Analytic Solution of Integral Equation Involving Spheroidal Wave Functions of Three Variables.

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Abstract

In this paper Laplace transform and convolution theorem are used to solve integral equation involving spheroidal wave functions(S.W.F.) of 3-variables of the form

$$g(x_1,x_2,x_3)=\int_0^{x_1}.\int_0^{x_2}\int_0^{x_3}(x_1-t_1)^{\rho-1}(x_2-t_2)^{q-1}(x_3-t_3)^{\beta-1}\phi_{\alpha_1,n_1}(x_1-t_1).\phi_{\alpha_2,n_2}(x_2-t_2)\phi_{\alpha_3,n_3}(x_3-t_3)f(t_1,t_2,t_3)dt_1dt_2dt_3 \quad (1)$$

where $\phi_{\alpha_i,n_i}(x-t), i=1,2,3$ are the spheroidal wave function:-

$$\phi_{\alpha_1,n_1}(x_1-t_1)=\frac{i^{n_1}\sqrt{2\pi}}{V_{\alpha_1,n_1}(x_1)}\sum_{k_1=0,1}^{\infty}a_{k_1}(x_1/\alpha_1 n_1)\frac{J_{k_1+\alpha_1+1/2}(x_1-t_1)}{(x_1-t_1)^{\alpha_1+1/2}}$$

$$\phi_{\alpha_2,n_2}(x_2-t_2)=\frac{i^{n_2}\sqrt{2\pi}}{V_{\alpha_2,n_2}(x_2)}\sum_{k_1=0,1}^{\infty}a_{k_2}(x_2/\alpha_2 n_2)\frac{J_{k_2+\alpha_2+1/2}(x_2-t_2)}{(x_2-t_2)^{\alpha_2+1/2}}$$

$$\phi_{\alpha_3,n_3}(x_3-t_3)=\frac{i^{n_3}\sqrt{2\pi}}{V_{\alpha_3,n_3}(x_3)}\sum_{k_1=0,1}^{\infty}a_{k_3}(x_3/\alpha_3 n_3)\frac{J_{k_3+\alpha_3+1/2}(x_3-t_3)}{(x_3-t_3)^{\alpha_3+1/2}}$$

which represents the function uniformly converges on (-∞,∞), where $V_{\alpha,n}(x)$ are eigen values and the coefficients $a_k(x,\alpha_n)$ satisfy the recursion formula and the asterisk over the summation sign indicates that the sum is taken over only even or odd values of (k) according as (n) is even or odd.

Introduction

Spheroidal wave functions are special functions in mathematical physics which have found many important and practical applications in science engineering where the prelate or the oblate spheroidal coordinate system is used (Bakshi & Bakshi, 2007; Leong, 2002**)**. In the evaluation of electromagnetic fields in spheroidal structures, spheroidal wave functions are frequently encountered, especially when boundary value problem in spheroidal structures are solved using full wave analysis (Leong, 2002; Verma 1977 and Wylie,1975) . The mechanics problem of calculating the time a particle to side under gravity down a given smooth curve, from any point on the curve to it's lower end, leads to an exercise in integration. The time $f(\tau)$ say ; for the particle to descend from the height t is given by an expression of the form

$$f(\tau)=\int_{0}^{\tau}\frac{y(t)dt}{(\tau-t)^{\frac{1}{2}}}\ ,\quad (a\le\tau\le b) \qquad \ldots (2)$$

where $y(t)$ embodies the shape of the given curve.

The converse problem, in which the time of descent from height τ is given and the particular curve which produces this time has to be found is less straight for ward, as it entails the determination of the function ϕ form (2), $f(\tau)$ now being assigned for $a\le\tau\le b$. From this point of view (2) is called integral equation, this description expressing the fact that the function to be determined appears under an integral sign. The equation (2) is one of many integral equations which result directly from a physical problem. The notation adopted in this section, and through out by y or $y(s)$ (Jerri,1985**)**.Some of this formulas have provide very useful in physical application such as electromagnetic diffraction.(Hamko, 2002).

In 1977, Gupta and Mishra have considered the solution of the convolution integral equation.

$$f(x)=\int_{0}^{x}(x-t)^{\alpha-1}\phi_{\alpha,n}(x-t)y(t)dt \qquad \ldots(3)$$

where $\phi_{\alpha,n}(x-t)$ is called spheroidal wave function given by the equation:

$$\phi_{\alpha,n}(x-t)=\frac{i^{n}\sqrt{2\pi}}{V_{\alpha,n}(x)}\sum_{k=0,1}^{\infty}a_{k}(x,\alpha_{n})\frac{J_{k+\alpha+1/2}(x-t)}{(x-t)^{\alpha+1/2}}$$

where $a_{k}(x,\alpha_{n})$ is the coefficient.

In 1987 Mishra and Chauhan have considered the solution of the convolution integral equation involving spheroidal wave function of two variables.

$$f(x_1,x_2)=\int_{0}^{x_1}\int_{0}^{x_2}(x_1-t_1)^{(r_1-p-1)}(x_2-t_2)^{(r_2-\beta-1)}\phi_{\alpha_1-n_1}(x_1,t_1)\phi_{\alpha_2-n_2}(x_2,t_2)y(t_1,t_2)dt_2dt_1 \qquad \ldots(4)$$

where

$$\phi_{\alpha_1,n_1}(t_1)=\sum_{k_1=0}^{\infty}\frac{Q_1E_{2k_1+1}t_1^{2k_1+1}}{\Gamma_{(2k_1+1+\alpha_1-n_1)}} \quad \text{and} \quad \phi_{\alpha_2,n_2}(t_2)=\sum_{k_1=0}^{\infty}\frac{Q_2E_{2k_2+1}t_2^{2k_2+1}}{\Gamma_{(2k_2+1+\alpha_2-n_2)}}$$

Basic definitions

Definition(1): (Gupta & Mishra, 1977**).**

A function $\phi_{\alpha,n}(x-t)$ of the convolution integral equation of the form

$$g(x)=\int_0^x (x-t)^{\alpha-1}\phi_{\alpha,n}(x-t)f(t)dt \ , \quad \phi_{\alpha,n}(x-t)=\frac{i^n\sqrt{2\pi}}{V_{\alpha,n}(x)}\sum_{k=0,1}^{\infty}a_k(x,\alpha_n)\frac{J_{k+\alpha+1/2}(x-t)}{(x-t)^{\alpha+1/2}}$$

is called spheroidal wave function.

Definition (2):- (Wylie , 1975).

The normalization property of the spheroidal wave function if α= -1/2, that means

$$\frac{\sqrt{2\pi}}{V_{-\frac{1}{2},n}(x)}\sum_{k=0,1}^{\infty\ *}(i)^k a_k(x/-\frac{1}{2},n)=\begin{bmatrix}1 & for\ k=n\\ 0 & for\ k\neq n\end{bmatrix}$$

for $x_i\to 0$ and $t_i\to\infty$ such that (x_i-t_i) remains finite, the function $\phi_{\alpha,n}(x-t)$ become

proportional to $(x-t)^{-(\alpha+1/2)}$ $J_{k+\alpha+1/2}$ (x-t)

hence $a_k(x,\alpha_n)=\begin{vmatrix}1, & (k=n)\\ 0, & (k\neq n)\end{vmatrix}$ by normalization property, that means α=-1/2.

Definition (3) :-(Murray, 1965)

Double Laplace transform:- Double Laplace transform are defined by

$\bar{f}(\alpha,\beta)=\int_0^{\alpha}\int_0^{\beta} f(x,y)e^{-(\alpha x+\beta y)}dxdy$. where α and β are two parameters and the given function f(x,y) contains two variables x and y.

