

Noetherian Fuzzy Rings

حلقات ضبابية نوثيرينية

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ABSTRACT :

In this paper ,we have introduced the concept of Noetherian fuzzy rings as a generalization of the Noetherian rings .We have given several basic properties and theorems about this concept .

المستخلص :

في هذا البحث قدمنا مفهوم الحلقات الضبابية النوثيرينية كتعميم للحلقات النوثيرينية الاعتيادية وأوضحنا بعض الخواص الأساسية والمبرهنات حول هذا الموضوع .

INTRODUCTION :

In 1965, (Zadeh L.A.) introduced the concept of fuzzy subset. Since that time many papers were introduced in different mathematical scopes of theoretical and practical applications.

In 1982, (Liu W.J.) formulated the term of fuzzy ring and fuzzy ideal of a ring R.

The main aim of this paper is to introduce the concept " the Noetherian fuzzy rings " . We are going to study and prove some properties about this concept.

1. Preliminary Concepts

Let R be a non empty set and $(R, +, \cdot)$ be a ring . **A fuzzy subset of R** is a function from R into $[0, 1]$, ([3], [4]).

Let A and B be fuzzy sets of R .We write $A \subseteq B$ if $A(x) \leq B(x)$ for all $x \in R$. If $A \subseteq B$ and there exist $x \in R$ such that $A(x) < B(x)$, then we write $A \subset B$,([3] and we say that A is a proper fuzzy subset of B, We write $A = B$ if and only if $A(x) = B(x)$, for all $x \in R$, [1].

We let ϕ denote $\phi(x) = 0$ for all $x \in R$, the **empty fuzzy subset of R** , ([1], [2]).

When we say fuzzy subset we mean a non empty fuzzy subset. We let $\text{Im}(A)$ denotes **the image of A**. We say that A is a **finite –valued** if $\text{Im}(A)$ is finite and $|\text{Im}(A)|$ denote **the cardinality of Im(A)** , ([1], [5]).

For each $t \in [0,1]$, the set $A_t = \{x \in R : A(x) \geq t\}$ is called **a level subset of R** and the set $A_* = \{x \in R : A(x) = A(0)\}$, ([3],[4]). Now, Let $(R, +, \cdot)$ be a ring with identity. Let X be a fuzzy subset of R , X is called **a fuzzy ring of R** if for all $x, y \in R$, $X(x-y) \geq \min\{X(x), X(y)\}$ and $X(x \cdot y) \geq \min\{X(x), X(y)\}$ and a **fuzzy subring A** of a fuzzy ring X is a fuzzy ring of R satisfying $A(x) \leq X(x)$ for all $x \in R$, ([1] ,[2]). A fuzzy subset A of R is called **a fuzzy ideal of R** if and only if for all $x, y \in R$, $A(x-y) \geq \min\{A(x), A(y)\}$ and $A(x \cdot y) \geq \max\{A(x), A(y)\}$, ([2],[4]). It is clear that every fuzzy ideal of R is a fuzzy ring of R but the converse is not true, [6] . Note that if A is a fuzzy ideal of R, $t \in [0, 1]$, then A_t and A_* are ideals of R , [7] .

2. Noetherian Fuzzy Rings

In this section, we shall introduce the concept " Noetherian fuzzy rings " and we shall see that this concept is equivalent to definition (2.1).

Definition 2.1 [8] :

An R-module M is said to be Noetherian (Artinian) iff for every ascending (descending) sequences of submodules M_n of M

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \dots ; (M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots)$$

There is an integer k such that $M_k = M_{k+i}$, $i=1,2,3,\dots$ we characterize the Noetherian module .

Remark 2.2 :

When we discuss the concept Noetherian fuzzy ring as a Noetherian ring, we cannot define this concept as: Any ascending chain of fuzzy ideals of a fuzzy ring of R are terminates .The following result explains this.

Theorem 2.3 :

For any fuzzy ring X of R , there exists an ascending chain of fuzzy ideals of X which does not terminate.

Proof:

Let $X(0)=t$. then $0 < t \leq 1$ choose $t_1 \in (0,1)$, such that $t_1 < t$.Hence there exist an infinite numbers t_2, t_3, \dots between t_1, t such that $t_1 < t_2 < t_3 < \dots < t$.

