0(a^*,b^*)]+(\zeta-1)^2J_0(a^*,b^*)\Big\}$$

...(19)

Now, the Efficiency of $\tilde{\alpha}$ relative to the $\hat{\alpha}$ denoted by R.Eff($\tilde{\alpha}$/α,R) is defined as

$$\text{R.Eff}(\tilde{\alpha}/\alpha,R)=\frac{\text{MSE}(\hat{\alpha})}{\text{MSE}(\tilde{\alpha}/\alpha,R)}$$; see [6], [9] ...(20)

4. Conclusions and Numerical Results

The computations of Relative Efficiency [R.Eff(·)] and Bias Ratio [B(·)] expression were used for the considered testimators $\tilde{\alpha}$. These computations were performed for the constants Δ = 0.01,0.05,0.1, n = 4,6,8,10,12, 16,20,30, k = 0.0(0.1)0.5 and ζ = 0.25(0.25)2. Some of these computations are displayed in tables (1)-(3) for some samples of these constants. The observation mentioned in the tables leads to the following results:

i. The Relative Efficiency [R.Eff(·)] of $\tilde{\alpha}$ are adversely proportional with small value of Δ especially when ζ = 1, i.e. Δ = 0.01 yield highest efficiency

ii. The Relative Efficiency [R.Eff(·)] of $\tilde{\alpha}$ has maximum value when $\alpha=\alpha_0$(ζ=1), for each k, n, Δ, and decreasing otherwise (ζ≠1). This feature shown the important usefulness of prior knowledge which given higher effects of proposed estimator as well as the important role of shrinkage technique and its philosophy.

iii. Bias ratio [B(·)] of $\tilde{\alpha}$ increases when ζ increases.

iv. Bias ratio [B(·)] of $\tilde{\alpha}$ are reasonably small when $\alpha=\alpha_0$ for each k, n, Δ, and increases otherwise. This property shown that the proposed estimator $\tilde{\alpha}$ is very closely to unbiasedness especially when $\alpha=\alpha_0$.

v. The Relative efficiency [R.Eff(·)] of $\tilde{\alpha}$ decreases function with increases value of k, for each n, Δ, ζ. This property employ the role of the prior information for proposed Shrinkage estimator via takes high weight for prior information which leads to maximum efficiency.

vi. The Effective Interval [the value of ζ that makes R.Eff.(·) greater than one] using proposed estimator $\tilde{\alpha}$ is [.75,1.5]. Here the pretest criterion is very important for guarantee that prior information is very closely to the actual value and prevent it faraway from it, which get optimal effect of the considered estimator to obtain high efficiency.

vii. The considered estimator $\tilde{\alpha}$ is better that the classical estimator especially when $\alpha \approx \alpha_0$, which is given the effective of $\tilde{\alpha}$ and important weight of prior knowledge as well as the increment of efficiency may be reach to tens time.

viii. The proposed estimator $\tilde{\alpha}$ has smaller MSE than some existing estimators introduced by authors, see for examples [4].

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Table (1)

Shown Bias Ratio [B(·)] and R.E.ff of $\hat{\alpha}$ w.r.t. Δ, n and ζ when k = 0.1

			ζ					
Δ	n	R.Eff. Bias	0.25	0.75	1	1.25	1.75	2
0.01	4	R.Eff(·) B(·)	1.0709 (–.6792)	8.6529 (– 0.2313)	154.1478 (0.0106)	7.3399 (0.2565)	0.8803 (0.7527)	0.4981 (1.0012)
	8	R.Eff(·) B(·)	0.3645 (– .675)	3.1770 (– .2256)	154.33 (0.011)	2.5422 (0.2550)	0.2954 (0.7510)	0.1667 (0.9998)
	16	R.Eff(·) B(·)	0.1565 (– .675)	1.4089 (– .2238)	157.0383 (.0102)	1.1106 (0.2533)	0.1269 (.7502)	0.0715 (0.9994)
	20	R.Eff(·) B(·)	0.1218 (– .6749)	1.0924 (– 0.2244)	159.2414 (0.0085)	0.8739 (0.2519)	0.0988 (0.7499)	0.0556 (0.9993)
0.05	4	R.Eff(·) B(·)	1.085 (– .6751)	9.5245 (– 0.2193)	126.9996 (0.0195)	6.9479 (0.2629)	0.8741 (0.7549)	0.4984 (1.0005)
	8	R.Eff(·) B(·)	0.3645 (– .6749)	3.3074 (– 0.2213)	143.0019 (0.0149)	2.4907 (0.2576)	0.2953 (0.7511)	0.1675 (0.9973)
	16	R.Eff(·) B(·)	0.1565 (– .6749)	1.3988 (– 0.2245)	146.3186 (0.0075)	1.1363 (0.2503)	0.1274 (0.7487)	0.0719 (0.9960)
	20	R.Eff(·) B(·)	0.1218 (– .6750)	1.0871 (– 0.2249)	135.8908 (0.0047)	0.9092 (0.2468)	0.0993 (0.7479)	0.0559 (0.9958)

Table (2)

