A method for finding the general solution of the LODEs and a system of two LODEs via two integral transforms

Lecturer Amal Khalaf Haydar,
Kufa University, College of Education for Girls, Math. Dep.

Abstract

In this paper we prove three theorems to introduce a new method for finding the general solution (without using any initial conditions) of the linear ordinary differential equations (LODEs) and for the systems of two linear ordinary differential equations with constant coefficients using the integral transform Laplace transform with the aid of the residues and then we prove another two theorems to show that the integral transform Sumudu transform can be used exactly equivalently to Laplace transform in finding the general solutions for these two types of differential equations .

المستخلص

في هذا البحث أثبتنا ثلاثة مبرهنات لنقدم طريقة لإيجاد الحل العام (بدون استخدام أي شروط حدودية) لكل من : المعادلة التفاضلية الاعتيادية الخطية ذات المعاملات الثابتة و نظام متكون من معادلتين تفاضليتين اعتياديتين خطيتين ذي معاملا ت ثابتة باستخدام التحويل التكاملي تحويل لابلاس ومساعدة الرواسب , ثم أثبتنا مبرهنتين اخريتين لبيان أن استخدام التحويل التكاملي تحويل سامودو يكون مكافئ إلى استخدام تحويل لابلاس في إيجاد الحل العام لهذين النوعين من المعادلات .

1 . Introduction

Laplace transform has played an important role in both pure and applied mathematics see for example [1] , [2] and [3], it is particularly effective in the study of initial value problems involving linear differential equations with constant coefficients and has enjoyed much success in this realm . We know that Laplace transform is defined for a function $f(t)$ by

$$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)dt \ . \tag{1.1}$$

It is known that a lot of work has been done on the theory and applications of Laplace transform , but very little on Sumudu transform , because it is little known . Our interest with the Sumudu transform also comes from the fact that this new transform can certainly treat most problems that are usually treated by Laplace transform , in addition the Sumudu transform may be used to solve problems without restoring to a new frequency domain , because it preserves scales and units properties .

The Sumudu transform is introduced in [4] and [5] as follows : Let A is a set of exponential order functions . Then for a given function $f(t)$ in A , the Sumudu transform is defined by

$$F_1(u) = S\,[f(t)] = \int_0^\infty f(ut)e^{-t}dt \ , \ u \in (-\tau_1, \tau_2) \tag{1.2}$$

where τ_1 , and / or $\tau_2 > 0$, also τ_1 and τ_2 need not simultaneously exist , and each may be infinite . Or by

$$F_1(u) = S\,[f(t)] = \frac{1}{u}\int_0^\infty e^{-\frac{t}{u}} f(t)dt, \tag{1.3}$$

provided the integral exists for some u .The connection between Laplace and Sumudu transforms is deeper , if $f(t) \in A$ having $F_1(u)$ and $F(s)$ for Sumudu and Laplace transforms respectively then

$$F_1(u) = \frac{F(1/u)}{u},$$
$$F(s) = \frac{F_1(1/s)}{s}. \qquad (1.4)$$

The pair of relations in (1.4) is called the duality relation .

In [5] some fundamental properties of the Sumudu transform are established and an integral production- depreciation problem solved using the Sumudu transform .

Theorem 1.1 [1] . If f is continuous and f' piecewise continuous on $[0,\infty)$, with f of exponential order α on $[0,\infty)$. If $F(s) = L[f(t)]$ for $\mathrm{Re}(s) = \mathrm{Re}(x+iy) = x > \alpha$, also

$$|F(s)| \le \frac{M}{|s|^p}, \quad p > 0, \qquad (1.5)$$

for all $|s|$ sufficiently large and some p , $M > 0$, and if $F(s)$ is analytic in C except for finitely many poles at $s_1, s_2, .., s_N$, then

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{ts}F(s)ds = \sum_{k=1}^{N} \mathrm{Re}\,s(e^{ts}F(s), s_k). \qquad (1.6)$$

For the Sumudu transform there is an analogue argument [6] that is as follows : Let $F_1(u) = S[f(t)]$. If we can find constants $\mu > 0, k > 0$ in Γ such that

i. $\left|\frac{1}{u}F_1(\frac{1}{u})\right| < \frac{\mu}{R^k}$, (1.7)

ii. $\frac{1}{u}F_1(\frac{1}{u})$ is meromorphic (i.e. only singularities are poles)

then

$$f(t) = S^{-1}[F_1(u)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{u}F_1(\frac{1}{u})\, e^{tu} du = \sum_{k=1}^{N} \mathrm{Re}\,s(\frac{1}{u}F_1(\frac{1}{u})\, e^{tu}, u_k), \qquad (1.8)$$

where R is the radius of the circular region Γ while all singularities $u_1, u_2, ...u_N$ lie in $\mathrm{Re}(u) < c$.