Theorem (1) :-(Murray, 1965)

Convolution theorem: If $\bar{f}_1(\alpha)$ and $\bar{f}_2(\alpha)$ are a transforms of $f_1(x)$ and $f_2(x)$, then $\bar{f}_1(\alpha)\,.\,\bar{f}_2(\alpha)$ is the transform of $\int_0^x f_1(u)f_2(x-u)du$.

Analytic Solution of Integral Equation Involving Spheroidal Wave Functions of Three Variables

The integral equation of 3-variables of the form (Gupta & Mishra ,1977)

$$g(x_1,x_2,x_3)=\int_0^{x_1}.\int_0^{x_2}\int_0^{x_3}(x_1-t_1)^{\rho-1}(x_2-t_2)^{q-1}(x_3-t_3)^{\beta-1}\phi_{\alpha_1,n_1}(x_1-t_1).\phi_{\alpha_2,n_2}(x_2-t_2).\phi_{\alpha_3,n_3}(x_3-t_3)f(t_1,t_2,t_3)dt_1dt_2dt_3$$

where $\phi_{\alpha_1,n_1}(x_1-t_1)$, $\phi_{\alpha_2,n_2}(x_2-t_2)$ and $\phi_{\alpha_3,n_3}(x_3-t_3)$ are the spheroidal wave functions given by

$$\phi_{\alpha_1,n_1}(x_1-t_1)=\frac{i^{n_1}\sqrt{2\pi}}{V_{\alpha_1,n_1}(x_1)}\sum_{k_1=0,1}^{\infty}a_{k_1}(x_1/\alpha_1n_1)\frac{J_{k_1+\alpha_1+1/2}(x_1-t_1)}{(x_1-t_1)^{\alpha_1+1/2}}$$

$$\phi_{\alpha_2,n_2}(x_2-t_2)=\frac{i^{n_2}\sqrt{2\pi}}{V_{\alpha_2,n_2}(x_2)}\sum_{k_1=0,1}^{\infty}a_{k_2}(x_2/\alpha_2n_2)\frac{J_{k_2+\alpha_2+1/2}(x_2-t_2)}{(x_2-t_2)^{\alpha_2+1/2}}$$

$$\phi_{\alpha_3,n_3}(x_3-t_3)=\frac{i^{n_3}\sqrt{2\pi}}{V_{\alpha_3,n_3}(x_3)}\sum_{k_1=0,1}^{\infty}a_{k_3}(x_3/\alpha_3n_3)\frac{J_{k_3+\alpha_3+1/2}(x_3-t_3)}{(x_3-t_3)^{\alpha_3+1/2}}$$

then the series of integral equation of 3-variables is written as

$$g(x_1,x_2,x_3)=\int_0^{x_1}\int_0^{x_2}.\int_0^{x_3}(x_1-t_1)^{\rho-1}(x_2-t_2)^{q-1}(x_3-t_3)^{\beta-1}\frac{i^{n_1+n_2+n_3}(2\pi)^{\frac{3}{2}}}{V_{\alpha_1,n_1}(x_1).V_{\alpha_2,n_2}(x_2).V_{\alpha_3,n_3}(x_3)}\sum_{k_1=0,1}^{\infty}a_{k_1}(x_1/\alpha_1n_1)*$$

$$\frac{J_{k_1+\alpha_1+\frac{1}{2}}(x_1-t_1)}{(x_1-t_1)^{\alpha_1+\frac{1}{2}}}*...*\sum_{k=0,1}^{\infty}a_{k_3}(x_3/\alpha_3n_3)\frac{J_{k_3+\alpha_3+\frac{1}{2}}(x_3-t_3)}{(x_3-t_3)^{\alpha_3+\frac{1}{2}}}f(t_1,t_2,t_3)dt_1dt_2dt_3 \qquad ...(5)$$

due to the uniformly converges of the series we have

$$g(x_1,x_2,x_3)=\frac{i^{n_1+n_2+n_3}(2\pi)^{\frac{n}{2}}}{V_{\alpha_1,n_1}(x_1).V_{\alpha_2,n_2}(x_2).V_{\alpha_{3n},n_3}(x_n)}\sum_{k_1=0,1}^{\infty}a_{k_1}(x_1/\alpha_1n_1).\sum_{k_2=0,1}^{\infty}a_{k_2}(x_2/\alpha_2n_2)$$

$$\sum_{k_3=0,1}^{\infty}a_{k3}(x_3/\alpha_3n_3)\int_0^{x_1}\int_0^{x_2}\int_0^{x_3}(x_1-t_1)^{\rho-\alpha_1-3/2}.(x_2-t_2)^{q-\alpha_2-3/2}.(x_3-t_3)^{\beta-\alpha_3-3/2}J_{k_1+\alpha_1+1/2}(x_1-t_1)J_{k_2+\alpha_2+1/2}(x_2-t_2)J_{k_3+\alpha_3+1/2}(x_3-t_3)*$$

$$f(t_1,t_2,t_3)dt_1dt_2dt_3 \qquad ...(6)$$

Taking Laplace transform of both sides of equation (6), and applying convolution theorem, yields

$$\bar{g}(s_1,s_2,s_3)=\frac{i^{n_1+n_2+n_3}(2\pi)^{\frac{3}{2}}}{V_{\alpha_1,n_1}(x_1).V_{\alpha_2,n_2}(x_2).V_{\alpha_3,n_3}(x_n)}\sum_{k_1=0,1}^{\infty}a_{k_1}(x_1/\alpha_1n_1).\sum_{k_2=0,1}^{\infty}a_{k_2}(x_2/\alpha_2n_2)$$

$$\sum_{k_3=0,1}^{\infty}a_{k3}(x_3/\alpha_3n_3)[\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}t_1^{\rho-\alpha_1-3/2}t_2^{q-\alpha_2-3/2}t_3^{\beta-\alpha_n-3/2}J_{k_1+\alpha_1+1/2}(t_1).J_{k_2+\alpha_2+1/2}(t_2).J_{k_3+\alpha_3+1/2}(t_3)*$$

$$Exp\ [-(s_1t_1+s_2t_2+s_3t_3)]*dt_1dt_2dt_3\,\bar{f}(s_1,s_2,s_3) \qquad \ldots (7)$$

where f and g are the symbols of Laplace transformation.

then

$$\bar{g}(s_1,s_2,s_3)=\frac{i^{n_1+n_2+n_3}(2\pi)^{\frac{3}{2}}}{V_{\alpha_1,n_1}(x_1)V_{\alpha_2,n_2}(x_2)V_{\alpha_3,n_3}(x_3)}\sum_{k_1=0,1}^{\infty}a_{k_1}(x_1/\alpha_1n_1).\sum_{k=0,1}^{\infty}a_{k_2}(x_2/\alpha_2n_2)\sum_{k_3=0,1}^{\infty}a_{k_3}(x_3/\alpha_3n_3)*$$

$$*\sum_{m_3=0}^{\infty}\frac{(-1)^{m_3}(\tfrac{1}{2})^{K_3+\alpha_3+\frac{1}{2}+2m_3}\Gamma(\beta+K_3+2m_3)}{m_3!\Gamma(K_3+\alpha_3+\frac{1}{2}+m_3+1)S_3^{\beta+K_3+2m_3}}\bar{f}(s_1,s_2,s_3) \qquad \ldots(8)$$

then we get

$$\bar{f}(s_1,s_2,s_3)=\frac{\bar{g}(s_1,s_2,s_3)\ V_{\alpha_1,n_1}(x_1).V_{\alpha_2,n_2}(x_2).V_{\alpha_3,n_3}(x_3)S_1^{p}S_2^{q}S_3^{\beta}}{i^{n_1+n_2+n_3}(2\pi)^{\frac{3}{2}}\sum_{K_1=0,1}^{\infty}a_{K_1}(x_1/\alpha_1n_1)\sum_{K_2=0,1}a_{k_2}(x_2/\alpha_2n_2)\sum_{K_3=0,1}^{\infty}a_{K_3}(x_3/\alpha_3n_3)}*$$