For each $i = 1, 2, 3, \dots$ put $A_i(a) = t_i$ if $a=0$, $A_i(a) = 0$ if $a \neq 0$.

It is clear that A_i is a fuzzy ideal of X and $A_1 \subset A_2 \subset A_3 \subset \dots$, which does not terminate.

Now ,we give the concept of Noetherian fuzzy rings.

Definition 2.4 :

A fuzzy ring X of R is called Noetherian (Artinian) if R is a Noetherian (Artinian) ring.

Lemma 2.5 [9].:

Any commutative Artinian ring with identity is Noetherian .

Remark 2.6 :

Let X be a fuzzy ring of R .If X is Artinian, then X is Noetherian.

Proof:

Since X is an Artinian fuzzy ring, then R is an Artinian ring by definition (2.4),which implies that R is a Noetherian ring by lemma (2.5) .

Hence X is a Noetherian fuzzy ring by definition (2.4).

The converse of remark (2.6) is not true as the following example shows.

Examples:

Let Z be the ring of integers , we know that Z is a Noetherian ring but not an Artinian ring, every fuzzy ring of Z is a Noetherian fuzzy ring, but not an Artinian fuzzy ring by definition (2.4).

Definition 2.7 [10] :

If L is a partially ordered set, L is said to satisfy the ascending (resp.descending) chain condition if every sequence of element $t_1 < t_2 < t_3 < \dots$ ($t_1 > t_2 > t_3 > \dots$) is finite.

Lemma 2.8 [10] :

Let X be a unital fuzzy ring of R , with $R(0) \geq X(0)$. If X is Noetherian (Artinian), then the set $K = \{X(a) : a \in R\}$ satisfies the descending (ascending) chain condition.

Now, we can prove the following properties.

Proposition 2.9 :

Let X be a Noetherian (Artinian) fuzzy ring over L .Then every ascending (descending) chain of the fuzzy ideals of X terminates.

Proof:

If X is a Noetherian fuzzy ring ,let $A_1 \subseteq A_2 \subseteq \dots$ be an ascending chain of fuzzy ideals of X .Suppose this chain does not terminate ,then for all $n \in \mathbb{N}$, there exist $p \in \mathbb{N}$ such that $A_n < A_{n+p}$.

Hence there exists $x_n \in R$ such that $A_n(x_n) < A_{n+p}(x_n)$.

$S = \{ A_n(x_n) : n \in \mathbb{N} \} \subseteq \text{Im } X$ which is a contradiction , since S is an infinite set and $\text{Im } X$ is a finite set . The chain must terminate.

The proof is analogous to the Artinian case.

Theorem 2.10 :

X is a Noetherian (Artinian) fuzzy ring over L if and only if X_t is a Noetherian (Artinian) ring, for all $t \in [0, 1]$.

Proof:

If X is a Noetherian fuzzy ring, to prove that X_t is a Noetherian ring, for all t .

If $t=0$, $X_0 = R$ (R is Noetherian ring by definition (2.4).

For $t \in (0, 1]$, let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of ideals in X_t .

For each $x \in I_1 \subseteq X_t \Rightarrow X(x) \geq t$

$x \in I_2 \subseteq X_t \Rightarrow X(x) \geq t$

Let $A_i : R \Rightarrow L$ for each $i=1,2,\dots$ such that :

$$A_i(x) = \begin{cases} X(x) & \text{if } x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

A_i is a fuzzy ideal of X for all $i=1,2,\dots$ and $A_1 \subseteq A_2 \subseteq \dots$

Hence by proposition (2.9),there exists $n \in \mathbb{N}$ such that $A_n = A_{n+p} = \dots$

It follows that $I_n = I_{n+1} = \dots$.Thus X_t is a Noetherian ring .

Similarly , X_t is an Artinian ring .

Conversely , if X_t is a Noetherian (Artinian) ring ,for all $t \in [0, 1]$,then $X_0 = R$ is a Noetherian (Artinian) ring ,which implies that X is a Noetherian (Artinian) fuzzy ring by definition (2.4) .