2. The general solution via Laplace transform

In this section we introduce a method for finding the general solution of the LODEs and the systems of two LODEs of order m with constant coefficients by using Laplace transform and the residues without any initial conditions via theorem 2.1 , theorem 2.2 and theorem 2.4 .

Now , we shall introduce the main theorem in this paper which represents a direct and simple consequence of a theorem in p. 177 in [7] .

Theorem 2.1. Given a function $f(z) = e^{tz}g(z)$, if there exists a positive integer m such that the function

$$\phi_k(z) = (z - z_k)^m f(z) = (z - z_k)^m e^{tz} g(z), \qquad (2.1)$$

is analytic at z_k and $\phi_k(z_k) \neq 0$, t is a constant . Then f has a pole of order m at z_k . Its residue there is given by

$$\mathrm{Re}\,s(f, z_k) = \frac{1}{(m-1)!} \lim_{z \to z_k} \frac{d^{m-1}}{dz^{m-1}} e^{tz}\psi_k(z) \quad \text{, if } m > 1, \qquad (2.2)$$

and

$$\mathrm{Re}\,s(f, z_k) = Ae^{tz_k} \text{, if } m = 1, \qquad (2.3)$$

where $\psi_k(z)$ is analytic function at z_k and A is a constant .

Proof . Using relation (3) p. 176 in [7] we have

$$\operatorname{Res}(f,z_k)=\frac{1}{(m-1)!}\lim_{z\to z_k}\frac{d^{m-1}}{dz^{m-1}}\phi_k(z),$$
$$=\frac{1}{(m-1)!}\lim_{z\to z_k}\frac{d^{m-1}}{dz^{m-1}}(z-z_k)^m e^{tz}g(z). \qquad (2.4)$$

Since the conditions in the theorem are satisfied when $g(z)$ has the form

$$g(z)=\frac{\psi_k(z)}{(z-z_k)^m}\quad ,\quad m=1,2,... \qquad (2.5)$$

such that

$$\phi_k(z)=e^{tz}\psi_k(z), \qquad (2.6)$$

where the function $\psi_k(z)$ is analytic at z_k and $\psi_k(z_k)\neq 0$, then

$$\operatorname{Res}(f,z_k)=\frac{1}{(m-1)!}\lim_{z\to z_k}\frac{d^{m-1}}{dz^{m-1}}e^{tz}\psi_k(z)\ .$$

Similarly , if $m=1$ then by using relation (4) p. 176 in [7] we get

$$\operatorname{Res}(f,z_k)=\lim_{z\to z_k}(z-z_k)\,e^{tz}g(z),$$
$$=\lim_{z\to z_k}e^{tz}\psi_k(z)=Ae^{tz_k}, \qquad (2.7)$$

where A is a constant .

Note 1 . If $m=1$ in theorem 2.1 then the point z_k is called a simple pole of the function $f(z)$.

Note 2 . : Suppose that the notation deg denotes the degree , for example $\deg(r(s))$ is the degree of the polynomial $r(s)$.

Theorem 2.2. Let $Y(s)$ and $\frac{k(s)}{h(s)}$ denote Lapace transforms of the functions $y(t)$ and $f(t)$ respectively and $\deg(h(s))=d$. Also let y, y' and Y satisfy the conditions in theorem 1.1. Then the general solution of the LODE of order m with constant coefficients

$$a_m y^{(m)}+a_{m-1}y^{(m-1)}+...+a_1y'+a_0y=f(t)\ , \qquad (2.8)$$

where a_i's are constants and $a_m\neq 0$, is given by

$$y(t)=\sum_{k=1}^{N}\operatorname{Res}\Big(\frac{e^{ts}p(s)}{h(s)\sum_{i=0}^{m}a_is^i},s_k\Big), \qquad (2.9)$$

where $p(s)$ is a polynomial of s with degree is less than or equal to $d+m-1$.

