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SOME PROPERTIES OF HALL SUBGROUP

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ABSTRACT

In the present paper , the order and index of the normal abelian Hall subgroups have been studied through using of some functions defined on group rings . Also , some properties of homomorphism have been studied.

Introduction:

Let G a group and H , K a subgroups of finite index in a group G . Also S a subset of group G , then $|S|$ will denote the order of S , and $|G{:}H|$ will denote the index of H in group G , and H is Hall subgroup if G is a finite group and $|G{:}H|$ is relatively prime to H .

We prove in this paper the following corollary : (Let G be a group containing a normal abelian Hall subgroup A of order m and index n in G .Then there exists a subgroup U of order n in G it follows that $G=U\,A$ and $U \cap A=1$)

Definition 1 : [1]

If H is a subgroup of finite index in a group G , and K is a subgroup of G containing H , then K is of finite index in G , and

$$|G{:}H| = |G{:}H|\,|K{:}H| \ .$$

Definition 2: [1]

Let A and B be subgroups of group G . If B is of of finite index in G , then $A \cap B$ is a subgroup of finite index in A , and

$$|A : A \cap B| \leq |G : B| \ .$$

Equality holds if and only if $G = A\,B$.

In particular, if $|G : A|$ is also finite , then

$$|G : A \cap B| \leq |G : A|\,|G : B|$$

with equality if and only if $G = A\,B$.

Definition 3: [1]

If N and K are subgroups of group G , and N is normal in G , then $N\,K$ is a subgroup of G , and

$$N\,K \,/\, N \cong K \,/\, (\,N \cap K\,)$$

Proposition (1) : [1]

If A and B are subgroups of finite index in a group G , and $|G : A|$ and $|G : B|$ are elatively prime, then $G = A\,B$.

Prove : By **(Definition 1)** , $|A : A \cap B|$ is divisible by both $|G : A|$ and $|G : B|$, and so by their least common multiple. Since $|G : A|$ and $|G : B|$ are relatively prime , their least common multiple is $|G : A| \cdot |G : B|$, and so this is at most $|G : A \cap B|$. The result now follows from **(Definition 2)** .

Proposition (2) : [2]

If A , B and C are subgroups of group G , and $A \subseteq C$, then

$$A\,B \cap C = A(\,B \cap C\,) \ .$$

Note : $A\,B$ is not necessarily a subgroup of G .

Prove : Let $a\,c \in A(\,B \cap C\,)$, where $a \in A$, and $c \in (\,B \cap C\,)$. Then $a\,c \in A\,B$, and $a\,c \in a\,C = C$. There fore

$$A(\,B \cap C\,) \subseteq A\,B \cap C \ .$$

On the other hand , if $a\,b \in A\,B \cap C$, where $a \in A$ and $b \in B$, then

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$b \in a^{-1} C = C$, and so

$$a\,b \in A(B \cap C) \;.$$

Thus $A\,B \cap C \subseteq A(B \cap C)$, and the result follows .

Definition 4: [2]

Let G be an arbitrary group, and let Ω be a set for which, for each $\alpha \in \Omega$ and each $x \in G$, we have defined an element $\alpha^x \in \Omega$ with the properties :

(a) The mapping $\bar{x} : \alpha \to \alpha^x$ is a permutation of the set Ω for each $x \in G$; and

(b) $\bar{x}\,\bar{y} = \overline{xy}$ for all $x , y \in G$. Then for each $\alpha \in \Omega$ the set

$$\alpha^G = \{ \alpha^x \mid x \in G \} \subseteq \Omega$$

is called the ***orbit*** (or transitivity set) of α , and the number of letters which α^G contains is the ***length*** of the orbit. The set

$$G_\alpha = \{ x \in G \mid \alpha^x = \alpha \} \subseteq G$$

is called the ***stabilizer*** (or stability subgroup) of α .

Proposition (3) : [3]

The mapping $x \to \bar{x} ; (x \in G)$, where $\bar{x}$ is defined in **Definition** 2 (a)) above defineds a homomorphism of G into S_Ω . The kernel of the homomorphism is $\bigcap_{\alpha \in \Omega} G_\alpha$.