Corollary 2.11 :

Let X be a fuzzy ring of R . X is a Noetherian (Artinian) if and only if X_t is a Noetherian (Artinian) ring ,for all $t \in [0,1]$ such that t is a minimal element of $\text{Im} X$

The proof of this corollary is clear .

Corollary 2.12 :

X_t is a Noetherian (Artinian) ring over L ,for all $t \in [0,X(0)]$. $\text{Im} X$ is a finite set ,then every an ascending chain of fuzzy ideals of X terminates .

Proof :

Let $A_1 \subseteq A_2 \subseteq \dots$ be an ascending chain of fuzzy ideals of X .

Let $\text{Im} X = \{ t_1, t_2, \dots, t_n \}$, then $(A_1)_{t_i} \subseteq (A_2)_{t_i} \subseteq \dots$ for all $i=1,2,\dots,n$ is an ascending chain in X_{t_i} which implies that there exists $m_i \in \mathbb{N}$ such that :

$$(A_{m_i})_{t_i} = (A_{m_i+1})_{t_i} = \dots$$

Let $m = \max \{ m_1, m_2, \dots \}$.Then $(A_m)_t = (A_{m+1})_t = \dots$ for all $t \in L$. Thus $A_m = A_{m+1} = \dots$, thus X satisfies ascending chain of fuzzy ideals of R .

Hence X satisfies the ascending chain of fuzzy ideals of X .

The proof is analogous in the Artinian case .

Lemma 2.13 [7] :

Let R be a ring with unity . R is Artinian iff every fuzzy ideal of R is finite valued .

Proposition 2.14 :

Let X be a fuzzy ring of R .Then X is an Artinian fuzzy ring ,if every fuzzy ideal A of X has sup property is a finite –valued .

Proof :

Since X is an Artinian fuzzy ring ,then R is an Artinian ring by definition (2.4).

Thus by lemma (2.13), every fuzzy ideal A of R is a finite –valued which implies that every fuzzy ideal A of X is has sup property is a finite –valued .

Proposition 2.15 :

Let A be a fuzzy ideal of R , if every fuzzy ideals of R is a finite –valued ,then every fuzzy ideals of R is a finitely generated .

Proof :

Since every fuzzy ideal A of R is a finite –valued, then R is Artinian by Proposition(2.14) and [10], which implies that R is Noetherian ring by Lemma (2.5) .

Hence every fuzzy ideal of R is finite generated by [10] .

The converse of this proposition is not true .

Proposition 2.16 :

Let X be a fuzzy ring of R .Then X is an Artinian ,if every fuzzy ideal A of X is finitely generated .

Proof :

Since X is Artinian ,then every fuzzy ideal A of X which has sup property is a finite –valued by proposition (2.14) .

By proposition (2.15) ,every fuzzy ideal A of R is finitely generated .

Corollary 2.17 :

Let X be a fuzzy ring of R . If every fuzzy ideal A of X is finitely generated ,then X is a Noetherian fuzzy ring .

Proof :

Since every fuzzy ideal A of R is finitely generated ,then R is an Artinian ring by Proposition(2.14) and [10] .By [9] , R is a Noetherian ring which implies that X is a Noetherian fuzzy ring by definition (2.4) .

Corollary 2.18 :

Let X be a fuzzy ring of R . If every fuzzy ideal A of R has a sup property is a finite –valued ,then X is Noetherian .

It follows directly by proposition (2.14) and Corollary (2.17) .

The converse of Corollary (2.17) and Corollary (2.18) are not true as the following example shows

Examples :

Let Z be the ring of integers , we know that Z is a Noetherian ring but not Artinian ring . Thus if $\text{Im}(I)$ is finite for all fuzzy ideal I of Z , then Z is Artinian ring ,by lemma (2.13) , which is not the case . Thus there exists a fuzzy ideal I of Z such that $\text{Im}(I)$ is not finite since the chain of ideals $\langle 2 \rangle \supset \langle 2^2 \rangle \supset \langle 2^3 \rangle \supset \dots$ is infinite .Let $I_n = \langle 2^n \rangle$.That is $I(x) = \begin{cases} 1 & \text{if } x \in \bigcap I_n \\ 1-1/i_n & \text{if } x \notin \bigcap I_n \end{cases}$ where i_n is

the smallest i such that $x \notin I_n$.that is construction of every fuzzy ideal I of R has a sup property is a finite –valued .