Proof . The differential equation (2.8**)** can be written as

$$a_0y(t)+\sum_{i=1}^{m}a_iy^{(i)}(t)=f(t). \qquad (2.10)$$

Taking Laplace transform to both sides of the differential equation (2.10) gives

$$a_0Y(s)+\sum_{i=1}^{m}a_iL[y^{(i)}(t)]=\frac{k(s)}{h(s)}. \qquad (2.11)$$

Applying the formula of Laplace transform of $y^{(i)}(t)$ where $i\geq 1$[2] in the differential equation (2.11) gives

$$Y(s)\sum_{i=0}^{m} a_i s^i - \sum_{i=1}^{m}\sum_{k=0}^{i-1} a_i y^{(k)}(0)\, s^{i-(k+1)} = \frac{k(s)}{h(s)}. \tag{2.12}$$

Hence , the transformed problem becomes

$$Y(s) = \frac{k(s)+h(s)r(s)}{h(s)\sum_{i=0}^{m} a_i s^i}, \tag{2.13}$$

where

$$r(s) = \sum_{i=1}^{m}\sum_{k=0}^{i-1} a_i y^{(k)}(0)\, s^{i-(k+1)}, \tag{2.14}$$

is a polynomial of s its degree is less than or equal to $m-1$. Since it is known from the tables of Laplace transforms , see for example [1] , [2], [3] and [5] that $\deg(k(s)) < \deg(h(s))$, therefore we have $\deg(k(s)) < \deg(h(s)r(s))$. Hence from equations (2.13) and (2.14) we have

$$Y(s) = \frac{p(s)}{h(s)\sum_{i=0}^{m} a_i s^i}, \tag{2.15}$$

where $p(s)$ is a polynomial of s its degree is less than or equal to $d+m-1$ which is less than the degree of the denominator $d+m$. Since y, y' and Y satisfy the conditions in theorem 1.1 then by taking the inverse Laplace transform L^{-1} of the equation (2.15) and using relation (1.6) gives the general solution of the differential equation (2.8) as

$$\begin{aligned} y(t) &= \sum_{k=1}^{N} \operatorname{Res}(e^{ts}Y(s), s_k), \\ &= \sum_{k=1}^{N} \operatorname{Res}\Big(\frac{e^{ts}p(s)}{h(s)\sum_{i=0}^{m} a_i s^i}, s_k\Big). \end{aligned} \tag{2.16}$$

Example 2.3. To find the general solution of the third order LODE

$$y''' - y'' + 4y' - 4y = 68\, e^t \sin 2t \;, \tag{2.17}$$

using the residues , we find from the differential equation (2.8) that $m = 3$, $a_3 = 1$, $a_2 = -1$, $a_1 = 4$, $a_0 = -4$ and $f(t) = 68\, e^t \sin 2t$. Since

$$L[f(t)] = 68\, L[e^t \sin 2t] = \frac{136}{s^2 - 2s + 5}. \tag{2.18}$$

Therefore $h(s) = s^2 - 2s + 5$. From equation (2.15) and theorem 2.1 we set

$$g(s) = Y(s) = \frac{p(s)}{(s^2-2s+5)(s-1)(s^2+4)}\;, \tag{2.19}$$

and

$$f(s) = e^{ts}Y(s) = \frac{e^{ts}p(s)}{(s^2-2s+5)(s-1)(s^2+4)}\;. \tag{2.20}$$

The function $f(s)$ has simple poles ($m=1$) at the points $1 \mp 2i$,1 and $\mp 2i$. Therefore from relation (2.3) we get

$$\operatorname{Res}(f, s_1 = 1+2i) = A_1 e^{ts_1} = A_1 e^{t+2ti}. \tag{2.21}$$

Similarly , from relation (2.3) yields for $k = 2,3,4,5$ that the residues of f are $A_2 e^{t-2ti}$, $A_3 e^t$, $A_4 e^{2ti}$ and $A_5 e^{-2ti}$ at the poles $s_2 = 1-2i$, $s_3 = 1$, $s_4 = 2i$ and $s_5 = -2i$ respectively , $A_1,\ldots,A_5$ are arbitrary constants . From theorem 2.2 we get

$$y(t)=\sum_{k=1}^{5} \operatorname{Res}(\frac{e^{ts}p(s)}{(s^2-2s+5)(s-1)(s^2+4)},s_k) \qquad (2.22)$$
$$=A_1e^{t+2ti}+A_2e^{t-2ti}+A_3e^{t}+A_4e^{2ti}+A_5e^{-2ti}.$$

Hence

$$y(t)=c_1e^t+c_2\cos 2t+c_3\sin 2t+c_4e^t\cos 2t+c_5e^t\sin 2t\,, \qquad (2.23)$$

where

$c_1=A_3$, $c_2=A_4+A_5$, $c_3=i(A_4-A_5)$, $c_4=A_1+A_2$ and $c_5=i(A_1-A_2)$.