$$\frac{1}{[\sum_{m_i=0}^{\infty}\frac{(-1)^{m_1}(\frac{1}{2})^{K_1+\alpha_1+\frac{1}{2}+2m_1}\Gamma(p+K_1+2m_1)}{m_1!\Gamma(K_1+\alpha_1+\frac{1}{2}+m_1+1)S_1^{K_1+2m_1}}*..\sum_{m_3=0}^{\infty}\frac{(-1)^{m_3}(\frac{1}{2})^{K_3+\alpha_3+\frac{1}{2}+2m_3}\Gamma(\beta+K_3+2m_3)}{m_3!\Gamma(K_3+\alpha_3+\frac{1}{2}+m_3+1)S_3^{K_3+2m_3}}} \ldots.(9)$$

$$\Rightarrow\ \bar{f}(s_1,s_2,s_3)=\frac{\bar{g}(s_1,s_2,s_3)\ V_{\alpha_1,n_1}(x_1)V_{\alpha_2,n_2}(x_2)V_{\alpha_3,n_3}(x_3)S_1^{P}S_2^{q}S_3^{\beta}}{i^{n_1+n_2+n_3}(2\pi)^{\frac{3}{2}}\sum_{K_1=0,1}^{\infty}a_{K_1}(x_1/\alpha_1n_1)\sum_{K_2=0,1}^{\infty}a_{k_2}(x_2/\alpha_2n_2)\sum_{K_3=0,1}^{\infty}a_{K_3}(x_3/\alpha_3n_3)}*$$

$$\frac{1}{[\frac{(\frac{1}{2})^{K_1+\alpha_1+\frac{1}{2}}\Gamma(p+K_1)}{0!\Gamma(K_1+\alpha_1+\frac{3}{2})S_1^{K_1}}-\frac{(\frac{1}{2})^{K_1+\alpha_1+\frac{5}{2}}\Gamma(p+K_1+2)}{1!\Gamma(K_1+\alpha_1+\frac{5}{2})S_1^{K_1+2}}+\frac{(\frac{1}{2})^{K_1+\alpha_1+\frac{9}{2}}\Gamma(p+K_1+4)}{2!\Gamma(K_1+\alpha_1+\frac{7}{2})S_1^{K_1+4}}-\ldots+\frac{(-1)^{m_1}(1/2)^{k_1+\alpha_1+\frac{1}{2}+2m_1}\Gamma(p+k_1+2m_1)}{m!\Gamma(k_1+\alpha_1+m_1+3/2)S_1^{k_1+2m_1}}}*$$

$$\frac{1}{[\frac{(\frac{1}{2})^{K_2+\alpha_2+\frac{1}{2}}\Gamma(q+K_2)}{0!\Gamma(K_2+\alpha_2+\frac{3}{2})S_2^{K_2}}-\frac{(\frac{1}{2})^{K_2+\alpha_2+\frac{5}{2}}\Gamma(q+K_2+2)}{1!\Gamma(K_2+\alpha_2+\frac{5}{2})S_2^{K_2+2}}+\frac{(\frac{1}{2})^{K_2+\alpha_2+\frac{9}{2}}\Gamma(q+K_2+4)}{2!\Gamma(K_2+\alpha_2+\frac{7}{2})S_2^{K_2+4}}-\ldots+\frac{(-1)^{m_2}(1/2)^{k_2+\alpha_{21}+\frac{1}{2}+2m_2}\Gamma(q+k_2+2m_2)}{m_2!\Gamma(k_2+\alpha_2+m_2+3/2)S_2^{k_2+2m_2}}}*$$

$$\frac{1}{\frac{(1/2)^{k_3+\alpha_3+1/2}\Gamma(\beta+k_3)}{0!\Gamma(k_3+\alpha_3+3/2)S_3^{k_3}}-\frac{(1/2)^{k_3+\alpha_3+5/2}\Gamma(\beta+k_3+2)}{1!\Gamma(k_3+\alpha_3+5/2)S_3^{k_3+2}}+\frac{(1/2)^{k_3+\alpha_3+9/2}\Gamma(\beta+\alpha_3+4)}{2!\Gamma(k_3+\alpha_3+7/2)S_3^{k_3+4}}-\ldots+\frac{(-1)^{m_3}(1/2)^{k_3+\alpha_3+1/2+2m_3}\Gamma(\beta+k_3+2m_3)}{m_3!\Gamma(k_3+\alpha_3+m_3+3/2)S_3^{k_3+2m_3}}}$$

(10)

Then we get

$$\bar{f}(s_1,s_2,s_3)=\frac{\theta}{\sum_{K_1=0,1}^{\infty}a_{K_1}(x_1,\alpha_1 n_1)\sum_{k_2=0,1}^{\infty}a_{k_2}(x_2,\alpha_2 n_2).\sum_{K_3=0,1}^{\infty}a_{K_3}(x_3,\alpha_3 n_3)}*$$

$$\frac{1}{\frac{(1/2)^{k_1+\alpha_1+1/2}\Gamma(p+k_1)}{0!\Gamma(k_1+\alpha_1+3/2)S_1^{k_1}}-\frac{(1/2)^{k_1+\alpha_1+5/2}\Gamma(p+k_1+2)}{1!\Gamma(k_1+\alpha_1+5/2)S_1^{k_1+2}}+\frac{(1/2)^{k_1+\alpha_1+9/2}\Gamma(p+k_1+4)}{2!\Gamma(k_1+\alpha_1+7/2)S_1^{k_1+4}}-...+\frac{(-1)^{m_1}(1/2)^{k_1+\alpha_1+1/2+2m_1}\Gamma(p+k_1+2m_1)}{m_1!\Gamma(k_1+\alpha_1+m_1+3/2)S_1^{k_1+2m_1}}}*$$

$$\frac{1}{\frac{(1/2)^{k_2+\alpha_2+1/2}\Gamma(q+k_2)}{0!\Gamma(k_2+\alpha_2+3/2)S_2^{k_2}}-\frac{(1/2)^{k_2+\alpha_2+5/2}\Gamma(q+k_2+2)}{1!\Gamma(k_2+\alpha_2+5/2)S_2^{k_2+2}}+\frac{(1/2)^{k_2+\alpha_2+9/2}\Gamma(q+k_2+4)}{2!\Gamma(k_2+\alpha_2+7/2)S_2^{k_2+4}}-...+\frac{(-1)^{m_2}(1/2)^{k_2+\alpha_2+1/2+2m_2}\Gamma(q+k_2+2m_2)}{m_2!\Gamma(k_2+\alpha_2+m_2+3/2)S_2^{k_2+2m_2}}}*$$

$$\frac{1}{\frac{(1/2)^{k_3+\alpha_3+1/2}\Gamma(\beta+k_3)}{0!\Gamma(k_3+\alpha_3+3/2)S_3^{k_3}}-\frac{(1/2)^{k_3+\alpha_3+5/2}\Gamma(\beta+k_3+2)}{1!\Gamma(k_3+\alpha_3+5/2)S_3^{k_3+2}}+\frac{(1/2)^{k_3+\alpha_3+9/2}\Gamma(\beta+k_3+4)}{2!\Gamma(k_3+\alpha_3+7/2)S_3^{k_3+4}}-...+\frac{(-1)^{m_3}(1/2)^{k_3+\alpha_3+1/2+2m_3}\Gamma(\beta+k_3+2m_3)}{m_3!\Gamma(k_3+\alpha_3+m_3+3/2)S_3^{k_3+2m_3}}}$$

where

$$\theta=\frac{\bar{g}(s_1,s_2,s_3)\ V_{\alpha_1,n_1}(x_1)V_{\alpha_2,n_2}(x_2)V_{\alpha_3,n_3}(x_3)S_1^{p}S_2^{q}S_3^{\beta}}{i^{n_1+n_2+n_3}(2\pi)^{\frac{3}{2}}} \qquad \ldots(11)$$