Lemma 2.19 [9] :

Let I be an ideal of the ring R .If I and R/I are both Noetherian (Artinian) rings, then R is Noetherian (Artinian) ring .

Proposition 2.20 :

Let X be a fuzzy ring of R .Let I be an ideal of R such that I and R/I are both Noetherian . Then X is a Noetherian fuzzy ring .

Proof :

Since I is an ideal of R such that I and R/I are both Noetherian ,then R is a Noetherian ring by lemma (2.19).

Hence X is a Noetherian fuzzy ring by definition (2.4) .

Lemma 2.21 [11] :

Let R be a ring .Then R is Noetherian ring iff the set of values of any fuzzy ideal on R is a well ordered subset of $[0,1]$.

Proposition 2.22 :

Let X be a fuzzy ring of R . X is a Noetherian if and only if the set of values of any fuzzy ideal of X is a well ordered subset of $[0,1]$.

The proof of a above proposition is clear by lemma (2.21) .

Lemma 2.23 [9] :

If R is a Noetherian (Artinian) ring, then any homomorphism image of R is also Noetherian (Artinian).

Proposition 2.24 :

Let X be a fuzzy ring of R_1 and Y be a fuzzy ring of R_2 . Let $f: R_1 \rightarrow R_2$ be homomorphism function and onto .Then if X is Noetherian (Artinian),then Y is Noetherian (Artinian) .

Proof :

X is Noetherian (Artinian),by definition (2.4), R_1 is Noetherian (Artinian) .

Then R_2 is Noetherian (Artinian) by lemma (2.23) .

Hence Y is Noetherian (Artinian),by definition (2.4) .

Now ,we shall introduce the concept of a quotient fuzzy ring and discuss some of its properties of it.

Definition 2.25 [12] :

Let X be a fuzzy ring of R and A be a fuzzy ideal of R .

Define X/A_* : $R/A_* \rightarrow [0,1]$ such that :

$$X/A(a+A_*) = \begin{cases} 1 & \text{if } a \in A_* \\ \sup \{X(a+b)\} & \text{if } a \notin A_*, b \in A_* \end{cases}$$

For all $a+A_* \in R/A_*$, X/A is called a **quotient fuzzy ring of X by A** .

We will state the following properties and theorems as the following .

Lemma 2.26 [12] :

Let X be a fuzzy ring of R and A is a fuzzy ideal in X , then X/A is a fuzzy ring of R/A_* .

Lemma 2.27 [12] :

Let X be a fuzzy ring of R and A is a fuzzy ideal in X , then $X/A (0+A_*) = 1, [0+A_* = A_*]$.

Theorem 2.28 :

Let X be a fuzzy ring of R and let A be a fuzzy ideal of X .Then X is Artinian if and only if X/A is an Artinian fuzzy ring and for every fuzzy ideal B of R , $\text{Im}(B/A_*) = \{B(x) | x \in A_*\}$ is a finite subset of $[0,1]$.

Proof :

Suppose that X is an Artinian which means that R is an Artinian by definition (2.4) .

Then by [9] , R/A_* is Artinian and $\text{Im}(B/A_*)$ is a finite subset of $[0,1]$ for a fuzzy ideal A of R .

To prove the converse , if X/\mathbf{A}^* is an Artinian fuzzy ring and for every fuzzy ideal B of R , $\text{Im}(B/\mathbf{A}^*) = \{B(x)|x \in \mathbf{A}^*\}$ is a finite subset of $[0,1]$.

Now, the fuzzy subset C of R/\mathbf{A}^* defined by $C(a+\mathbf{A}^*) = \sup \{B(a+x) | x \in \mathbf{A}^*\}$ is a fuzzy ideal of R/\mathbf{A}^* .