Since the differential equation (2.17) is of the third order then its general solution must contain only three arbitrary constants , therefore if we find y',y'' and y''' from equation (2.23) and substitution in (2.17) we get $c_4=-2$ and $c_5=-8$. Hence the general solution of the differential equation (2.17) is

$$y(t)=c_1e^t+c_2\cos 2t+c_3\sin 2t-2e^t\cos 2t-8e^t\sin 2t\,, \qquad (2.24)$$

where c_1,c_2 and c_3 are arbitrary constants .

Theorem 2.4. Let $X(s),Y(s),\frac{k_1(s)}{h_1(s)}$ and $\frac{k_2(s)}{h_2(s)}$ be Laplace transforms of the functions $x(t),y(t),f_1(t)$ and $f_2(t)$ respectively , $\deg(h_1(s))=d_1$ and $\deg(h_2(s))=d_2$, also let x,y,x',y',X and Y satisfy the conditions in theorem 1.1. Then for the system of two LODEs of order m with constant coefficients of the form

$$\begin{aligned}&a_mx^{(m)}+b_my^{(m)}+a_{m-1}x^{(m-1)}+b_{m-1}y^{(m-1)}+...+a_0x+b_0y=f_1(t)\,,\\&a'_mx^{(m)}+b'_my^{(m)}+a'_{m-1}x^{(m-1)}+b'_{m-1}y^{(m-1)}+...+a'_0x+b'_0y=f_2(t)\,,\end{aligned} \qquad (2.25)$$

where a_i,a'_i,b_i and b'_i are constants for $i=0,1,...,m$. Then we have

i. $X(s)$ and $Y(s)$ have the same poles and their orders are the same .
ii. The general solution of the system (2.25) is given by the pair

$$\begin{aligned}x(t)&=\sum_{j=1}^{N} \operatorname{Res}[\frac{e^{ts}p(s)}{h_1(s)h_2(s)\sum_{i=0}^{2m}e_is^i},s_j],\\y(t)&=\sum_{j=1}^{N} \operatorname{Res}[\frac{e^{ts}q(s)}{h_1(s)h_2(s)\sum_{i=0}^{2m}e_is^i},s_j],\end{aligned} \qquad (2.26)$$

where $p(s)$ and $q(s)$ are polynomials of s each with a degree is less than or equal to d_1+d_2+2m-1 and

$$e_i=\sum_{k=0}^{i}[a_kb'_{i-k}-a'_kb_{i-k}], \qquad (2.27)$$

$i=0,1,...,2m$ and $a_i=b_i=a'_i=b'_i=0$ for $i=m+1,m+2,...,2m.$

Proof . The system (2.25**)** can be written as

$$\begin{aligned}&a_0x(t)+\sum_{i=1}^{m}a_ix^{(i)}(t)+b_0y(t)+\sum_{i=1}^{m}b_iy^{(i)}(t)=f_1(t)\,,\\&a'_0x(t)+\sum_{i=1}^{m}a'_ix^{(i)}(t)+b'_0y(t)+\sum_{i=1}^{m}b'_iy^{(i)}(t)=f_2(t)\,.\end{aligned} \qquad (2.28)$$

By taking Laplace transform to both sides of the two differential equations in the system (2.28) and using $L[x^{(i)}(t)]$ and $L[y^{(i)}(t)]$ for $i\geq 1$ yields the system

$$(\sum_{i=0}^{m} a_i s^i)X(s)+(\sum_{i=0}^{m} b_i s^i)Y(s)=\frac{k_1(s)+h_1(s)r_1(s)}{h_1(s)},$$
$$(\sum_{i=0}^{m} a_i' s^i)X(s)+(\sum_{i=0}^{m} b_i' s^i)Y(s)=\frac{k_2(s)+h_2(s)r_2(s)}{h_2(s)}, \tag{2.29}$$

where

$$r_1(s)=\sum_{i=1}^{m}\sum_{k=0}^{i-1}[a_i x^{(k)}(0)+b_i y^{(k)}(0)]\, s^{i-(k+1)}, \tag{2.30}$$

and

$$r_2(s)=\sum_{i=1}^{m}\sum_{k=0}^{i-1}[a_i' x^{(k)}(0)+b_i' y^{(k)}(0)]\, s^{i-(k+1)}, \tag{2.31}$$

are polynomials of s each with a degree is less than or equal to $m-1$. Solving the two equations in the system (2.29) simultaneously and some simplifications gives the pair