Shown B(·) and R.E.ff of $\tilde{\alpha}$ w.r.t. Δ, n and ζ when k = 0.5

			ζ					
Δ	n	R.Eff. Bias	0.25	0.75	1	1.25	1.75	2
0.01	4	R.Eff(·) B(·)	1.6951 (– .3962)	3.8090 (– .1567)	6.1659 (.0528)	3.6144 (0.2823)	0.8073 (0.7634)	0.4785 (1.0060)
	8	R.Eff(·) B(·)	0.9142 (– .3751)	3.0175 (– .1281)	6.1733 (.0551)	1.8748 (.2751)	0.2899 (.7552)	0.1660 (.9991)
	16	R.Eff(·) B(·)	0.4507 (– .3749)	2.3840 (.1188)	6.2815 (.0512)	0.9593 (.2667)	0.1264 (.7512)	0.0717 (0.9968)
	20	R.Eff(·) B(·)	0.3596 (– .3749)	2.0044 (– .12181)	6.3697 (.0424)	0.7976 (.2597)	0.0987 (.7499)	0.0558 (.09964)
0.05	4	R.Eff(·) B(·)	1.8755 (– 0.3757)	4.1987 (– 0.0967)	5.0799 (0.0976)	2.9411 (0.3145)	0.7664 (0.7746)	0.4708 (1.0025)
	8	R.Eff(·) B(·)	0.9143 (– 0.3749)	3.5922 (– 0.1067)	5.7201 (0.0745)	1.7171 (0.2879)	0.2878 (0.7555)	0.1682 (0.9864)
	16	R.Eff(·) B(·)	0.4507 (– 0.3749)	2.2386 (– 0.1224)	5.8527 (0.0374)	1.0484 (0.2517)	0.1285 (0.7436)	0.0734 (0.9802)
	20	R.Eff(·) B(·)	0.3596 (– 0.3750)	1.9106 (– 0.1243)	5.4356 (0.0238)	0.9335 (0.2341)	0.1010 (0.7397)	0.0572 (0.9791)

Table (3)

Shown B(·) and R.E.ff of $\tilde{\alpha}$ w.r.t. Δ, n and ζ when k = 0.01

			ζ					
Δ	n	R.Eff. Bias	0.25	0.75	1	1.25	1.75	2
0.01	4	R.Eff(·) B(·)	0.9058 (– 0.7429)	8.1152 (– 0.2481)	15414.7760 (0.0011)	7.9558 (0.2506)	0.8882 (0.7503)	0.4999 (1.0001)
	8	R.Eff(·) B(·)	0.3023 (– 0.7425)	2.7187 (– 0.2476)	15433.2995 (0.0011)	2.6558 (0.2505)	0.2962 (0.7501)	0.16667 (0.99998)
	16	R.Eff(·) B(·)	0.1296 (– 0.7425)	1.1671 (– 0.2474)	15703.8332 (0.0010)	1.1398 (0.2503)	0.12698 (0.7500)	0.0714 (0.9999)
	20	R.Eff(·) B(·)	0.1008 (– 0.7425)	0.9073 (– 0.2474)	15924.1373 (85×10^{-4})	0.8875 (0.2502)	0.0988 (0.74999)	0.0556 (0.9999)
0.05	4	R.Eff(·) B(·)	0.9068 (– 0.7425)	8.1940 (– 0.2469)	12699.9569 (0.0019)	7.9145 (0.2513)	0.8877 (0.7505)	0.4999 (1.0001)
	8	R.Eff(·) B(·)	0.3023 (– 0.7445)	2.7282 (– 0.2471)	14300.19 (0.0015)	2.6503 (0.2508)	0.2962 (0.7501)	0.1668 (0.9997)
	16	R.Eff(·) B(·)	0.1296 (– 0.7425)	1.1664 (– 0.2474)	14631.8639 (75×10^{-4})	1.1425 (0.2500)	0.1270 (0.7499)	0.0715 (0.9996)
	20	R.Eff(·) B(·)	0.1008 (– 0.7425)	0.9069 (– 0.2475)	13589.0783 (20×10^{-4})	0.8911 (0.2497)	0.09882 (0.7498)	0.0567 (0.99)

حول مقدر الاختبار المعنوي لمعلمة الشكل في توزيع رالي العام

عباس نجم سلمان
قسم الرياضيات - كلية التربية (ابن الهيثم) - جامعة بغداد

المستخلص:

يتعلق موضوع هذا البحث بمقدر الاختبار الأولي المقلص ذي المرحلتين لتقدير معلمة الشكل المجهولة (α) لتوزيع رالي العام ذي المعلمتين (GR) عندما تكون معلمة القياس (λ) معلومة وعند توافر معلومات مسبقة حول المعلمة المجهولة (α) بشكل قيم أولية (α_0) عن طريق استعمال عامل تقلص موزون $\psi(\cdot)$ ومجال اختبار أولي R. هذه المعلومات المسبقة حول المعلمة المجهولة يتم الحصول عليها من خلال الخبرات السابقة أو من خلال الدراية او العلم للحالات المشابهة. اشتقت معادلات التحيز، متوسط مربعات الخطأ (MSE) والكفاءة النسبية [R.Eff(·)] للمقدر المقترح. أعطيت النتائج العددية والاستنتاجات للمعادلات المذكورة لبيان فائدة وكفاءة المقدر المقترح وشرحت النتائج العددية وعرضت من خلال الجداول المتظمنة في البحث. قدمت مقارنات بين المقدر المقترح مع المقدر الكلاسيكي $\hat{\alpha}$ ومع بعض الدراسات الاخرى باستخدام المؤشرات الاحصائية معدل مربعات الخطأ (MSE) أوالكفاءة النسبية [R.Eff(·)].