Case 1: Taking firstly $K_1 = K_2 = K_3 = 0, 2, 4, \ldots$ in the equation (11), then:-
If $K_1 = K_2 = K_3 = 0$

$$\bar{f}(s_1,s_2,s_3)=\frac{\theta}{a_o(x_1/\alpha_1 n_1)a_0(x_2/\alpha_2 n_2)a_o(x_3/\alpha_3 n_3)}*$$

$$\frac{1}{[\frac{(1/2)^{\alpha_1+1/2}\Gamma(p)}{0!\Gamma(\alpha_1+3/2)S_1^2}-\frac{(1/2)^{\alpha_1+5/2}\Gamma(p+2)}{1!\Gamma(\alpha_1+5/2)S_1^4}+\frac{(1/2)^{\alpha_1+9/2}\Gamma(p+4)}{2!\Gamma(\alpha_1+7/2)S_1^6}-...+\frac{(-1)^{m_1}(1/2)^{\alpha_1+1/2+2m_1}\Gamma(p+2m_1)}{m_1!\Gamma(\alpha_1+m_1+3/2)S_1^{2m_1}}]}*$$

$$\frac{1}{[\frac{(1/2)^{\alpha_2+1/2}\Gamma(q)}{0!\Gamma(\alpha_2+3/2)S_2^2}-\frac{(1/2)^{\alpha_2+5/2}\Gamma(q+2)}{1!\Gamma(\alpha_2+5/2)S_2^4}+\frac{(1/2)^{\alpha_2+9/2}\Gamma(q+4)}{2!\Gamma(\alpha_2+7/2)S_2^6}-...+\frac{(-1)^{m_2}(1/2)^{\alpha_2+1/2+2m_2}\Gamma(q+2m_2)}{m_2!\Gamma(\alpha_2+m_2+3/2)S_2^{2m_2}}]}*$$

$$\frac{1}{[\frac{(1/2)^{\alpha_3+1/2}\Gamma(\beta)}{0!\Gamma(\alpha_3+3/2)s_3^2}-\frac{(1/2)^{\alpha_3+5/2}\Gamma(\beta+2)}{1!\Gamma(\alpha_3+5/2)s_3^4}+\frac{(1/2)^{\alpha_3+9/2}\Gamma(\beta+4)}{2!\Gamma(\alpha_3+7/2)s_3^6}-...+\frac{(-1)^{m_3}(1/2)^{\alpha_3+1/2+2m_3}\Gamma(\beta+2m_3)}{m_3!\Gamma(\alpha_3+m_3+3/2)s_3^{2m_3}}]}$$

If $K_1=K_2=K_3=2$

$$\bar{f}(s_1,s_2,s_3)=\frac{\theta}{a_2(x_1/\alpha_1 n_1)a_2(x_2/\alpha_2 n_2)a_2(x_3/\alpha_3 n_3)}*$$

$$\frac{1}{[\frac{(1/2)^{\alpha_1+5/2}\Gamma(p+2)}{0!\Gamma(\alpha_1+7/2)S_1^2}-\frac{(1/2)^{\alpha_1+9/2}\Gamma(p+4)}{1!\Gamma(\alpha_1+9/2)S_1^4}+\frac{(1/2)^{\alpha_1+13/2}\Gamma(p+6)}{2!\Gamma(\alpha_1+11/2)S_1^6}-....+\frac{(-1)^{m_3}(1/2)^{\alpha_1+5/2+2m_1}\Gamma(p+2m_1+2)}{m_1!\Gamma(\alpha_1+m_1+7/2)S_1^{2m_1+2}}]}*$$

$$\frac{1}{[\frac{(1/2)^{\alpha_2+5/2}\Gamma(q+2)}{0!\Gamma(\alpha_2+7/2)S_2^2}-\frac{(1/2)^{\alpha_2+9/2}\Gamma(q+4)}{1!\Gamma(\alpha_2+9/2)S_2^4}+\frac{(1/2)^{\alpha_2+13/2}\Gamma(q+6)}{2!\Gamma(\alpha_2+11/2)S_2^6}-...+\frac{(-1)^{m_2}(1/2)^{\alpha_2+5/2+2m_2}\Gamma(q+2m_2+2)}{m_2!\Gamma(\alpha_2+m_2+7/2)S_2^{2m_2+2}}]}*$$

$$\frac{1}{[\frac{(1/2)^{\alpha_3+5/2}\Gamma(\beta+2)}{0!\Gamma(\alpha_3+7/2)S_3^2}-\frac{(1/2)^{\alpha_3+9/2}\Gamma(\beta+4)}{1!\Gamma(\alpha_3+9/2)S_3^4}+\frac{(1/2)^{\alpha_3+13/2}\Gamma(\beta+6)}{2!\Gamma(\alpha_3+11/2)S_3^6}-....+\frac{(-1)^{m_3}(1/2)^{\alpha_3+5/2+2m_3}\Gamma(\beta+2m_3+2)}{m_3!\Gamma(\alpha_3+m_3+7/2)S_3^{2m_3+2}}]}$$

If $K_1 = K_2 = K_3 = 4$

$$\frac{1}{\left[\frac{(1/2)^{\alpha_1+9/2}\Gamma(p+4)}{0!\Gamma(\alpha_1+11/2)S_1^{4}}-\frac{(1/2)^{\alpha_1+13/2}\Gamma(p+6)}{1!\Gamma(\alpha_1+13/2)S_1^{6}}+\frac{(1/2)^{\alpha_1+17/2}\Gamma(p+8)}{2!\Gamma(\alpha_1+13/2)S_1^{8}}-....+\frac{(-1)^{m_1}(1/2)^{\alpha_1+9/2+2m_1}\Gamma(p+2m_1+4)}{m_1!\Gamma(\alpha_1+m_1+11/2)S_1^{2m_1+4}}\right]}$$

$$\frac{1}{\left[\frac{(1/2)^{\alpha_2+9/2}\Gamma(q+4)}{0!\Gamma(\alpha_2+11/2)S_2^{4}}-\frac{(1/2)^{\alpha_2+13/2}\Gamma(q+6)}{1!\Gamma(\alpha_2+13/2)S_2^{6}}+\frac{(1/2)^{\alpha_2+17/2}\Gamma(q+8)}{2!\Gamma(\alpha_2+13/2)S_2^{8}}-...+\frac{(-1)^{m_2}(1/2)^{\alpha_2+9/2+2m_2}\Gamma(q+2m_2+4)}{m_2!\Gamma(\alpha_2+m_2+11/2)S_2^{2m_2+4}}\right]}*$$

$$\frac{1}{\left[\frac{(1/2)^{\alpha_3+9/2}\Gamma(\beta+4)}{0!\Gamma(\alpha_3+11/2)S_3^{4}}-\frac{(1/2)^{\alpha_3+13/2}\Gamma(\beta+6)}{1!\Gamma(\alpha_3+13/2)S_3^{6}}+\frac{(1/2)^{\alpha_3+17/2}\Gamma(\beta+6)}{2!\Gamma(\alpha_3+13/2)S_3^{8}}-....+\frac{(-1)^{m_3}(1/2)^{\alpha_3+9/2+2m_3}\Gamma(\beta+2m_3+4)}{m_3!\Gamma(\alpha_3+m_3+11/2)S_3^{2m_3+4}}\right]} ....(12)$$