$$X(s)=\frac{1}{h_1(s)h_2(s)\sum_{i=0}^{2m} e_i s^i}[\{k_1(s)h_2(s)+h_1(s)h_2(s)r_1(s)\}\sum_{i=0}^{m} b_i' s^i-\{k_2(s)h_1(s)$$
$$+h_1(s)h_2(s)r_2(s)\}\sum_{i=0}^{m} b_i s^i],$$
$$Y(s)=\frac{1}{h_1(s)h_2(s)\sum_{i=0}^{2m} e_i s^i}[\{k_2(s)h_1(s)+h_1(s)h_2(s)r_2(s)\}\sum_{i=0}^{m} a_i s^i-\{k_1(s)h_2(s)$$
$$+h_1(s)h_2(s)r_1(s)\}\sum_{i=0}^{m} a_i' s^i], \tag{2.32}$$

where e_i is defined as in equation (2.27) . It is clear from the system (2.32) that $X(s)$ and $Y(s)$ have the same poles and their orders are the same . Since

$$\deg(k_i(s))<\deg(h_i(s)),\quad i=1,2, \tag{2.33}$$

then the system (2.32) can be written , by using relations (2.30) and (2.31) , as

$$X(s)=\frac{p(s)}{h_1(s)h_2(s)\sum_{i=0}^{2m} e_i s^i},$$
$$Y(s)=\frac{q(s)}{h_1(s)h_2(s)\sum_{i=0}^{2m} e_i s^i}, \tag{2.34}$$

where $p(s)$ and $q(s)$ are polynomials of s each with degree is less than or equal to d_1+d_2+2m-1 , because

$$\begin{aligned}\deg(p(s))&=\deg(h_1(s))+\deg(h_2(s))+\deg(r_1(s))+m\\&=\deg(h_1(s))+\deg(h_2(s))+\deg(r_2(s))+m\\&\le d_1+d_2+2m-1.\end{aligned} \tag{2.35}$$

It is clear that $\deg(q(s))=\deg(p(s))$. Since x, y, x', y', X and Y satisfy the conditions in theorem 1.1 then by taking L^{-1} to both sides of the two equations in the system (2.34) and using the relation (1.6) then the general solution of the system (2.25) is the system (2.26) .

Example 2.5. Suppose we have the system of two LODEs

$$\begin{aligned} &x'' + 4x + y = \sin^2 t, \\ &y'' + y - 2x = \cos^2 t. \end{aligned} \tag{2.36}$$

By comparing it with the system (2.25) we find that $m = 2$, $a_2 = 1$, $a_0 = 4$, $b_0 = 1$, $b_2' = 1$, $a_0' = -2$, $b_0' = 1$, $b_2 = a_1 = b_1 = a_2' = a_1' = b_1' = 0$, $f_1(t) = \sin^2(t)$ and $f_2(t) = \cos^2(t)$. Since

$$L[f_1(t)] = L[\sin^2 t] = \frac{2}{s(s^2+4)}, \tag{2.37}$$

and

$$L[f_2(t)] = L[\cos^2 t] = \frac{s^2+2}{s(s^2+4)}, \tag{2.38}$$

then $h_1(s) = h_2(s) = s(s^2+4)$. From relation (2.27) with $a_i = b_i = a_i' = b_i' = 0$ for $i = 3,4$ we have

$$\sum_{i=0}^{2m} e_i s^i = \sum_{i=0}^{4}\sum_{k=0}^{i} [a_k b_{i-k}' - a_k' b_{i-k}]\, s^i = s^4 + 5s^2 + 6 = (s^2+2)(s^2+3)\,. \tag{2.39}$$

From the first equation of the system (2.34) and theorem 2.1 we set

$$g(s) = X(s) = \frac{p(s)}{s^2(s^2+4)^2(s^2+2)(s^2+3)}, \tag{2.40}$$

and

$$f(s) = e^{ts} X(s) = \frac{e^{ts} p(s)}{s^2(s^2+4)^2(s^2+2)(s^2+3)}. \tag{2.41}$$

The function $f(s)$ has simple poles ($m=1$) at the points $\sqrt{2}i$, $-\sqrt{2}i$, $\sqrt{3}i$ and $-\sqrt{3}i$ and poles of order $m = 2$ at the points $2i$, $-2i$ and 0 . From relation (2.3) yields for $k = 1,2,3,4$ that the residues of f are $A_1 e^{\sqrt{2}\,ti}$, $A_2 e^{-\sqrt{2}\,ti}$, $A_3 e^{\sqrt{3}\,ti}$ and $A_4 e^{-\sqrt{3}\,ti}$ at the poles $s_1 = \sqrt{2}i$, $s_2 = -\sqrt{2}i$, $s_3 = \sqrt{3}i$ and $s_4 = -\sqrt{3}i$ respectively . For $k = 5$ then from relation (2.2) yields that

$$\begin{aligned} \operatorname{Re}s(f, s_5 = 2i) &= \frac{1}{(2-1)!}\lim_{s\to s_5}\frac{d}{ds} e^{ts}\psi_5(s) \\ &= \lim_{s\to 2i}[e^{ts}\psi_5'(s) + te^{ts}\psi_5(s)] \\ &= A_5 e^{2ti} + A_6 t e^{2ti}. \end{aligned} \tag{2.42}$$