Let $a \in R$, suppose $B(a) \neq B(x)$ for any $x \in \mathbf{A}^*$. Then either $B(a) > B(x)$ or $B(a) < B(x)$, $0 \neq x \in \mathbf{A}^*$. If $B(a) > B(x)$, $B(a+x) \geq \min \{B(a) , B(x)\} = B(x)$ and $B(x) = B(a+x-a) \geq \min \{B(a+x) , B(a)\} = B(a+x)$. Hence $B(a+x) = B(x)$.

If $B(a) < B(x)$, then we can show that $B(a+x) = B(a)$.

$$\begin{aligned} \text{Hence } C(a+\mathbf{A}^*) &= \sup \{B(a+x) | x \in \mathbf{A}^*\} = \sup \{B(a) , B(a+x) | x \in \mathbf{A}^* \setminus \{0\}\} \\ &= \sup \{B(a) , B(a) | x \in \mathbf{A}^* \setminus \{0\}\} = B(a) . \end{aligned}$$

Thus we find that $B(a) = B(x)$ for some $x \in \mathbf{A}^*$ or $B(a) = C(a+\mathbf{A}^*)$.

Since $B(\mathbf{A}^*)$ and $C(R/\mathbf{A}^*)$ are finite subsets of $[0,1]$, it follows that $\text{Im } B$ is a finite subset of $[0,1]$. By ([9] , [13]) , R is an Artinian ring .

Hence X is an Artinian fuzzy ring of R by definition (2.4).

Proposition 2.29 :

Let X be a fuzzy ring of R and let A be a fuzzy ideal of X . The quotient fuzzy ring of X by A is X/A . Then X is a Noetherian (Artinian) fuzzy ring if and only if X/A and \mathbf{A}^* are Noetherian (Artinian) .

Proof :

If X is a Noetherian (Artinian) fuzzy ring of R , then R is a Noetherian (Artinian) ring by definition (2.4) which implies that \mathbf{A}^* and R/\mathbf{A}^* are Noetherian (Artinian) ring by [14] .

Hence X/A is a Noetherian (Artinian) fuzzy ring of R/\mathbf{A}^* by definition (2.4).

Conversely , X/A is a Noetherian (Artinian) fuzzy ring of R/\mathbf{A}^* , then R/\mathbf{A}^* is a Noetherian (Artinian) ring and since \mathbf{A}^* is a Noetherian (Artinian) ideal implies that R is a Noetherian (Artinian) ring by [14] .

Thus X is a Noetherian (Artinian) fuzzy ring of R by definition (2.4) .

Definition 2.30 [12] :

Let X be a fuzzy ring of R_1 and Y be a fuzzy ring of R_2 . Let $f: R_1 \oplus R_2 \rightarrow [0,1]$ definite by $T(a,b) = \min \{X(a), Y(b)\}$ for all $(a,b) \in R_1 \oplus R_2$. T is called a **fuzzy external direct sum of denoted by $X \oplus Y$** .

We give some result about the fuzzy direct sum of X and Y .

Lemma 2.31 [13] :

If X and Y are fuzzy rings of R_1 and R_2 respectively , then $T = X \oplus Y$ is a fuzzy ring of $R_1 \oplus R_2$.

Theorem 2.32 :

Let X be a fuzzy ring of R_1 and Y be a fuzzy ring of R_2 . Then $T = X \oplus Y$ is a Noetherian (Artinian) fuzzy ring of $R_1 \oplus R_2$ if and only if X and Y are Noetherian (Artinian) fuzzy ring of R_1 and R_2 respectively .

Proof :

Firstly , if $T = X \oplus Y$ is a Noetherian (Artinian) fuzzy ring of $R_1 \oplus R_2$. To prove that X and Y are Noetherian (Artinian) fuzzy rings of R_1 and R_2 respectively .

$T = X \oplus Y$ is a Noetherian (Artinian) fuzzy ring of $R_1 \oplus R_2$, then $X_t = (X \oplus Y)_t$ is a Noetherian (Artinian) ring where t is minimal element of $\text{Im } T = \text{Im}(X \oplus Y)$, by corollary (2.11) .