Prove: The mapping is into S_Ω by (Def.(2) (a)) , and a homomorphism is

$$\{ x \in G \mid \alpha^x = \alpha \text{ for all } \alpha \in \Omega \} =$$
$$= \bigcap_{\alpha \in \Omega} G_\alpha \;.$$

Proposition (4) : [2]

Let H be a subgroup of a group G . We define Ω to be the set of all right cosets $H\,a\,(a \in G)$, and define : $(Ha)^x = H a\, x \;(H a , H a x \in \Omega ;\; x \in G)$

Then the stabilizer of $H\,a$ is $a^{-1} H\,a$, and the kernel K of the homomorphism define in **(Proposition (3))** is $\bigcap_{a \in G} a^{-1} H\,a$, which is the largest subgroup of H normal in G .

Prove : The stabilizer of $H\,a$ is

$$\{ y \in G \mid H\,a\,y = H\,a \} =$$
$$= \{ y \in G \mid y \in a^{-1} H\,a \} =$$
$$= a^{-1} H\,a \;.$$

Hence , by **(Proposition(3))** the kernel of the homomorphism is as described . If $N \subseteq H$, and N is a normal subgroup of G , then

$$N = a^{-1} N\,a \subseteq a^{-1} H\,a$$

for all $a \in G$, and so $N \subseteq K$ as required .

Proposition (5) : [2]

If G is a finite group , and n is positive integer relatively prime to the order of G , then for each $x \in G$, there is a unique $y \in G$ such that $y^n = x$. In particular , if $y^n = z^n$ for two elements y and x in G , then $y = z$.

Prove: We first show that , if $y, z \in G$ and $y^n = z^n$ then $y = z$.

Let $m = |G|$ since m and n are relatively prime, there exist integers s and t such that $m s + n t = 1$. Then

$$y = y^{m s + n t} = y^{n t} = z^{n t} =$$
$$= z^{m s + n t} = z$$

because the order of y and z both divde m .

It now follows that the set $\{ y^n \mid y \in G \}$ contains $|G|$ distinct elements of G , and so comprises the whole of G . Thus $x = y^n$ for some unique y in G .

Let C denote the field of complex numbers. Let G be a group, and consider the set R_G of all formal sums: $\sum_{x \in G} \alpha_x\, x (\alpha_x \in C)$ in which all but a finite number of coefficients α_x are zero. We define addition and multiplication in R_G by

$$\left(\sum_{x \in G} \alpha_x\, x \right) + \left(\sum_{x \in G} \beta_x\, x \right) =$$
$$= \sum_{x \in G} (\alpha_x + \beta_x)\, x$$

And

$$\left(\sum_{x \in G} \alpha_x\, x \right)\left(\sum_{x \in G} \beta_x\, x \right) =$$

$= \sum_{x \in G} \gamma_x \, x$,

Where $\gamma_x = \sum_{z \in G} \alpha_{xz^{-1}} \, \beta_z$ (Note That γ_x is a finite sum of elements in C because β_z is zero for all but a finite number of $z \in G$).

Definition 5: [3]

An element $\sum_{x \in G} \alpha_x x$ in R_G , which, for some $u \in G$, has $\alpha_u = 1$ and $\alpha_x = 0$ for $x \neq u$, is written as u and is said to be an element of R_G lying in G . It is readily shown that R_G is an associative ring with unity element 1 (the identity of G) , and that R_G is commutative if and only if G is abelian. We call R_G the ***group ring*** of G (over C).

Proposition (6) : [5]

Let G be a finite group of order $m\,n$, where m is relatively prime to n . Let A be normal abelian subgroup of order m , and let H and K be sub groups of order n in G . Then there is an isomorphism θ of H on to K such that :

$$A x = A x^{\theta} \; (x \in H \; ; x^{\theta} \in K).$$

Moreover, for some $c \in A$,

$c\,x\,c^{-1} = x^{\theta}$ for all $x \in H$ .Thus, H is conjugate to K in G .