Similarly , using relation (2.2) for $k = 6,7$ then the residues of f are $A_7 e^{-2ti} + A_8 t e^{-2ti}$ and $A_9 + A_{10}t$ at the poles $s_6 = -2i$ and $s_7 = 0$ respectively . From the first equation of the system (2.26) we get that

$$\begin{aligned} x(t) &= \sum_{j=1}^{7} \operatorname{Re}s\Big(\frac{e^{ts} p(s)}{s^2(s^2+4)^2(s^2+2)(s^2+3)}, s_j\Big) \\ &= A_1 e^{\sqrt{2}ti} + A_2 e^{-\sqrt{2}ti} + A_3 e^{\sqrt{3}ti} + A_4 e^{-\sqrt{3}ti} + A_5 e^{2ti} + A_6 t e^{2ti} + A_7 e^{-2ti} + A_8 t e^{-2ti} + A_9 + A_{10} t\,, \end{aligned} \tag{2.43}$$

where $A_1, ..., A_{10}$ are arbitrary constants . Thus after simplification we get

$$\begin{aligned} x(t) &= c_1\cos\sqrt{2}t + c_2\sin\sqrt{2}t + c_3\cos\sqrt{3}t + c_4\sin\sqrt{3}t + c_5\cos 2t + c_6\sin 2t + c_7 t\cos 2t \\ &\quad + c_8 t\sin 2t + c_9 + c_{10}t \end{aligned} \tag{2.44}$$

where

$c_1 = A_1 + A_2$, $c_2 = i(A_1 - A_2)$, $c_3 = A_3 + A_4$, $c_4 = i(A_3 - A_4)$, $c_5 = A_5 + A_7$, $c_6 = i(A_5 - A_7)$, $c_7 = A_6 + A_8$, $c_8 = i(A_6 - A_8)$, $c_9 = A_9$, $c_{10} = A_{10}$.

Similarly , from the second equation of the system (2.26) and using (i) of theorem 2.4 then $y(t)$ can be written as

$$y(t) = d_1\cos\sqrt{2}t + d_2\sin\sqrt{2}t + d_3\cos\sqrt{3}t + d_4\sin\sqrt{3}t + \mathrm{d}_5\cos 2t + d_6\sin 2t + d_7 t\cos 2t + d_8 t\sin 2t + d_9 + d_{10}t \qquad (2.45)$$

where

$d_1 = A_1' + A_2'$, $d_2 = i(A_1' - A_2')$, $d_3 = A_3' + A_4'$, $d_4 = i(A_3' - A_4')$, $d_5 = A_5' + A_7'$, $d_6 = i(A_5' - A_7')$, $d_7 = A_6' + A_8'$, $d_8 = i(A_6' - A_8')$, $d_9 = A_9'$, $d_{10} = A_{10}'$.

Since the system (2.36) is of the second order then its general solution must contain only four arbitrary constants , therefore substitution x, y, x'' and y'' in the system (2.36) yields a system of equations , that solving it gives values many of the constants c_i's and d_i's . Then the general solution of the system (2.36) is as follows

$$\begin{aligned} x(t) &= c_1\cos\sqrt{2}t + c_2\sin\sqrt{2}t + c_3\cos\sqrt{3}t + c_4\sin\sqrt{3}t + \frac{1}{2}\cos 2t, \\ y(t) &= -2c_1\cos\sqrt{2}t - 2c_2\sin\sqrt{2}t - c_3\cos\sqrt{3}t - c_4\sin\sqrt{3}t - \frac{1}{2}\cos 2t + \frac{1}{2} \end{aligned} \qquad (2.46)$$

where $c_1,...,c_4$ are arbitrary constants .

3 . The general solution via Sumudu transform

In this section we shall show that the Sumudu transform can be used exactly equivalent to Laplace transform for getting the general solutions of the differential equation (2.8) and the system (2.25) via theorem 3.1 and theorem 3.3 **.**

Theorem 3.1. For the LODE (2.8) suppose that $Y_1(u)$ is the sumudu transform of the function $y(t)$ and $L[f(t)] = \frac{k(s)}{h(s)}$. Also let $\frac{1}{u}Y_1(\frac{1}{u})$ is meromorphic (i.e. only singularities are poles) [6] and satisfies inequality (1.7) . Then the general solution $y(t)$ of the differential equation (2.8) by using the Sumudu transform is completely determined via equation (2.9) at the point $s = u$.