$$\bar{f}(s_1,s_1,s_3)=\frac{\theta}{\sum_{\lambda_1=0}^{\infty} D_0\lambda_1 S_1^{-2\lambda_1}\sum_{\lambda_2=0}^{\infty} D_1\lambda_2 S_2^{-2\lambda_2}\sum_{\lambda_3=0}^{\infty} D_2\lambda_3 S_3^{-2\lambda_3}} \qquad (13)$$

where $D_o=\dfrac{a_o(X_1/\alpha_1 n_1)(\frac{1}{2})^{\alpha_1+\frac{1}{2}}\Gamma(p)}{0!\Gamma(\alpha_1+\frac{3}{2})}\cdot\dfrac{a_0(X_2/n_2)(1/2)^{\alpha_2+1/2}\Gamma(q)}{0!\Gamma(\alpha_2+3/2)}\dfrac{a_o(X_3/\alpha_3 n_3)(\frac{1}{2})^{\alpha_3+\frac{1}{2}}\Gamma(\beta)}{0!\Gamma(\alpha_3+\frac{3}{2})}$

$$D_4=[a_4(X_1/\alpha_1 n_1)\frac{(\frac{1}{2})^{\alpha_1+\frac{9}{2}}\Gamma(p+4)}{0!\Gamma(\alpha_1+\frac{11}{2})}-a_2(X_1/\alpha_1 n_1)\frac{(\frac{1}{2})^{\alpha_1+\frac{9}{2}}\Gamma(p+4)}{1!\Gamma(\alpha_1+\frac{9}{2})}+a_0(X_1/\alpha_1 n_1)\frac{(\frac{1}{2})^{\alpha_1+\frac{9}{2}}\Gamma(p+4)}{2!\Gamma(\alpha_1+\frac{7}{2})}]*$$

$$[a_4(X_2/\alpha_2 n_2)\frac{(\frac{1}{2})^{\alpha_2+\frac{9}{2}}\Gamma(q+4)}{0!\Gamma(\alpha_2+\frac{11}{2})}-a_2(X_2/\alpha_2 n_2)\frac{(\frac{1}{2})^{\alpha_2+\frac{9}{2}}\Gamma(q+4)}{1!\Gamma(\alpha_2+\frac{9}{2})}+a_0(X_2/\alpha_2 n_2)\frac{(\frac{1}{2})^{\alpha_2+\frac{9}{2}}\Gamma(q+2)}{2!\Gamma(\alpha_2+\frac{7}{2})}]*$$

$$[a_4(X_3/\alpha_3 n_3)\frac{(\frac{1}{2})^{\alpha_3+\frac{9}{2}}\Gamma(\beta+4)}{0!\Gamma(\alpha_3+\frac{11}{2})}-a_2(X_3/\alpha_3 n_3)\frac{(\frac{1}{2})^{\alpha_3+\frac{9}{2}}\Gamma(\beta+4)}{1!\Gamma(\alpha_3+\frac{9}{2})}+a_o(X_3/\alpha_3 n_3)\frac{(\frac{1}{2})^{\alpha_3+\frac{9}{2}}\Gamma(\beta+2)}{2!\Gamma(\alpha_3+\frac{7}{2})}]$$

$$D_{2r_1}D_{2r_3}=[a_{2r_1}(X_1/\alpha_1 n_1)\frac{(\frac{1}{2})^{\alpha_1+2r_1+\frac{1}{2}}\Gamma(p+2r_1)}{0!\Gamma(\alpha_1+2r_1+\frac{3}{2})}-a_{2r_1-2}(X_1/\alpha_1 n_1)\frac{(\frac{1}{2})^{\alpha_1+2r_1+\frac{1}{2}}\Gamma(p+2r_1)}{1!\Gamma(\alpha_1+2r_1+\frac{1}{2})}+\ldots+(-1)^{r_1}a_o(X_1/\alpha_1 n_1)\frac{(\frac{1}{2})^{\alpha_1+2r_1+\frac{1}{2}}\Gamma(p+2r_1)}{r_1!\Gamma(\alpha_1+r_1+\frac{3}{2})}]$$

$$[a_{2r_2}(X_2/\alpha_2 n_2)\frac{(\frac{1}{2})^{\alpha_{23}+2r_2+\frac{1}{2}}\Gamma(q+2r_2)}{0!\Gamma(\alpha_2+2r_2+\frac{3}{2})}-a_{2r_2-2}(X_2/\alpha_2 n_2)\frac{(\frac{1}{2})^{\alpha_{23}+2r_{32}+\frac{1}{2}}\Gamma(q+2r_2)}{1!\Gamma(\alpha_2+2r_2+\frac{1}{2})}+\ldots+(-1)^{r_2}a_o(X_2/\alpha_2 n_2)\frac{(\frac{1}{2})^{\alpha_2+2r_{23}+\frac{1}{2}}\Gamma(q+2r_2)}{r_2!\Gamma(\alpha_2+r_2+\frac{3}{2})}]$$

$$[a_{2r_3}(X_3/\alpha_3 n_3)\frac{(\frac{1}{2})^{\alpha_3+2r_3+\frac{1}{2}}\Gamma(\beta+2r_3)}{0!\Gamma(\alpha_3+2r_3+\frac{3}{2})}-a_{2r_n-2}(X_n/\alpha_n n_n)\frac{(\frac{1}{2})^{\alpha_3+2r_{3n}+\frac{1}{2}}\Gamma(\beta+2r_3)}{1!\Gamma(\alpha_3+2r_3+\frac{1}{2})}+\ldots+(-1)^{r_n}a_o(X_3/\alpha_3 n_3)\frac{(\frac{1}{2})^{\alpha_3+2r_3+\frac{1}{2}}\Gamma(\beta+2r_3)}{r_3!\Gamma(\alpha_3+r_3+\frac{3}{2})}].$$

Thus

$$\bar{f}(s_1,s_2,s_3)=\frac{\bar{g}(s_1,s_2,s_3)\ V_{\alpha_1,n_1}(x_1)V_{\alpha_2,n_2}(x_2)V_{\alpha_3,n_3}(x_3)S_1^{P}S_2^{q}S_3^{\beta}}{i^{n_1+n_2+n_3}(2\pi)^{\frac{3}{2}}\sum_{\lambda_1=0}^{\infty}D_0\lambda_1 S_1^{-2\lambda_1}\sum_{\lambda_2=0}^{\infty}D_1\lambda_2 S_2^{-2\lambda_2}\sum_{\lambda_3=0}^{\infty}D_2\lambda_3 S_3^{-2\lambda_3}} \qquad \ldots(14)$$