Prove : Since m is relatively prime to n , $G = A H = A K$,by **(Proposition (1))** and : $A \cap H = A \cap K = 1$. Thus H and K are each complete sets of coset representatives for A in G . Therefore we can define a one-to-one mapping θ of H onto K by the condition

$A x = A x^{\theta} \; (x \in H \; ; x^{\theta} \in K)$. Since

$$\begin{aligned} A(xy)^{\theta} &= A\,x\,y = \\ &= (A x)(A y) = \\ &= (A x^{\theta})(A y^{\theta}) = \\ &= A(x^{\theta} y^{\theta}) \end{aligned}$$

for any $x, y \in H$, it follows that θ is an isomorphism of H onto K .

We now construct c . Since $u^{\theta} u^{-1} \in A$ for all $u \in H$, and A is abelian , we can define $b \in A$ by

$$b = \prod_{u \in H} u^{\theta} u^{-1}$$

For all $x \in H$, we have

$$x b x^{-1} = \prod_{u \in H} \left\{ x (x^{\theta})^{-1} (xu)^{\theta} (xu)^{-1} \right\} =$$

$$= \left\{ x (x^{\theta})^{-1} \right\}^{n} \prod_{u \in H} \left\{ (xu)^{\theta} (xu)^{-1} \right\} =$$

$$= \left\{ x (x^{\theta})^{-1} \right\}^{n} b \; .$$

Because m is relatively prime to n , we can use **(Proposition(5))** to find $c \in A$ such that $b = c^{n}$ Then

$$(x\,c\,x^{-1})^{n} = x\,c^{n}\,x^{-1} = \\ = \left\{ x (x^{\theta})^{-1} c \right\}^{n} \; .$$

Thus by **(Proposition (5))**,

$x\,c^{-1}\,x^{-1} = x (x^{\theta})^{-1} c$; that is ,

$x^{\theta} = c\,x\,c^{-1}$. In particular,

$K = c\,H\,c^{-1}$.

Definition 6: [4]

The set $M(n, G)$ of all $n \times n$ monomial matrices over G is a group in which $D(n, G)$ is a normal subgroup , Moreover ,

$M(n, G) = S(n)\, D(n, G)$ and $S(n) \cap D(n, G) = 1$.

Definition 7: [2]

Let G be a group with a subgroup H of finite index n in G . Let θ be a homomorphism of H into a group S . Then we define $\breve{\theta}$ as a function of G into the group ring of S by:

$$\breve{\theta}(x) = \begin{cases} \theta(x) & if \;\; x \in H \\ 0 & otherwise \end{cases}$$

Definition 8:

Let $r_1, r_2, \ldots, r_n$ be a set of left coset representatives for H in G . We shall now define the ***monomial representation*** θ^{G} of G induced from θ over the given set of coset representatives. For each $x \in G$, We define.

$$\theta^G(x) = \begin{pmatrix} \breve{\theta}(r_1^{-1} x r_1) & \breve{\theta}(r_1^{-1} x r_2) \cdots \breve{\theta}(r_1^{-1} x r_n) \\ \breve{\theta}(r_2^{-1} x r_1) & \breve{\theta}(r_2^{-1} x r_2) \cdots \breve{\theta}(r_2^{-1} x r_n) \\ \vdots & \\ \breve{\theta}(r_n^{-1} x r_1) & \breve{\theta}(r_n^{-1} x r_2) \cdots \breve{\theta}(r_n^{-1} x r_n) \end{pmatrix}$$

$$= \left[\breve{\theta} \left(r_i^{-1} x r_j \right) \right] \quad \text{........(*)}$$

The matrix $\theta^G(x)$ lies in $M(n, S)$

Proposition (7) : [4]

Let θ^G be the function of the group G into $M(n, S)$ as defined by (*). Then θ^G is a homomorphism of G into $M(n, S)$. If the kernel of θ is N , then the kernel of θ^G is $\bigcap_{x \in G} x^{-1} N x$

Prove : For any $x, y \in G$, We have

$$\theta^G(x) \ \theta^G(y) =$$
$$= \left[\breve{\theta}(r_i^{-1} x r_j) \right] \left[\breve{\theta}(r_i^{-1} y r_j) \right] =$$
$$= \sum_{k=1}^{n} \breve{\theta}(r_i^{-1} x r_k) \ \breve{\theta}(r_k^{-1} y r_j)$$