Proof . Since

$$Y_1(u) = S[y(t)], \qquad (3.1)$$

then the general solution $y(t)$ of the LODE (2.8) is gotten by taking the inverse Sumudu transform S^{-1} to both sides of the equation (3.1) . Since $\frac{1}{u}Y_1(\frac{1}{u})$ is meromorphic and satisfies inequality (1.7) then by taking $S^{-1}$and using relation (1.8) we get

$$y(t) = S^{-1}[Y_1(u)] = \sum_{k=1}^{N} \mathrm{Re}\,s(\frac{1}{u}Y_1(\frac{1}{u})\,e^{tu}, u_k)\,. \qquad (3.2)$$

From the second relation of the duality relation (1.4) we have

$$Y(u) = \frac{1}{u}Y_1(\frac{1}{u}) \ , \qquad (3.3)$$

where

$$Y(u) = L[y(t)]\big|_{\mathrm{s=u}} \ , \qquad (3.4)$$

is the Laplace transform of $y(t)$ in the LODE (2.8) at $s = u$. Therefore by using relations (3.3) and (2.15) we get that the general solution (2.9) of the differential equation (2.8) can be written , by using the Sumudu transform , as

$$y(t)=\sum_{k=1}^{N}\operatorname{Re}s(Y(u)\,e^{tu},u_k)$$
$$=\sum_{k=1}^{N}\operatorname{Re}s(\frac{e^{tu}\,p(u)}{h(u)\sum_{i=0}^{m}a_i u^i},u_k)\,, \tag{3.5}$$

where $p(u)$ is defined as in theorem 2.2 . It is clear that the second equation of (3.5) is equation (2.9) at $s=u$.

Note : According to our method : To find the general solution of any LODE using the Sumudu transform , theorem 3.1 asserts that we can use the same steps in example 2.3 only by replacing the variable s by u as shown in the following example :

Example 3.2. To find the general solution of the fourth order LODE

$$y^{(4)}+2y^{(2)}+y=t\ , \tag{3.6}$$

By using theorem 3.1 we have $m=4$, $a_4=1, a_3=0,\ a_2=2, a_1=0$, $a_0=1$ and $f(t)=t$ by comparing with the differential equation (2.8) . Since

$$L[f(t)]\big|_{s=u}=L[t]\big|_{s=u}=\frac{1}{u^2}\ , \tag{3.7}$$

then $h(u)=u^2$. From equation (2.15) at $s=u$ and theorem 2.1 we set

$$g(u)=\frac{1}{u}Y_1(\frac{1}{u})=L[y(t)]\big|_{s=u}=Y(u)=\frac{p(u)}{u^2(1+u^2)^2}\,, \tag{3.8}$$

and

$$f(u)=e^{tu}\frac{1}{u}Y_1(\frac{1}{u})=e^{tu}Y(u)=\frac{e^{tu}\,p(u)}{u^2(1+u^2)^2}\ . \tag{3.9}$$

The function $f(u)$ has poles of order $m=2$ at 0 , i and $-i$. Therefore by using relation (2.2) and equation (2.9) at $s=u$ (or from the second equation of (3.5)) we have

$$y(t)=\sum_{k=1}^{5}\operatorname{Re}s(\frac{e^{tu}\,p(u)}{u^2(1+u^2)^2},u_k)$$
$$=A_1+A_2t+A_3e^{ti}+A_4te^{ti}+A_5e^{-ti}+A_6te^{-ti}, \tag{3.10}$$

where $A_1,,A_6$ are arbitrary constants .Hence

$$y(t)=c_1\cos t+c_2\sin t+c_3t\cos t+c_4t\sin t+c_5+c_6t\,, \tag{3.11}$$

where $c_1=A_3+A_5$, $c_2=i(A_3-A_5)$, $c_3=A_4+A_6$, $c_4=i(A_4-A_6)$, $c_5=A_1$ and $c_6=A_2$.

Since the differential equation (3.6) is of the fourth order then its general solution must contain only four arbitrary constants therefore substitution $y,y^{(2)}$ and $y^{(4)}$ in the differential equation (3.6) gives $c_5=0$ and $c_6=1$. Therefore the general solution of the LODE (3.6) is

$$y(t)=c_1\cos t+c_2\sin t+c_3t\cos t+c_4t\sin t+t\,, \tag{3.12}$$

where $c_1,...,c_4$ are arbitrary constants

Theorem 3.3. For the system of two LODE (2.25) suppose that $X_1(u)$ and $Y_1(u)$ represent the Sumudu transforms of the functions $x(t)$ and $y(t)$ respectively , $L[f_1(t)]=\frac{k_1(s)}{h_1(s)}$ and $L[f_2(t)]=\frac{k_2(s)}{h_2(s)}$. Also let $\frac{1}{u}X_1(\frac{1}{u})$ and $\frac{1}{u}Y_1(\frac{1}{u})$ are meromorphic and satisfy inequality (1.7).