Let t_1 is minimal element of $\text{Im } X$ and t_2 is minimal element of $\text{Im } Y$, then $t = \min \{ t_1 , t_2 \}$ where $t, t_1, t_2 \in [0,1]$.

$$T_t = (X \oplus Y)_t = X_t \oplus Y_t , \text{ by [12, Lemma (2.4.1.10)]} .$$

Since $T_t = R_1 \oplus R_2 = X_t \oplus Y_t$, then $R_1 = X_t$, $R_2 = Y_t$. Thus $R_1 \oplus R_2$ a Noetherian (Artinian) ring .By [14], R_1 and R_2 are Noetherian (Artinian) rings ,implies that X_t and Y_t are Noetherian (Artinian) rings Hence X and Y are Noetherian (Artinian) fuzzy rings of R_1 and R_2 respectively by corollary (2.11). Conversely ,if X and Y are Noetherian (Artinian) fuzzy rings of R_1 and R_2 respectively ,then X_{t_1} and Y_{t_2} are Noetherian (Artinian) rings where t_1 is minimal element of $\text{Im}X$ and t_2 is minimal element of $\text{Im}Y$ by corollary (2.11) .

But $X_{t_1} = R_1$ and $Y_{t_2} = R_2$, then $X_t = R_1$ and $Y_t = R_2$ where $t = \min \{ t_1, t_2 \}$, $t, t_1, t_2 \in [0,1]$.

By [14], $X_t \oplus Y_t = R_1 \oplus R_2$ is a Noetherian (Artinian) ring . But $T_t = (X \oplus Y)_t = X_t \oplus Y_t$ is a Noetherian (Artinian) ring of $R_1 \oplus R_2$ by [12, Lamme (2.4.1.10)].

Hence T is a Noetherian (Artinian) fuzzy ring of $R_1 \oplus R_2$ by corollary (2.11) .

References

- 1) Fuzzy Sets, Zadeh L.A., 1965, Information and Control, vol.8, pp.338 – 353.
- 2) Fuzzy Invariant Subgroups and Fuzzy Ideal, Liu W.J., 1982, Fuzzy Sets and Systems, vol.8, pp.133 – 139.
- 3) Factorization of Fuzzy Ideals in Dedekind Domains, Lee K.H. and Mordeson J.N., 1997, the journal of fuzzy mathematics, vol.5, No.3, pp.741 – 745.
- 4) On Fuzzy Ideals of a Ring I, Mukherjee T.K. and Sen M.K., 1987, Fuzzy Sets and Systems, vol.21, pp. 99 – 104.
- 5) Fuzzy Principal Ideals and Simple Field Extensions, AL- Khamees Y. and Mordeson J.N., 1998, Fuzzy Sets and Systems, vol.96, pp.247 – 253.
- 6) Some Results of Fuzzy Rings, Hussein W.R. 1999, M.Sc.Thesis, University of Baghdad, College of Education Ibn-AL-Haitham.
- 7) Fuzzy Ideals of Artinian Rings, Malik D.S., 1990, Fuzzy Sets and Systems, vol.37, pp. 111 – 115.
- 8) On Fuzzy Submodules and Their Radicals ,Mukherjee T.K. Sen M.K. and Roy D., 1996, the journal of fuzzy mathematics, vol.4, No.3, pp.549 – 558.
- 9) A First Course in Rings and Ideals, David M., 1970, Addison-Wesleg Publishing Company, London Don Mills, Ontario, University of New Hampshire.
- 10) Fuzzy Modules Over Fuzzy Rings in Connection with Fuzzy Ideals of Fuzzy Rings , Martines L., 1996, the journal of fuzzy mathematics, vol.4, No.3, pp.843 – 857.
- 11) Rings with Chain Conitions , Mukherjee T.K. and Sen M.K , 1991, Fuzzy Sets and Systems, vol.39 pp. 117 – 123 .
- 12) On Almost Quasi – Frobenius Fuzzy Rings, Hameed A.T., 2000, M.Sc.Thesis, University of Baghdad, College of Education Ibn-AL-Haitham.
- 13) Fuzzy Intersection Equations and Peimary Representations, Mordeson J.N., 1996, Fuzzy Sets and Systems, vol.83, pp.93 – 98.
- 14) Modern Algebra ,Quzi-Zameeruddin S.S., 1972, Vikas Publishing House Putted .