But $\breve{\theta}(r_i^{-1} x r_k) \ \breve{\theta}(r_k^{-1} y r_j)$ is nonzero only if we have both $(r_i^{-1} x r_k)$ and $(r_k^{-1} y r_j)$ lying in H that is, both $(x^{-1} r_i)$ and $(y r_j)$ lying in the same coset $r_k H$. For given i and j there is at most one such k . There is such a k exactly when $(x^{-1} r_i)^{-1}(y r_j) = r_i^{-1} x y r_j$ lies in H . Thus

$$\sum_{k=1}^{n} \breve{\theta}(r_i^{-1} x r_k) \ \breve{\theta}(r_k^{-1} y r_j) =$$
$$= \breve{\theta}(r_i^{-1} x y \ r_k) \ .$$

Hence $\theta^G(x) \ \theta^G(y) = \theta^G(x y)$ for all $x, y \in G$, and so θ^G is a homomorphism . Finally,

$\theta^G(x) = diag(1, 1, \ldots, 1)$ if and only if $(r_i^{-1} x r_i) \in N$ for each i . Thus the kernel of θ^G is

$$\bigcap_{i=1}^{n} r_i N r_i^{-1} = \bigcap_{x \in G} x^{-1} N x$$

Definition 9: [4]

Let G be a group , H a subgroup of G , and S a subset of G . Then $|S|$ will denote the order of S , and $|G : H|$ will denote the index of H in G . If G is a finite group, then H is a ***Hall subgroup*** of G if $|G : H|$ is relatively prime to $|H|$.

Proposition (8) : [5]

Let G be a group possessing a normal abelian Hall subgroup A of order m and index n in G . If H is a Hall subgroup of order n in G , and K is a subgroup of G such that n divides $|K|$, then for some $x \in G$, $x^{-1} H x \subseteq K$.

Prove : The subgroup $N = K \bigcap A$ is normalized by both A and K , and so it is a normal subgroup of $G = K A$. The group G/N contains two Hall subgroup K/N and $H N/N$ of order n , using **(Definition (3))**, and an abelian normal subgroup A/N of index n . Therefore, by **(Proposition (6))** , there is $x \in G$ such that

$$x^{-1} H x \subseteq x^{-1} H N x = K$$

Corollary :

Let G be a group containing a normal abelian Hall subgroup A of order m and index n in G . Then there exists a subgroup U of order n in G it follows that $G = U A$ and $U \bigcap A = 1$

Prove:

Let θ be the identity homomorphism of A onto it self ; that is $\theta(a) = a$ for all $a \in A$. Writing $G^* = \theta^G(G)$ and $A^* = \theta^G(A)$,we have $G^* \cong G$ and $A^* \cong A$, by **(Proposition (7))** . Now

$$P = S(n) \bigcap G^* D(n, A)$$

is a group of permutation matrices , and

$$P\,D(n, A) =$$
$$= \{ S(n) \cap G^* D(n, A) \} D(n, A)$$
$$= S(n)\, D(n, A) \cap G^* D(n, A)$$
$$= G^* D(n, A)$$

Using **(Proposition (2))** and **(Definition (6))**. Therefore

$$| P | = | P : P \cap D(n, A) | =$$
$$= | P\,D(n, A) : D(n, A) | =$$
$$= | G^* : G^* \cap D(n, A) | =$$
$$= | G^* : A^* | = n$$

Thus P is a Hall subgroup of $P\,D(n, A)$. Since $D(n, A)$ is abelian and of order $|A|^n$, and n divides $|G^*|$, it follows from **(Proposition (8))** that G^* contains a subgroup of order n conjugate to P in $G^* D(n, A)$. Thus G, which is isomorphic to G^*, has a subgroup of order n as asserted.

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بعض خصائص الزمر الجزئية (من النمط– هول)

مثنى عبدالواحد محمود

الخلاصة:

درسنا في بحثنا هذا رتبة ودليل الزمر الجزئية (من النمط – هول) الابدالية السوية من خلال استخدامنا لبعض الدوال المعرفة على الزمر الحلقية، وايضاً باستخدام بعض صفات الهومومورفيزم، ونتائج هذا البحث تتمثل بالنتيجة الاخيرة.