Then the general solution of the system (2.25) by using the Sumudu transform is completely determined via the system (2.26) at the point $s=u$ and relation (2.27) .

Proof . Since

$$X_1(u) = S[x(t)], \quad Y_1(u) = S[y(t)], \tag{3.13}$$

then the general solution of the system (2.25) is gotten by taking S^{-1} to both sides of the two differential equations in the system (3.13) . Since $\frac{1}{u}X_1(\frac{1}{u})$ and $\frac{1}{u}Y_1(\frac{1}{u})$ are meromorphic and satisfy inequality (1.7) then by taking S^{-1} and using relation (1.8) we get

$$x(t) = S^{-1}[X_1(u)] = \sum_{j=1}^{N} \operatorname{Re}s(\frac{1}{u}X_1(\frac{1}{u})\,e^{tu}, u_j), \quad y(t) = S^{-1}[Y_1(u)] = \sum_{j=1}^{N} \operatorname{Re}s(\frac{1}{u}Y_1(\frac{1}{u})\,e^{tu}, u_j). \tag{3.14}$$

From the second relation of the duality relation (1.4) we have

$$X(u) = \frac{1}{u}X_1(\frac{1}{u}), \quad Y(u) = \frac{1}{u}Y_1(\frac{1}{u}), \tag{3.15}$$

where

$$X(u) = L[x(t)]\big|_{s=u}, \quad Y(u) = L[y(t)]\big|_{s=u}, \tag{3.16}$$

are the Laplace transforms of $x(t)$ and $y(t)$ respectively in the system (2.25) at $s=u$. Therefore by using the two systems (3.15) and (2.34) we get that the general solution (2.26) of the system (2.25) can be written , by using the Sumudu transform , as follows

$$x(t) = \sum_{j=1}^{N} \operatorname{Re}s(X(u)\,e^{tu}, u_j) = \sum_{j=1}^{N} \operatorname{Re}s(\frac{e^{tu}\,p(u)}{h_1(u)h_2(u)\sum_{i=0}^{2m} e_i u^i}, u_j), \quad y(t) = \sum_{j=1}^{N} \operatorname{Re}s(Y(u)\,e^{tu}, u_j) = \sum_{j=1}^{N} \operatorname{Re}s(\frac{e^{tu}\,q(u)}{h_1(u)h_2(u)\sum_{i=0}^{2m} e_i u^i}, u_j), \tag{3.17}$$

where e_i , $p(u)$ and $q(u)$ are defined as in theorem 2.4 .

4. Conclusion

In this paper we introduced a new method for finding the general solution of LODEs and for systems of two LODEs using Laplace transform and the residues and then we show that the Sumudu transform can be used exactly equivalently to Laplace transform in finding the general solutions for these two types of differential equations . This equivalence comes from the relation between the inverse Sumudu transform and the residues as well as the important relation between these two integral transforms that is duality relation .

References //

1. Schiff , J. L., the Laplace Transform : Theory and Applications , Springer-Verlag New York , Berlin , (1999) .
2. Rainville , E.D. , Bedient , P.E. , Bedient , R.E. , Elementary Differential Equations , Eighth Edition , Prentice Hall , Inc., (1997).
3. Dass , H. K., Advanced Engineering Mathematics , Ram Nagar , New Delhi , (2009).
4. Watugala , G. K., Sumudu Transform : a New Integral Transform to Solve Differential Equations and Control Engineering Problems , Int. J. Math. Educ. Sci. Technol.,Vol.24 , No.1 ,(1993)35-43 .
5. Belgacem , F.B. M., Karaballi , A.A., Kalla , S. L., Analytical Investigations of the Sumudu Transorm and Applications to Integral Production Equations, Math. Probl. Engr., Vol.3 , (2003), 103-118 .
6. Weerakoon , S. , Complex Inversion Formula for Sumudu Transform , Int. J. Math. Edu. Sci. Techn., Vol.29, No.4, (1998) 618-621 .
7. Churchill , R. V., Brown, J. W. , Verhey, R. F., Complex Variables and Applications , Third Ed. , McGraw-Hill Kogakusha , Ltd.,Tokyo , (1974) .