As $D_o \neq 0$ the series in (14) can be reciprocated. Hence

$$\bar{f}(s_1,s_2,s_3)=\frac{\bar{g}(s_1,s_2,s_3)\ V_{\alpha_1,n_1}(x_1)V_{\alpha_2,n_2}(x_2)V_{\alpha_3,n_3}(x_3)S_1^{p}S_2^{q}S_3^{\beta}}{i^{n_1+n_2+n_2}(2\pi)^{\frac{n}{2}}\sum_{u_1=0}^{\infty}E_{2u_1}S_1^{-2u_1}\sum_{u_2=0}^{\infty}E_{2u_2}S_2^{-2u_2}\sum_{u_3=0}^{\infty}E_{2u_3}S_3^{-2u_3}} \quad \text{…. (15)}$$

where, $E_{2u_1}D_o+E_{2u_1-2}D_2+\ldots+E_oD_{2u_1}=0$

$E_{2u_2}D_o+E_{2u_{21}-2}D_2+\ldots+E_oD_{2u_2}=0$

and $E_{2u_3}D_o+E_{2u_3-2}D_2+\ldots+E_oD_{2u_3}=0$

As the leading coefficient E_o of the series is not zero, the function has no zero for $\left|\frac{1}{S_1,S_3}\right|<\varepsilon$ for some ε>0, meaning there by the convergence is true for $\left|S_1,S_3\right|>\frac{1}{\varepsilon}$, thus:

$$\bar{f}(s_1,s_2,s_3)=S_1^{-r_1+p}S_2^{-r_2+q}S_3^{-r_3+\beta}\left(\sum_{u_1=0}^{\infty}Q_1E_{2u_1}S_1^{-2u_1}\sum_{u_2=0}^{\infty}Q_2E_{2u_2}S_2^{-2u_2}\sum_{u_3=0}^{\infty}Q_3E_{3u_3}S_3^{3u_3}\right)S_1^{r_1}S_2^{r_2}S_3^{r_3}\,\bar{g}(s_1,s_2,s_3)\text{....(16)}$$

where

$$Q_1=\frac{V_{\alpha_1,n_1}(x_1)}{i^{n_1}\sqrt{2\pi}}\quad,\ Q_2=\frac{V_{\alpha_2,n_2}(x_2)}{i^{n_2}\sqrt{2\pi}}\ \text{ and }\ Q_3=\frac{V_{\alpha_3,n_3}(x_3)}{i^{n_3}\sqrt{2\pi}}$$

which can be intrepreted as convolution integral provided

min(r_1-p)>0, p<0 and r_1 is a +ve integer ,then r_1>p.

min(r_2-q)>0, q<0 and r_2 is a +ve integer ,then r_2>q.

min(r_3-β)>0, β>0 and r_3 is a +ve integer ,then r_3>β.

<u>Case (2)</u>: Taking $K_1=K_2=K_3=1,3,5,...$ in the equation (11), then

$$\bar{f}(s_1,s_2,s_3)=\frac{\theta}{a_1(x_1/\alpha_1 n_1).a_1(x_2/\alpha_2 n_{.2})a_1(x_3/\alpha_3 n_3)} * \frac{\theta}{a_2(x_1/\alpha_1 n_1).a_2(x_2/\alpha_2 n_{.2})a_2(x_3/\alpha_3 n_3)}\,\frac{n!}{r!\left(n-r\right)!}$$

$$\frac{1}{[\frac{(1/2)^{\alpha_1+3/2}\Gamma(p+1)}{0!\Gamma(\alpha_1+5/2)S_1^1}-\frac{(1/2)^{\alpha_1+7/2}\Gamma(p+3)}{1!\Gamma(\alpha_1+7/2)S_1^3}+\frac{(1/2)^{\alpha_1+11/2}\Gamma(p+5)}{2!\Gamma(\alpha_1+9/2)S_1^5}+....+\frac{(-1)^{m_1}(1/2)^{\alpha_1+2m_1+3/2}\Gamma(p+2m_1+1)}{m_1!\Gamma(\alpha_1+m_1+5/2)S_1^{2m_1+1}}]}*$$

$$\frac{1}{[\frac{(1/2)^{\alpha_2+3/2}\Gamma(q+1)}{0!\Gamma(\alpha_2+5/2)S_2^1}-\frac{(1/2)^{\alpha_2+7/2}\Gamma(q+3)}{1!\Gamma(\alpha_2+7/2)S_2^3}+\frac{(1/2)^{\alpha_2+11/2}\Gamma(q+5)}{2!\Gamma(\alpha_2+9/2)S_2^5}+....+\frac{(-1)^{m_2}(1/2)^{\alpha_2+2m_2+3/2}\Gamma(q+2m_2+1)}{m_2!\Gamma(\alpha_2+m_2+5/2)S_2^{2m_2+1}}]}*$$

$$\frac{1}{[\frac{(1/2)^{\alpha_3+3/2}\Gamma(\beta+1)}{0!\Gamma(\alpha_3+5/2)S_3^1}-\frac{(1/2)^{\alpha_3+7/2}\Gamma(\beta+3)}{1!\Gamma(\alpha_3+7/2)S_3^3}+\frac{(1/2)^{\alpha_3+11/2}\Gamma(\beta+5)}{2!\Gamma(\alpha_3+9/2)S_3^5}+...+\frac{(-1)^{m_3}(1/2)^{\alpha_3+2m_3+3/2}\Gamma(\beta+2m_3+1)}{m_3!\Gamma(\alpha_3+m_3+5/2)S_3^{2m_3+1}}]}*$$

$$+[\frac{\theta}{a_3(x_1/\alpha_1 n_1)a_3(x_2/\alpha_2 n_2)a_3(x_3/\alpha_3 n_3)}]$$

$$\frac{1}{[\frac{(1/2)^{\alpha_1+7/2}\Gamma(p+3)}{0!\Gamma(\alpha_1+9/2)S_1^3}-\frac{(1/2)^{\alpha_1+11/2}\Gamma(p+5)}{1!\Gamma(\alpha_1+11/2)S_1^5}+\frac{(1/2)^{\alpha_1+15/2}\Gamma(p+7)}{2!\Gamma(\alpha_1+13/2)S_1^7}-...+\frac{(-1)^{m_1}(1/2)^{\alpha_1+7/2+2m_1}\Gamma(p+2m_1+3)}{m_1!\Gamma(\alpha_1+m_1+9/2)S_1^{2m_1+3}}]}*$$

$$\frac{1}{[\frac{(1/2)^{\alpha_2+7/2}\Gamma(q+3)}{0!\Gamma(\alpha_2+9/2)S_2^3}-\frac{(1/2)^{\alpha_2+11/2}\Gamma(q+5)}{1!\Gamma(\alpha_2+11/2)S_2^5}+\frac{(1/2)^{\alpha_2+15/2}\Gamma(q+7)}{2!\Gamma(\alpha_3+13/2)S_2^7}-...+\frac{(-1)^{m_2}(1/2)^{\alpha_3+7/2+2m_3}\Gamma(q+2m_2+3)}{m_2!\Gamma(\alpha_2+m_2+9/2)S_2^{2m_2+3}}]}*$$

$$\frac{1}{[\frac{(1/2)^{\alpha_3+7/2}\Gamma(\beta+3)}{0!\Gamma(\alpha_3+9/2)S_3^3}-\frac{(1/2)^{\alpha_3+11/2}\Gamma(\beta+5)}{1!\Gamma(\alpha_3+11/2)S_3^5}+\frac{(1/2)^{\alpha_3+15/2}\Gamma(\beta+7)}{2!\Gamma(\alpha_3+13/2)S_3^7}-...+\frac{(-1)^{m_3}(1/2)^{\alpha_3+7/2+2m_3}\Gamma(\beta+2m_3+3)}{m_3!\Gamma(\alpha_3+m_3+9/2)S_3^{2m_3+3}}]}*$$

$$+[\frac{\theta}{a_5(x_1/\alpha_1 n_1)a_5(x_2/\alpha_2 n_2)a_5(x_3/\alpha_3 n_3)}]*$$

$$\frac{1}{[\frac{(\frac{1}{2})^{\alpha_1+11/2}\Gamma(p+5)}{0!\Gamma(\alpha_1+13/2)S_1^5}-\frac{(\frac{1}{2})^{\alpha_1+15/2}\Gamma(p+7)}{1!\Gamma(\alpha_1+15/2)S_1^7}+\frac{(\frac{1}{2})^{\alpha_1+19/2}\Gamma(p+9)}{2!\Gamma(\alpha_1+17/2)S_1^9}-....+\frac{(-1)^{m_1}(\frac{1}{2})^{\alpha_1+11/2+2m_1}\Gamma(p+2m_1+5)}{m_1!\Gamma(\alpha_1+m_1+13/2)S_1^{2m_1+5}}]}*$$

$$\frac{1}{[\frac{(\frac{1}{2})^{\alpha_2+11/2}\Gamma(q+5)}{0!\Gamma(\alpha_2+13/2)S_2^5}-\frac{(\frac{1}{2})^{\alpha_2+15/2}\Gamma(q+7)}{1!\Gamma(\alpha_2+15/2)S_2^7}+\frac{(\frac{1}{2})^{\alpha_2+19/2}\Gamma(q+9)}{2!\Gamma(\alpha_2+17/2)S_2^9}-...+\frac{(-1)^{m_2}(\frac{1}{2})^{\alpha_2+11/2+2m_2}\Gamma(q+2m_2+5)}{m_2!\Gamma(\alpha_2+m_2+13/2)S_2^{2m_{21}+5}}]}*$$

$$\frac{1}{[\frac{(\frac{1}{2})^{\alpha_3+11/2}\Gamma(\beta+5)}{0!\Gamma(\alpha_3+13/2)S_3^5}-\frac{(\frac{1}{2})^{\alpha_3+15/2}\Gamma(\beta+7)}{1!\Gamma(\alpha_3+15/2)S_3^7}+\frac{(\frac{1}{2})^{\alpha_3+19/2}\Gamma(\beta+9)}{2!\Gamma(\alpha_3+17/2)S_3^9}-...+\frac{(-1)^{m_3}(\frac{1}{2})^{\alpha_3+11/2+2m_3}\Gamma(\beta+2m_3+5)}{m_3!\Gamma(\alpha_3+m_3+13/2)S_3^{2m_3+5}}]}......(17)$$

$$\bar{f}(s_1,s_1,s_3)=\frac{\theta}{(\sum_{\lambda_1=0}^{\infty}D_1\lambda_{1+1}S_1^{-(2\lambda_1+1)}\sum_{\lambda_2=0}^{\infty}D_3\lambda_{2+1}S_2^{-(2\lambda_2+1)}\sum_{\lambda_3=0}^{\infty}D_5\lambda_{3+1}S_3^{-(2\lambda_3+1)})^{-1}}..............(18)$$

$$where\ D_1=[a_1(X_1/\alpha_1 n_1)\frac{(\frac{1}{2})^{\alpha_1+3/2}\Gamma(p+1)}{0!\Gamma(\alpha_1+5/2)}.a_1(X_2/\alpha_2 n_2)\frac{(1/2)^{\alpha+3/2}\Gamma(q+1)}{0!\Gamma(\alpha_2+5/2)}.a_1(X_3/\alpha_3 n_3)\frac{(\frac{1}{2})^{\alpha_3+3/2}\Gamma(\beta+1)}{0!\Gamma(\alpha_3+5/2)}]$$

$$[a_3(X_2/\alpha_2 n_2)\frac{(\frac{1}{2})^{\alpha_{21}+7/2}\Gamma(q+3)}{0!\Gamma(\alpha_2+9/2)}-a_1(X_2/\alpha_2 n_2)\frac{(\frac{1}{2})^{\alpha_2+7/2}\Gamma(q+3)}{1!\Gamma(\alpha_2+7/2)}]*$$

$$[a_3(X_3/\alpha_3 n_3)\frac{(\frac{1}{2})^{\alpha_{31}+7/2}\Gamma(\beta+3)}{0!\Gamma(\alpha_3+9/2)}-a_1(X_3/\alpha_3 n_3)\frac{(\frac{1}{2})^{\alpha_3+7/2}\Gamma(\beta+3)}{1!\Gamma(\alpha_3+7/2)}]$$

$$D_5=[a_5(X_1/\alpha_1 n_1)\frac{(\frac{1}{2})^{\alpha_1+11/2}\Gamma(p+5)}{0!\Gamma(\alpha_1+13/2)}-a_3(X_1/\alpha_1 n_1)\frac{(\frac{1}{2})^{\alpha_1+11/2}\Gamma(p+5)}{1!\Gamma(\alpha_1+11/2)}+a_1(X_1/\alpha_1 n_1)\frac{(\frac{1}{2})^{\alpha_1+11/2}\Gamma(p+5)}{2!\Gamma(\alpha_1+9/2)}]*$$

$$[a_5(X_2/\alpha_2 n_2)\frac{(1/2)^{\alpha_2+11/2}\Gamma(q+5)}{0!\Gamma(\alpha_2+13/2)}-a_3(X_2/\alpha_2 n_2)\frac{(1/2)^{\alpha_2+11/2}\Gamma(q+5)}{1!\Gamma(\alpha_2+11/2)}+a_1(X_2/\alpha_2 n_2)\frac{(1/2)^{\alpha_2+11/2}\Gamma(q+5)}{2!\Gamma(\alpha_2+9/2)}]*$$

$$[a_5(X_3/\alpha_3 n_3)\frac{(1/2)^{\alpha_3+11/2}\Gamma(\beta+5)}{0!\Gamma(\alpha_3+13/2)}-a_3(X_3/\alpha_3 n_3)\frac{(1/2)^{\alpha_3+11/2}\Gamma(\beta+5)}{1!\Gamma(\alpha_3+11/2)}+a_1(X_3/\alpha_3 n_3)\frac{(1/2)^{\alpha_3+11/2}\Gamma(\beta+5)}{2!\Gamma(\alpha_3+9/2)}]$$

$$D_{2r_1+1}D_{2r_3+1}=[a_{2r_1+1}(X_1/\alpha_1 n_1)\frac{(1/2)^{\alpha_1+2r_1+3/2}\Gamma(p+2r_1+1)}{0!\Gamma(\alpha_1+2r_1+5/2)}-a_{2r_1-1}(X_1/\alpha_1 n_1)\frac{(1/2)^{\alpha_1+2r_1+3/2}\Gamma(p+2r_1+1)}{1!\Gamma(\alpha_1+2r_1+3/2)}+\ldots+(-1)^{r_1}a_1(X_1/\alpha_1 n_1)\frac{(1/2)^{\alpha_1+2r_1+3/2}\Gamma(p+2r_1+1)}{r_1!\Gamma(\alpha_1+r_1+5/2)}]*$$

$$[a_{2r_{2+1}}(X_2/\alpha_2 n_2)\frac{(1/2)^{\alpha_{23}+2r_2+3/2}\Gamma(q+2r_2+1)}{0!\Gamma(\alpha_2+2r_2+5/2)}-a_{2r_2-1}(X_2/\alpha_2 n_2)\frac{(1/2)^{\alpha_2+2r_2+3/2}\Gamma(q+2r_3+1)}{1!\Gamma(\alpha_2+2r_2+3/2)}+\ldots+(-1)^{r_2}a_1(X_2/\alpha_2 n_2)\frac{(1/2)^{\alpha_2+2r_2+3/2}\Gamma(q+2r_2+1)}{r_2!\Gamma(\alpha_2+r_2+5/2)}]*$$

$$[a_{2r_{3+1}}(X_3/\alpha_3 n_3)\frac{(1/2)^{\alpha_3+2r_3+3/2}\Gamma(\beta+2r_3+1)}{0!\Gamma(\alpha_3+2r_3+5/2)}-a_{2r_n-1}(X_3/\alpha_3 n_3)\frac{(1/2)^{\alpha_3+2r_3+3/2}\Gamma(P+2r_3+1)}{1!\Gamma(\alpha_3+2r_3+3/2)}+\ldots+(-1)^{r_3}a_1(X_3/\alpha_3 n_3)\frac{(1/2)^{\alpha_3+2r_3+3/2}\Gamma(\beta+2r_3+1)}{r_3!\Gamma(\alpha_3+r_3+5/2)}]$$

Then

$$\bar{f}(s_1,s_2,s_3)=\frac{\bar{g}(s_1,s_2,s_3)\,V_{\alpha_1/n_1}(x_1)V_{\alpha_2/n_2}(x_2)V_{\alpha_3/n_3}(x_3)S_1^{P}S_2^{q}S_3^{\beta}}{i^{n_1+n_2+n_3}(2\pi)^{\frac{n}{2}}(\sum_{\lambda_1=0}^{\infty}D_1\lambda_{1+1}S_1^{-(2\lambda_1+1)}.\sum_{\lambda_2=0}^{\infty}D_3\lambda_{2+1}S_2^{-(2\lambda_2+1)}\sum_{\lambda_3=0}^{\infty}D_5\lambda_{3+1}S_3^{-(2\lambda_3+1)}} \quad (19)$$

Then we get

$$\bar{f}(s_1,s_2,s_3)=\frac{\bar{g}(s_1,s_2,s_3)\;V_{\alpha_1,n_1}(x_1)V_{\alpha_2,n_2}(x_2)V_{\alpha_3,n_3}(x_3)}{i^{n_1+n_2+n_3}(2\pi)^{\frac{n}{2}}\sum_{u_1=0}^{\infty}E_{2u_1+1}S_1^{-(2u_1+1)}\sum_{u_2=0}^{\infty}E_{2u_2+1}S_2^{-(2u_2+1)}\sum_{u_3=0}^{\infty}E_{2u_3+1}S_3^{-(2u_3+1)}}\ldots(20)$$

where

$E_1D_1 = 1$

$$E_{2u_1+1}D_1 + E_{2u_1-1}D_3 + \ldots + D_{2u_1+1}E_1 = 0,$$

$$E_{2u_2+1}D_1 + E_{2u_{21}-1}D_3 + \ldots + D_{2u_2+1}E_1 = 0$$

and $$E_{2u_3+1}D_1 + E_{2u_3-1}D_3 + \ldots + D_{2u_3+1}E_1 = 0$$

$$\bar{f}(s_1,s_2,s_3) = S_1^{-r_1+p}.S_2^{-r_2+q}S_3^{-r_3+\beta}\left(\sum_{u_1=0}^{\infty} Q_1E_{2u_1+1}S_1^{-(2u_1+1)}\sum_{u_2=0}^{\infty} Q_2E_{2u_2+1}S_2^{-(2u_2+1)}\sum_{u_3=0}^{\infty} Q_3E_{2u_3+1}S_3^{-(2u_3+1)}\right)S_1^{r_1}S_2^{r_2}S_3^{r_3}\bar{g}(s_1,s_2,s_3) \ldots (21)$$

where $Q_1 = \dfrac{V_{\alpha_1,n_1}(x_1)}{i^{n_1}\sqrt{2\pi}}$, $Q_2 = \dfrac{V_{\alpha_2,n_2}(x_2)}{i^{n_2}\sqrt{2\pi}}$ and $Q_3 = \dfrac{V_{\alpha_3,n_3}(x_3)}{i^{n_3}\sqrt{2\pi}}$

<u>Theorem (2):-</u> The convolution integral equation of 3- variables

$$f(x_1,x_2,x_3) = \int_0^{x_1}\int_0^{x_2}\int_0^{x_3}(x_1-t_1)^{r_1-p_1-1}(x_2-t_2)^{r_2-q-1}(x_3-t_3)^{r_3-\beta-1}\psi_{r_1-p}(x_1-t_1)\psi_{r_2-q}(x_2-t_2)\psi_{r_3-\beta}(x_3-t_3)D_{t_1}^{r_1}[g(t_1)] - D_{t_n}^{r_n}[g(t_n)]dt_n \ldots\ldots dt_1$$

where $\psi_{r_1-p}(t_1) = \sum_{u_1=0}^{\infty}\dfrac{Q_1E_{2u_1+1}\,t_1^{2u_1+1}}{\Gamma(2u_1+1+r_1-p)}$, $\psi_{r_2-q}(t_2) = \sum_{u_2=0}^{\infty}\dfrac{Q_2E_{2u_2+1}\,t_2^{2u_2+1}}{\Gamma(2u_2+1+r_2-q)}$ and

$$\psi_{r_3-\beta}(t_3) = \sum_{u_n=0}^{\infty}\frac{Q_3E_{2u_3+1}\,t_3^{2u_3+1}}{\Gamma(2u_3+1+r_3-\beta)}$$

<u>Particular Case</u>

If t_1,t_2,t_3 are taken to be zero this reduces to the convolution integral equation involving Bessel functions of 3- variables.

<u>Conclusion</u>

The analytic solution of integral equation involving Spheroidal Wave Functions of three variables was introduced by using Laplace transform and convolution theorem. we get two cases the first case $K_1=K_2=K_3= 0, 2, 4,\ldots$ and in the second case $K_1=K_2=K_3 = 1, 3, 5,\ldots$ we get two distinct results which is indicated in both equations (16)and (21). After apply Laplace transform and convolution theorem we got the Analytic Solution of it.

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الحل التحليلي لمعادلة تكاملية تحتوي ثلاث دوال موجية لثلاثة متغيرات

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الخلاصة

في هذا البحث استخدمنا تحويل لابلاس و مبرهنة (Convolution theorem) لايجاد الحل التحليلي لمعادلة تكاملية تشمل ثلاث دوال موجية لثلاثة متغيرات، صيغتها هي

$$g(x_1,x_2,x_3)=\int_0^{x_1}.\int_0^{x_2}\int_0^{x_3}(x_1-t_1)^{\rho-1}(x_2-t_2)^{q-1}(x_3-t_3)^{\beta-1}\phi_{\alpha_1,n_1}(x_1-t_1).\phi_{\alpha_2,n_2}(x_2-t_2)\phi_{\alpha_3,n_3}(x_3-t_3)f(t_1,t_2,t_3)dt_1dt_2dt_3$$

بشرط $\phi_{\alpha_i,n_i}(x-t), i=1,2,3$ دوال موجية و يعبر كالاتي:-

$$\phi_{\alpha_1,n_1}(x_1-t_1)=\frac{i^{n_1}\sqrt{2\pi}}{V_{\alpha_1,n_1}(x_1)}\sum_{k_1=0,1}^{\infty}a_{k_1}(x_1/\alpha_1 n_1)\frac{J_{k_1+\alpha_1+1/2}(x_1-t_1)}{(x_1-t_1)^{\alpha_1+1/2}}$$

$$\phi_{\alpha_2,n_2}(x_2-t_2)=\frac{i^{n_2}\sqrt{2\pi}}{V_{\alpha_2,n_2}(x_2)}\sum_{k_1=0,1}^{\infty}a_{k_2}(x_2/\alpha_2 n_2)\frac{J_{k_2+\alpha_2+1/2}(x_2-t_2)}{(x_2-t_2)^{\alpha_2+1/2}}$$

$$\phi_{\alpha_3,n_3}(x_3-t_3)=\frac{i^{n_3}\sqrt{2\pi}}{V_{\alpha_3,n_3}(x_3)}\sum_{k_1=0,1}^{\infty}a_{k_3}(x_3/\alpha_3 n_3)\frac{J_{k_3+\alpha_3+1/2}(x_3-t_3)}{(x_3-t_3)^{\alpha_1+1/2}}$$

و يعتبرهذه دوال متقاربة بشكل منتظم على فترة $(-\infty,\infty)$.